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ORIGINAL ARTICLE

Several new inequalities on operator means of non-negative maps and Khatri–Rao products of positive definite matrices

Zeyad Abdel Aziz Al-Zhour *

Department of Basic Sciences and Humanities, College of Engineering, University of Dammam, P.O. Box 1982, Dammam 34151, Saudi Arabia

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Abstract In this paper, we provide some interested operator inequalities related with non-negative linear maps by means of concavity and convexity structure, and also establish some new attractive inequalities for the Khatri–Rao products of two or more positive definite matrices. These results lead to inequalities for Hadamard product and Ando's and α-power geometric means, as a special case.

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1. Introduction

The geometric mean of two or more positive (semi) definite matrices arises naturally in several areas such as in Electrical Network Theory, Statistics, Engineering, and many fields of pure and applied Mathematics; and it has several properties (equalities and inequalities) of the geometric mean of positive scalars (Ando and Hiai, 1998; Bhatia and Kittaneh, 2000; Xiao and Zhang, 2003). Let R^+ be positive real numbers and for every $x,y \in R^+$, then the function $M{:}R^+ \times R^+ \to R^+$ is said to be a *mean* if the following properties hold:

(i) $M(x,x) = x$ (1-1)
(ii) $M(x,y) = M(y,x)$ (1-2)
(iii) If $x < y$, then $x < M(x,y) < y$ (1-3)
(iv) If $x_1 < x_2$ and $y_1 < y_2$, then $M(x_1,y_1) < M(x_2,y_2)$ (1-4)
(v) $M(x,y)$ is continuous. (1-5)
(vi) If $k \in R^+$, then $M(kx,ky) = kM(x,y)$. (1-6)

For positive real numbers x and y, the geometric mean $G(x,y) = \sqrt{xy}$, the arithmetic mean $A(x,y) = \frac{x+y}{2}$ and the harmonic mean $H(x,y) = \frac{x^{-1}+y^{-1}}{2}$ are the familiar means and sometimes called the Pythagorean means. Note that there are many other means for two or more positive numbers as well, such as the logarithmic mean, power mean, Identric mean, Horn mean, generalizations of power mean, generalizations of Horn means,Young means , Heinz mean, binomial means, Lehmer means, power difference means, Stolarsky means, Heron means, Karcher Mean and Geometric Bonferroni mean, (Alic et al., 1997; Ando, 1983; Ando et al., 2004; Fiedler and Ptak, 1997; Furuichi et al., 2005; Furuta, 2006; Mond and Pečarić,

* Tel.: +966 595884164.
E-mail addresses: zeyad1968@yahoo.com, zalzhour@ud.edu.sa.
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1997; Mond et al., 1996; Qi and Guo, 2003; Sagae and Tanabe, 1994; Xiao and Zhang, 2003; Lim and Palfia, 2012; Lim and Yamazaki, 2013; Xia et al., 2013; Bhatia and Kosaki, 2007).

Before starting on the geometric means of positive definite matrices, we need to study some important basic concepts and results on matrices. Let us first introduce the definitions of Kronecker, Hadamard, Tracy–Singh and Khatri–Rao products of matrices which are defined, respectively, by (Al-Zhour and Kilicman, 2006; Al-Zhour, 2012; Cao et al., 2002; Kilicman and Al-Zhour, 2005; Liu, 2002; Liu, 1999; Zhang, 1999).

(i) $A \otimes B = (a_{ij}B)_{ij}$ (1-7)

(ii) $A \circ C = (a_{ij}c_{ij})_{ij} = C \circ A$ (1-8)

(iii) $A\Theta B = (A_{ij}\Theta B)_{ij} = ((A_{ij} \otimes B_{kl})_{kl})_{ij}$ (1-9)

(iv) $A * B = (A_{ij} \otimes B_{ij})_{ij}$ (1-10)

where $A = [a_{ij}]$ and $C = [c_{ij}]$ are matrices of order $m \times n\left(m = \sum_{i=1}^{t} m_i, n = \sum_{j=1}^{c} n_j\right)$ and $B = [b_{kl}]$ is a matrix of order $p \times q\left(p = \sum_{i=1}^{t} p_i, q = \sum_{j=1}^{c} q_j\right)$; and $A = [A_{ij}], B = [B_{kl}]$ are partitioned matrices (where A_{ij} and B_{kl} are sub-matrices of order $m_i \times n_j$ and $p_k \times q_l$, respectively).

Note that if A and B are non-partitioned matrices, then $A\Theta B$ is reduced to $A \otimes B$ and $A*B$ is reduced to $A \circ B$(Liu, 1999).

Let A and B be Hermitian matrices, then the relation $A > B$ means that $A - B > 0$ is a positive definite matrix and the relation $A \geqslant B$ means $A - B \geqslant 0$ is a positive semi-definite matrix. If $A > 0$, then $A^{1/2}$ is called the positive definite square root of A. Zhang (1999) showed that if $A > 0$ and $B > 0$, then the relation $A \geqslant B$ implies $A^{-1} \leqslant B^{-1}, A^2 \geqslant B^2$ and $A^{1/2} \geqslant B^{1/2}$.

Here the symbol $M_{m,n}$ stands to the set of all $m \times n$ matrices over the field M and when $m = n$, we write M_m instead of $M_{m,n}$. The symbols A^T, A^*, A^{-1} stand to, respectively, the transpose, conjugate transpose and inverse of matrix A. The Symbols H_n and H_n^+ are, respectively, the space of n-square Hermitian and n-square positive definite matrices. The linear map φ from H_n to H_m is said to be *positive* if it transforms H_n^+ to H_m^+. The positive linear map φ is said to be *unital* or *normalized* if it transforms the identity I_n to the identity matrix I_m and *monotone* if $A \leqslant B$ implies $\varphi(A) \leqslant \varphi(B)$. For more details, see (Ando, 1979).

The following formula is very important for getting our results which is studied by many researchers (Al-Zhour and Kilicman, 2006; Al-Zhour, 2012; Cao et al., 2002; Liu, 2002; Liu, 1999; Zhang, 1999) :

$$\prod_{i=1}^{k} * A_i = Z_1^T \left(\prod_{i=1}^{k} \Theta A_i \right) Z_2, \quad (1\text{-}11)$$

where $A_i \in M_{m(i),n(i)} (1 \leqslant i \leqslant k, k \geqslant 2)$ are compatibly partitioned matrices, $(m = \prod_{i=1}^{k} m(i)$ and $n = \prod_{i=1}^{k} n(i)$, $r = \sum_{j=1}^{t} \prod_{i=1}^{k} m_j(i), s = \sum_{j=1}^{t} \prod_{i=1}^{k} n_j(i), m(i) = \sum_{j=1}^{t} m_j(i), \quad n(i) = \sum_{j=1}^{t} n_j(i))$, Z_1 and Z_2 are real matrices with entries zeros and ones of order $m \times r$ and $n \times s$, respectively such that $Z_1^T Z_1 = I_r, Z_2^T Z_2 = I_s$, where I_r and I_s are identity matrices of order $r \times r$ and $s \times s$, respectively.

In particular, if $m(i) = n(i)$, then there exists $m \times r\left(m = \prod_{i=1}^{k} m(i), r = \sum_{j=1}^{t} \prod_{i=1}^{k} m_j(i)\right)$ matrix Z of zeros and ones such that $Z^T Z = I_r$, and

$$\prod_{i=1}^{k} * A_i = Z^T \left(\prod_{i=1}^{k} \Theta A_i \right) Z. \quad (1\text{-}12)$$

Let A_i and $B_i (1 \leqslant i \leqslant k, k \geqslant 2)$ be compatibly partitioned matrices, then (Al-Zhour and Kilicman, 2006; Al-Zhour, 2012; Liu, 2002; Liu, 1999) :

(i) $\left(\prod_{i=1}^{k} \Theta A_i \right) \left(\prod_{i=1}^{k} \Theta B_i \right) = \left(\prod_{i=1}^{k} \Theta (A_i B_i) \right)$ (1-13)

(ii) $\left(\prod_{i=1}^{k} \Theta A_i \right)^* = \prod_{i=1}^{k} \Theta A_i^*$ and $\left(\prod_{i=1}^{k} * A_i \right)^* = \prod_{i=1}^{k} * A_i^*$ (1-14)

(iii) If A_i are positive (semi) definite matrices and r any real number, then

$$\left(\prod_{i=1}^{k} \Theta A_i \right)^r = \prod_{i=1}^{k} \Theta A_i^r \quad (1\text{-}15)$$

(iv) $\left(\prod_{i=1}^{k} (A_i \Theta B_i) \right) = \left(\prod_{i=1}^{k} A_i \right) \Theta \left(\prod_{i=1}^{k} B_i \right)$ (1-16)

Now, let us study some means on matrices. Let A and $B \in M_n$, then the *arithmetic mean* is defined as follows (see, e.g., Ando, 1979; Alic et al., 1997):

$$A \sim B = \frac{1}{2}(A + B). \quad (1\text{-}17)$$

Similarly, when A and $B > 0$ of order $n \times n$, then the *harmonic mean* is given by (Ando et al., 2004; Beesack and Pečarić, 1985; Bhatia and Kittaneh, 2000; Cao et al., 2002; Furuichi et al., 2005; Furuta, 2006; Fiedler and Ptak, 1997) :

$$A!B = \left\{ \frac{1}{2}(A^{-1} + B^{-1}) \right\}^{-1} \quad (1\text{-}18)$$

Researchers have tried to define a geometric mean on two or more positive definite matrices, but there is still no satisfactory definition because the *geometric mean* $A\#B$ of two positive $n \times n$ matrices A and B should satisfy at least the desirable properties (i)–(viii) that mentioned in (Kilicman and Al-Zhour, 2005), which are, respectively: commutative property, positive property, symmetry property, arithmetic-geometric-harmonic inequality, distributive property, mixed property, inverse property and eigenvalue property. For example, Kilicman and Al-Zhour (2005) discussed a family of candidates of geometric means of positive definite matrices and proved that all considered definitions failed to satisfy at least one of the desirable properties that are mentioned above. Ando (1979) defined the geometric mean for two positive $n \times n$ matrices A and B as follows:

$$A\#B = A^{1/2} D^{1/2} A^{1/2} : \quad D = A^{-1/2} B A^{-1/2}, \quad (1\text{-}19)$$

which is called *Ando's geometric mean* and satisfied the first seven properties that are mentioned in (Kilicman and Al-Zhour, 2005) and many other desirable properties such as:

(a) $A\#A = A$ (1-20)

(b) $A^p \# A^q = A^{(p+q)/2}$, for all $-\infty < p, q < \infty$ (1-21)

(c) $(A\#B)A^{-1}(A\#B) = B$ (1-22)

(d) $(AB^{-1}A)\#B = A$ (1-23)

(e) $A^{-1/2}(A\#B)B^{-1/2}$ is a unitary matrix. (1-24)

Ando and Hiai (1998) generalized Ando's geometric mean to the α -*power mean* that still satisfied properties from (i) to (vii) that are mentioned in (Kilicman and Al-Zhour, 2005) as follows:

$$A \underset{\alpha}{\#} B = A^{1/2} D^{\alpha} A^{1/2} : \quad D = A^{-1/2} B A^{-1/2}, \tag{1-25}$$

where α is any real number; and A and B are positive definite matrices. This definition also satisfies the following new properties:

$$(a) \quad A \underset{\alpha}{\#} A = A \tag{1-26}$$

$$(b) \quad A^{p} \underset{\alpha}{\#} A^{q} = A^{(1-\alpha)p+\alpha q}, \text{ for all } -\infty < p, q < \infty. \tag{1-27}$$

Micic et al. (2000) also generalized the α-power mean to the *operator mean* as follows:

$$A \sigma B = A^{1/2} f(D) A^{1/2} : \quad D = A^{-1/2} B A^{-1/2}, . \tag{1-28}$$

where $f(t)$ is any non-negative operator monotone function on $[0,\infty)$ and A and B are positive definite matrices. In fact, the α-power means are determined by the operator monotone function $f(t) = t^{\alpha}$ when $0 < \alpha \leqslant 1$ or by the operator monotone function $f(t) = t^{1/\alpha}$ when$1 \leqslant \alpha < \infty$.

Ando et al. (2004) found other desirable properties that should be required for a reasonable geometric mean of three positive definite matrices.

Hu et al. (2005) presented several kinds of mixed means for three or more positive definite matrices, and proved some related mixed mean inequalities. Lim (2008) described the maximal and minimal positive definite solutions of the non-linear matrix equation $X = T - BX^{-1}B$ in terms of Ando's geometric mean $A\#B$.

Jung et al. (2009) established some new properties of α-power mean and used this mean in the solution of non-linear matrix equation $X^{n} = f(X)$.

Recently, Lee et al. (2011) defined a family of weighted geometric means of n-tuples positive definite matrices and showed that these weighted geometric means satisfied multidimensional versions of all properties that one would expect of a two-variable weighted geometric mean. Fujii et al. (2010) presented the Cauchy–Schwraz and Holder inequalities involving geometric means of positive definite matrices. Kim et al. (2011) defined a new family of matrix means such as a resolvent mean which is defined of m positive definite matrices $\mathrm{A} = (A_1, A_2, \ldots, A_m)$ with weight vector $\omega = (w_1, w_2, \ldots, w_m)$ as follows:

$$\mathfrak{R}_{\mu}(\mathrm{A}, \omega) = \left[\sum_{i=1}^{m} w_i (A_i + \mu I)^{-1} \right]^{-1} - \mu I, \quad \mu \geqslant 0$$

and this mean satisfies several desirable properties that are mentioned in (Kim et al., 2011). Note that for $\mu = \infty$, the resolvent mean is the weighted arithmetic mean.

Ito et al. (2011) described some geometric properties of positive definite matrices cone with respect to the Thompson metric. More Recently, Lim (2012) introduced a new class of (metric) geometric means of positive definite matrices varying over Hermitian unitary matrices and gave some basic properties comparable to those geometric means. Finally, Bhatia and Grover (2012) presented the norm inequalities related to the geometric mean of positive definite matrices.

Here in this paper, we recover Ando's geometric mean to the case of operator mean and derive some desirable properties which play a central role for establishing our results. Several inequalities related to operator means and Khatri–Rao products are established by applying concavity and convexity structures. Finally, the results lead to inequalities for Hadamard Product, and Ando's and α-power geometric means, as a special case.

2. Further properties and connections

In this section, we study some interested properties and connections which are very important to obtain our results in next section.

Lemma 2.1. *Let $A_i > 0$ and $B_i > 0 (i = 1,2)$ be $n \times n$ compatible partitioned matrices. Then for any real number α,*

$$\text{(i)} \quad (A_1 \Theta B_1) \underset{\alpha}{\#} (A_2 \Theta B_2) \quad (A_1 \underset{\alpha}{\#} A_2) \Theta (B_1 \underset{\alpha}{\#} B_2) \tag{2-1}$$

$$\text{(ii)} \quad (A_1 \underset{\alpha}{\#} B_1) \Theta (A_2 \underset{\alpha}{\#} B_2) \quad (A_1 \Theta A_2) \underset{\alpha}{\#} (B_1 \Theta B_2). \tag{2-2}$$

Proof. (i) In order to see if this indeed is true, let $D_1 = A_1^{-1/2} A_2 A_1^{-1/2}$ and $D_2 = B_1^{-1/2} B_2 B_1^{-1/2}$. Then

$$\begin{aligned} (A_1 \Theta B_1) \underset{\alpha}{\#} (A_2 \Theta B_2) &= (A_1 \Theta B_1)^{1/2} ((A_1 \Theta B_1)^{-1/2} (A_2 \Theta B_2) (A_1 \Theta B_1)^{-1/2})^{\alpha} (A_1 \Theta B_1)^{1/2} \\ &= \left(A_1^{1/2} \Theta B_1^{1/2}\right) \left(\left(A_1^{-1/2} \Theta B_1^{-1/2}\right) (A_2 \Theta B_2) \left(A_1^{-1/2} \Theta B_1^{-1/2}\right) \right)^{\alpha} \left(A_1^{1/2} \Theta B_1^{1/2}\right) \\ &= \left(A_1^{1/2} \Theta B_1^{1/2}\right) \left\{ \left(A_1^{-1/2} A_2 A_1^{-1/2}\right)^{\alpha} \Theta \left(B_1^{-1/2} B_2 B_1^{-1/2}\right)^{\alpha} \right\} \left(A_1^{1/2} \Theta B_1^{1/2}\right) \\ &= \left(A_1^{1/2} \Theta B_1^{1/2}\right) (D_1^{\alpha} \Theta D_2^{\alpha}) \left(A_1^{1/2} \Theta B_1^{1/2}\right) \\ &= \left(A_1^{1/2} D_1^{\alpha} A_1^{1/2}\right) \Theta \left(B_1^{1/2} D_2^{\alpha} B_1^{1/2}\right) \\ &= (A_1 \underset{\alpha}{\#} A_2) \Theta (B_1 \underset{\alpha}{\#} B_2). \end{aligned}$$

Similarly, we can prove part (ii).

Theorem 2.2. *Let $A_i > 0 \quad (i = 1,2)$ be $n \times n$ compatible partitioned matrices. Then*

$$(A_1 \Theta A_2)^{p} \underset{\alpha}{\#} (A_1 \Theta A_2)^{q} = (A_1 \Theta A_2)^{(1-\alpha)p+\alpha q}, \tag{2-3}$$

where α is any real number and for all $-\infty < p,q < \infty$.

Proof. Due to Lemma 2.1 and 1-27, we have

$$\begin{aligned} (A_1 \Theta A_2)^{p} \underset{\alpha}{\#} (A_1 \Theta A_2)^{q} &= (A_1^{p} \Theta A_2^{p}) \underset{\alpha}{\#} (A_1^{q} \Theta A_2^{q}) = \left(A_1^{p} \underset{\alpha}{\#} A_1^{q}\right) \Theta \left(A_2^{p} \underset{\alpha}{\#} A_2^{q}\right) \\ &= A_1^{(1-\alpha)p+\alpha q} \Theta A_2^{(1-\alpha)p+\alpha q} = (A_1 \Theta A_2)^{(1-\alpha)p+\alpha q}. \end{aligned}$$

Theorem 2.3. *Let $A_i > 0 (1 \leqslant i \leqslant k, k \geqslant 2)$ be $n \times n$ compatible partitioned matrices. Then*

$$\left(\prod_{i=1}^{k} \Theta A_i^{p} \right) \underset{\alpha}{\#} \left(\prod_{i=1}^{k} \Theta A_i^{q} \right) = \left(\prod_{i=1}^{k} \Theta A_i^{(1-\alpha)p+\alpha q} \right), \tag{2-4}$$

where α is any real number and for all $-\infty < p,q < \infty$.

Proof. The proof is straightforward by using Theorem 2.2 and induction on k.

Theorem 2.4. *Let $A_i >$ and $B_i > 0 (1 \leqslant i \leqslant k, k \geqslant 2)$ be $n \times n$ compatible partitioned matrices. Then*

(i) $$\prod_{i=1}^{k}\#_{\alpha}(A_i\Theta B_i)=\left(\prod_{i=1}^{k}\#_{\alpha}A_i\right)\Theta\left(\prod_{i=1}^{k}\#_{\alpha}B_i\right).\quad (2\text{-}5)$$

(ii) $$\prod_{i=1}^{k}\Theta(A_i\#_{\alpha}B_i)=\left(\prod_{i=1}^{k}\Theta A_i\right)\#_{\alpha}\left(\prod_{i=1}^{k}\Theta B_i\right).\quad (2\text{-}6)$$

Proof. The proof follows immediately by induction on k.

Theorem 2.5. *Let $A_i > 0$ and $B_i > 0$ $(1 \leqslant i \leqslant k, k \geqslant 2)$ be $n\times n$ compatible partitioned matrices and let $f(t)$ be a non-negative operator monotone function on $[0,\infty)$ such that $f\left(\prod_{i=1}^{k}\Theta D_i\right)=\prod_{i=1}^{k}\Theta f(D_i)$ for any matrices D_i $(1 \leqslant i \leqslant k, k \geqslant 2)$. Then*

(i) $$\prod_{i=1}^{k}\sigma(A_i\Theta B_i)=\left(\prod_{i=1}^{k}\sigma A_i\right)\Theta\left(\prod_{i=1}^{k}\sigma B_i\right).\quad (2\text{-}7)$$

(ii) $$\prod_{i=1}^{k}\Theta(A_i\sigma B_i)=\left(\prod_{i=1}^{k}\Theta A_i\right)\sigma\left(\prod_{i=1}^{k}\Theta B_i\right).\quad (2\text{-}8)$$

Proof. The proof is straightforward by induction on k and Eqs. 1-13,1-15 and 1-16.

3. Several types of inequalities on operator means and Khatri–Rao products

For many years mathematicians have been interested in inequalities involving geometric means of positive semi-definite matrices (Ando, 1983; Ando, 1979; Ando et al., 2004; Hu et al., 2005; Furuichi et al., 2005; Furuta, 2006; Hernandez et al., 2001; Kilicman and Al-Zhour, 2005; Micic et al., 2000; Mond et al., 1996; Qi and Guo, 2003; Sagae and Tanabe, 1994; Satnoianu, 2002; Xiao and Zhang, 2003, Lim and Yamazaki, 2013; Fujii et al., 2010; Bhatia and Grover, 2012). In this section, we present many attractive inequalities involving geometric means and Khatri–Rao products of positive definite matrices based on the properties of convexity and concavity structures.

Definition 3.1. Let $A_i, B_i \in H_{n_i}^{+}(i=1,2,\ldots,k)$ and $0 < \lambda < 1$. Then the map φ from $H_{n_1}^{+}\times\cdots\times H_{n_k}^{+}$ to H_m is said to be :

(i) *Convex* if

$$\varphi(\lambda A_1+(1-\lambda)B_1,\ldots,\lambda A_k+(1-\lambda)B_k)\leqslant\lambda\varphi(A_1,\ldots,A_k)+(1-\lambda)\varphi(B_1,\ldots,B_k).\quad (3\text{-}1)$$

(ii) *Concave* if the map $(A_1,\ldots,A_k)\mapsto -\varphi(A_1,\ldots,A_k)$ is convex

(iii) *Affine* if

$$\varphi(\lambda A_1+(1-\lambda)B_1,\ldots,\lambda A_k+(1-\lambda)B_k)=\lambda\varphi(A_1,\ldots,A_k)+(1-\lambda)\varphi(B_1,\ldots,B_k).\quad (3\text{-}2)$$

Definition 3.2. Let f be a real valued continuous function . Then

(i) f is *Supermultiplicative* if

$$f(xy)\geqslant f(x)f(y).\quad (3\text{-}3)$$

(ii) f is *Submultiplicative* if

$$f(xy)\leqslant f(x)f(y).\quad (3\text{-}4)$$

Lemma 3.3. *Let φ be a normalized positive linear map and σbe an operator mean which has the representation function f which is not affine (f is an operator-monotone on $(0,\infty)$). If A and B are positive definite matrices, then the following statements are equivalent:*

(i) $$\varphi(A\sigma B)\leqslant\varphi(A)\sigma\varphi(B).\quad (3\text{-}5)$$

(ii) $$\varphi(f(A))\leqslant f(\varphi(A)).\quad (3\text{-}6)$$

Proof. It suffices to show that (ii) implies (i). Consider the map ψ defined by

$$\psi(X)=\varphi(A)^{-1/2}\varphi(A^{1/2}XA^{1/2})\varphi(A)^{-1/2}.\quad (3\text{-}7)$$

It follows from the assumption of (ii) that $\psi\,(f(A^{-1/2}BA^{-1/2}))\leqslant f(\psi(A^{-1/2}BA^{-1/2}))$. Therefore we have

$$\begin{aligned}\varphi(A\sigma B)&=\varphi(A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2})\\&=\varphi(A)^{1/2}\psi(f(A^{-1/2}BA^{-1/2}))\varphi(A)^{1/2}\\&\leqslant\varphi(A)^{1/2}f(\psi(A^{-1/2}BA^{-1/2}))\varphi(A)^{1/2}\\&=\varphi(A)^{1/2}f(\varphi(A)^{1/2}\varphi(B)\varphi(A)^{-1/2})\varphi(A)^{1/2}\\&=\varphi(A)\sigma\varphi(B).\end{aligned}$$

Theorem 3.4. *let φ be a positive linear map and let A and B be positive definite matrices. Then*

$$\varphi(A\,\#_{\alpha}\,B)\leqslant\varphi(A)\,\#_{\alpha}\,\varphi(B).\quad (3\text{-}8)$$

Proof. Consider the map ψ defined by

$$\psi(X)=\varphi(B)^{-1/2}\varphi(B^{1/2}XB^{1/2})\varphi(B)^{-1/2}.$$

By a nice technique in the proof of Lemma 3.3, set $f(t) = t^{\alpha}$ for any real number α, we get 3-8.

The following results as in Theorems 3.5 and 3.6 are referring to Ando (1979).

Theorem 3.5. *Let $A\in H_n^{+}$. Then the map*

(i) $A\mapsto A^p$ is concave if $0 < p \leqslant 1$ and is convex if $1\leqslant p\leqslant 2$ or $-1\leqslant p < 0$.

(ii) $A\mapsto\log[A]$ is concave, while the map $A\mapsto A\log[A]$ is convex.

Theorem 3.6. *Let φ be a normalized positive linear map from H_n to H_m and $A > 0$. Then*

(i) $$\varphi(A)\leqslant\varphi(A^p)^{1/p}\text{ if }1\leqslant p<\infty,\quad (3\text{-}9)$$

(ii) $$\varphi(A)\geqslant\varphi(A^p)^{1/p}\text{ if }\frac{1}{2}\leqslant p\leqslant 1,\quad (3\text{-}10)$$

(iii) $$\varphi(A)\geqslant\varphi(A^{-p})^{-1/p}\text{ if }1\leqslant p<\infty,\quad (3\text{-}11)$$

(iv) $$\varphi(\log[A])\leqslant\log[\varphi(A)],\quad (3\text{-}12)$$

(v) $$\varphi(A\log[A])\geqslant\varphi(A)\log[\varphi(A)].\quad (3\text{-}13)$$

Theorem 3.7. *Let $A_i \in H_n^+(1 \leqslant i \leqslant k, k \geqslant 2)$ be commutative compatible partitioned matrices. Then*

$$\text{(i)} \quad \prod_{i=1}^{k}\Theta A_i \leqslant \left(\prod_{i=1}^{k}\Theta A_i^p\right)^{1/p} \quad \text{if } 1 \leqslant p < \infty, \tag{3-14}$$

$$\text{(ii)} \quad \prod_{i=1}^{k}\Theta A_i \geqslant \left(\prod_{i=1}^{k}\Theta A_i^p\right)^{1/p} \quad \text{if } \frac{1}{2} \leqslant p \leqslant 1, \tag{3-15}$$

$$\text{(iii)} \quad \prod_{i=1}^{k}\Theta A_i \geqslant \left(\prod_{i=1}^{k}\Theta A_i^{-p}\right)^{-1/p} \quad \text{if } 1 \leqslant p < \infty, \tag{3-16}$$

$$\text{(iv)} \quad \prod_{i=1}^{k}\Theta \log[A_i] \leqslant \log\left[\prod_{i=1}^{k}\Theta A_i\right], \tag{3-17}$$

$$\text{(v)} \quad \prod_{i=1}^{k}\Theta A_i \log[A_i] \geqslant \left(\prod_{i=1}^{k}\Theta A_i\right)\log\left[\prod_{i=1}^{k}\Theta A_i\right]. \tag{3-18}$$

Proof. The proof is straightforward by setting $\varphi(A_1,\ldots,A_k) = \prod_{i=1}^{k}\Theta A_i$ in Theorem 3.6.

Corollary 3.8. *Let $A_i \in H_n^+(1 \leqslant i \leqslant k, k \geqslant 2)$ be commutative compatible partitioned matrices. Then*

$$\text{(i)} \quad \prod_{i=1}^{k} * A_i \leqslant \left(\prod_{i=1}^{k} * A_i^p\right)^{1/p} \quad \text{if } 1 \leqslant p < \infty, \tag{3-19}$$

$$\text{(ii)} \quad \prod_{i=1}^{k} * A_i \geqslant \left(\prod_{i=1}^{k} * A_i^p\right)^{1/p} \quad \text{if } \frac{1}{2} \leqslant p \leqslant 1, \tag{3-20}$$

$$\text{(iii)} \quad \prod_{i=1}^{k} * A_i \geqslant \left(\prod_{i=1}^{k} * A_i^{-p}\right)^{-1/p} \quad \text{if } 1 \leqslant p < \infty, \tag{3-21}$$

$$\text{(iv)} \quad \prod_{i=1}^{k} * \log[A_i] \leqslant \log\left[\prod_{i=1}^{k} * A_i\right], \tag{3-22}$$

$$\text{(v)} \quad \prod_{i=1}^{k} * A_i \log[A_i] \geqslant \left(\prod_{i=1}^{k} * A_i\right)\log\left[\prod_{i=1}^{k} * A_i\right]. \tag{3-23}$$

Proof. The proof follows immediately by applying 1-11 and 1-12 on Theorem 3.7.

Theorem 3.9. *Let A_i and $B_i \in H_n^+$, ($1 \leqslant i \leqslant k$,$k \geqslant 2$) be compatible partitioned matrices. Let σ be an operator mean with supermultiplicative representing function f. Then*

$$\text{(i)} \quad \prod_{i=1}^{k}\Theta(A_i\sigma B_i) \leqslant \left(\prod_{i=1}^{k}\Theta A_i\right)\sigma\left(\prod_{i=1}^{k}\Theta B_i\right). \tag{3-24}$$

$$\text{(ii)} \quad \prod_{i=1}^{k} * (A_i\sigma B_i) \leqslant \left(\prod_{i=1}^{k} * A_i\right)\sigma\left(\prod_{i=1}^{k} * B_i\right). \tag{3-25}$$

Proof. (i) Set $X_i = A_i^{-1/2}B_iA_i^{-1/2} \quad (i = 1,2,\ldots,k)$, then it follows from supermultiplicative of f that

$$f\left(\prod_{i=1}^{k}\Theta X_i\right) \geqslant \prod_{i=1}^{k}\Theta f(X_i). \tag{3-26}$$

Now

$$\begin{aligned}\prod_{i=1}^{k}\Theta(A_i\sigma B_i) &= \left(A_1^{1/2}f(X_1)A_1^{1/2}\right)\Theta\cdots\Theta\left(A_k^{1/2}f(X_k)A_k^{1/2}\right)\\ &= \left(A_1^{1/2}\Theta\cdots\Theta A_k^{1/2}\right)(f(X_1)\Theta\cdots\Theta f(X_k))\left(A_1^{1/2}\Theta\cdots\Theta A_k^{1/2}\right)\\ &= \left(\prod_{i=1}^{k}\Theta A_i\right)^{1/2}\left(\prod_{i=1}^{k}\Theta f(X_i)\right)\left(\prod_{i=1}^{k}\Theta A_i\right)^{1/2}\\ &\leqslant \left(\prod_{i=1}^{k}\Theta A_i\right)^{1/2} f\left(\prod_{i=1}^{k}\Theta X_i\right)\left(\prod_{i=1}^{k}\Theta A_i\right)^{1/2}\\ &= \left(\prod_{i=1}^{k}\Theta A_i\right)\sigma\left(\prod_{i=1}^{k}\Theta B_i\right).\end{aligned}$$

(ii) It follows immediately by applying 1-11 and 1-12 in Part (i) of Theorem 3.9.

Corollary 3.10. *Let φ be a positive linear map, then for any compatible partitioned matrices A_i and $B_i \in H_n^+(i = 1,2)$*

$$\varphi((A_1\Theta B_1)\mathop{\#}_{\alpha}(A_2\Theta B_2)) \leqslant \varphi(A_1\mathop{\#}_{\alpha}A_2)\Theta\varphi(B_1\mathop{\#}_{\alpha}B_2). \tag{3-27}$$

Proof. It follows by replacing A by $A_1\Theta B_1$ and B by $A_2\,\Theta B_2$ in Theorem 3.4.

Corollary 3.11. *Let φ be a positive linear map, then for any compatible partitioned matrices A_i and $B_i \in H_n^+ \quad (1 \leqslant i \leqslant k, k \geqslant 2)$*

$$\begin{aligned}&\varphi((A_1\Theta B_1)\mathop{\#}_{\alpha}(A_2\Theta B_2)\mathop{\#}_{\alpha}\cdots\mathop{\#}_{\alpha}(A_k\Theta B_k))\\ &\quad\leqslant \varphi(A_1\mathop{\#}_{\alpha}A_2\mathop{\#}_{\alpha}\cdots\mathop{\#}_{\alpha}A_k)\Theta\varphi(B_1\mathop{\#}_{\alpha}B_2\mathop{\#}_{\alpha}\cdots\mathop{\#}_{\alpha}B_k).\end{aligned} \tag{3-28}$$

Proof. The proof is straightforward by using Corollary 3.10 and induction on k.

Corollary 3.12. *Let A_i and $B_i \in H_n^+(i = 1,2)$ be compatible partitioned matrices. Then*

$$(A_1 * B_1)\mathop{\#}_{\alpha}(A_2 * B_2) \geqslant (A_1\mathop{\#}_{\alpha}A_2) * (B_1\mathop{\#}_{\alpha}B_2). \tag{3-29}$$

Proof. Due to Corollary 3.10, Lemma 2.1 and using 1-11 and 1-12, then there is a normalized positive linear map φ such that

$$\begin{aligned}(A_1\mathop{\#}_{\alpha}A_2) * (B_1\mathop{\#}_{\alpha}B_2) &= \varphi((A_1\Theta B_1)\mathop{\#}_{\alpha}(A_2\Theta B_2))\\ &= \varphi((A_1\mathop{\#}_{\alpha}A_2)\Theta(B_1\Theta B_2))\\ &\leqslant \varphi(A_1\Theta B_1)\mathop{\#}_{\alpha}\varphi(A_2\Theta B_2) = (A_1 * B_1)\mathop{\#}_{\alpha}(A_2 * B_2).\end{aligned}$$

Corollary 3.13. *Let A_i and $B_i \in H_n^+ \quad (1 \leqslant i \leqslant k, k \geqslant 2)$ be compatible partitioned matrices. Then*

$$\prod_{i=1}^{k}\mathop{\#}_{\alpha}(A_i * B_i) \geqslant \left(\prod_{i=1}^{k}\mathop{\#}_{\alpha}A_i\right) * \left(\prod_{i=1}^{k}\mathop{\#}_{\alpha}B_i\right). \tag{3-30}$$

Proof. The proof is straightforward by using Corollary 3.12 and induction on k.

Corollary 3.14. *Let $A_i \in H_n^+(1 \leqslant i \leqslant k, k \geqslant 2)$ be compatible partitioned matrices. Then for $-\infty < p,q < \infty$,*

$$\left(\prod_{i=1}^{k} * A_i^p\right) \# \left(\prod_{i=1}^{k} * A_i^q\right) \geqslant \prod_{i=1}^{k} * A_i^{(p+q)/2}. \tag{3-31}$$

Proof. Due to Theorem 2.3, Theorem 3.4 and using 1-11 and 1-12, then there is a normalized positive linear map φ such that

$$\varphi\left[\left(\prod_{i=1}^{k}\Theta A_i^p\right)\#\left(\prod_{i=1}^{k}\Theta A_i^q\right)\right] = \varphi\left(\prod_{i=1}^{k}\Theta A_i^{(p+q)/2}\right)$$
$$\leqslant \varphi\left(\prod_{i=1}^{k}\Theta A_i^p\right)\#\varphi\left(\prod_{i=1}^{k}\Theta A_i^q\right) = \left(\prod_{i=1}^{k} * A_i^p\right)\#\left(\prod_{i=1}^{k} * A_i^q\right).$$

Theorem 3.15. *Let A and $B \in H_n^+$ be compatible partitioned matrices such that $A*B = B*A$. Then*

$$A * B \geqslant (A\#B) * (A\#B). \tag{3-32}$$

Proof. Since $A*B = B*A$, then

$$(A * B)\#(B * A) = (A * B)\#(A * B) = (A * B).$$

Since $A\#B = B\#A$ and from Corollary 3.12, then we have

$$(A * B)\#(B * A) = (A * B) \geqslant (A\#B) * (B\#A)$$
$$= (A\#B) * (A\#B).$$

Theorem 3.16. *Let A_i and $B_i \in H_{n_i}^+ (1 \leqslant i \leqslant k)$ be compatible partitioned matrices and let φ_i be a concave map from $H_{n_i}^+$ to $H_{m_i}^+ \quad (1 \leqslant i \leqslant k)$. Then the map*

$$(A_1,\ldots,A_k)\mapsto\prod_{i=1}^{k}\Theta\varphi_i(A_i)^{-1} \tag{3-33}$$

is convex.

Proof. It suffices to show the convexity when $\lambda = 1/2$. Since the map under consideration is continuous, then

$$\prod_{i=1}^{k}\Theta\varphi_i(\lambda A_i + (1-\lambda)B_i)^{-1} = \prod_{i=1}^{k}\Theta\varphi_i\left(\tfrac{1}{2}(A_i+B_i)\right)^{-1}$$
$$\leqslant \prod_{i=1}^{k}\Theta\left(\tfrac{1}{2}\{\varphi_i(A_i)+\varphi_i(B_i)\}\right)^{-1} \quad \text{(Concavity of } \varphi_i)$$
$$\leqslant \prod_{i=1}^{k}\Theta(\varphi_i(A_i)\#\varphi_i(B_i))^{-1} \quad = \prod_{i=1}^{k}\Theta(\varphi_i(A_i)^{-1}\#\varphi_i(B_i)^{-1})$$
$$= \left\{\prod_{i=1}^{k}\Theta\varphi_i(A_i)^{-1}\right\}\#\left\{\prod_{i=1}^{k}\Theta\varphi_i(B_i)^{-1}\right\} \quad (\textit{Theorem}(2.4))$$
$$\leqslant \tfrac{1}{2}\left\{\prod_{i=1}^{k}\Theta\varphi_i(A_i)^{-1} + \prod_{i=1}^{k}\Theta\varphi_i(B_i)^{-1}\right\}.$$

Corollary 3.17. *Let $A_i \in H_{n_i}^+ \quad (1 \leqslant i \leqslant k)$ be compatible partitioned matrices and let $0 \leqslant p_i \leqslant 1 \quad (1 \leqslant i \leqslant k)$. Then the map*

$$(A_1,\ldots,A_k)\mapsto\prod_{i=1}^{k}\Theta A_i^{-p_i} \tag{3-34}$$

is convex on $H_{n_1}^+ \times \cdots \times H_{n_k}^+$.

Proof. The proof is straightforward by applying Theorems 3.16 and 3.5.

Corollary 3.18. *Let $A_i \in H_{n_i}^+ (1 \leqslant i \leqslant k)$ be compatible partitioned matrices and let $0 \leqslant p_i \leqslant 1 \quad (1 \leqslant i \leqslant k)$ such that $\sum_{i=1}^{k} p_i \leqslant 1$. Then the map*

$$(A_1,\ldots,A_k)\mapsto\prod_{i=1}^{k}\Theta A_i^{p_i} \tag{3-35}$$

is concave on $H_{n_1}^+ \times \cdots \times H_{n_k}^+$.

Proof. The proof is by induction on k. If $k = 1$, then the result is true by Theorem 3.5. Suppose that Eq. 3-35 is true for the case $k-1$. If $p_k = 1$, then $p_i = 0 \quad (1 \leqslant i \leqslant k-1)$ and the map becomes $(A_1,\ldots,A_k)\mapsto I_1\Theta\cdots\Theta I_{n_{k-1}}\Theta A_k$, which is concave. If $p_k = 0$, then the map becomes

$$(A_1,\ldots,A_k)\mapsto A_1^{p_1}\Theta\cdots\Theta A_k^{p_{k-1}}\Theta I_{n_k},$$

which is concave. Now suppose $0 < p_k < 1$. Then the map

$$(A_1,\ldots,A_{k-1})\mapsto\prod_{i=1}^{k-1}\Theta A_i^{p_i/(1-p_k)}$$

is concave by the induction assumption. Now with $f(\lambda) = \lambda^{p_k}$, the map

$$(A_1,\ldots,A_k)\mapsto\prod_{i=1}^{k}\Theta A_i^{p_i}$$

is concave.

Corollary 3.19. *Let $A_i \in H_{n_i}^+ (1 \leqslant i \leqslant k)$ be compatible partitioned matrices and let $1 \leqslant q \leqslant 2, \quad 0 \leqslant p_i \leqslant 1 \quad (1 \leqslant i \leqslant k)$ such that $\sum_{i=1}^{k} p_i \leqslant q-1$. Then the map*

$$(A_0,A_1,\ldots,A_k)\mapsto A_0^q\Theta\left(\prod_{i=1}^{k}\Theta A_i^{-p_i}\right) \tag{3-36}$$

is convex on $H_{n_1}^+ \times \cdots \times H_{n_k}^+$.

Proof. The map

$$\varphi(A_0,A_1,\ldots,A_k) = A_0^{2-q}\Theta\left(\prod_{i=1}^{k}\Theta A_i^{p_i}\right)$$

is concave, while the map

$$\Psi(A_0,A_1,\ldots,A_k) = A_0\Theta\left(\prod_{i=1}^{k}\Theta I_{n_i}\right)$$

is affine, and the Corollary 3.19 follows by using the following result (Ando, 1979): If φ and ψ are maps from H_n^+ to H_m^+; and if φ is concave and ψ is affine. Then the map $A \mapsto \psi(A)\varphi(A)^{-1}\psi(A)$ is convex.

Remark 3.20. All results obtained in Section 3 is quite general. These results lead to inequalities involving the Hadamard and Kronecker product for non-partitioned matrices $A_i (i = 1,2, \ldots k, k \geqslant 2)$; and Ando's mean by setting $\alpha = 1/2$, as a special case.

4. Conclusion

Several new attractive and interested inequalities related to operator means associated with non-negative linear maps and Khatri–Rao products of positive definite matrices are established by using means of concavity and convexity theorems. Some important special cases of these inequalities are also discussed. The satisfactory definition of geometric mean of positive definite matrices which satisfy properties from (i) to (viii) and properties from (1) to (12) that are mentioned, respectively, Kilicman and Al-Zhour (2005) and Kim et al. (2011), and many other desirable new properties still need further researches.

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References

Alic, M., Mond, B., Pečarić, J., Volence, V., 1997. The arithmetic–geometric–harmonic mean and related matrix inequalities. Linear Algebra Appl. 264, 55–62.

Al-Zhour, Z., Kilicman, A., 2006. Extension and generalization inequalities involving the Khatri–Rao product of several positive matrices. J. Ineq. Appl. 2006, 1–12.

Al-Zhour, Z., 2012. New holder-type inequalities for the Tracy–Singh and Khatri–Rao products of positive matrices. Int. J. Comp. Eng. Res. 3 (3), 50–54.

Ando, T., Hiai, F., 1998. Holder type inequalities for matrices. Math. Ineq. Appl. 1, 1–30.

Ando, T., 1979. Concavity of certain maps on positive definite matrices and applications to Hadamard products. Linear Algebra Appl. 26, 203–241.

Ando, T., 1983. On arithmetic–geometric–harmonic mean inequalities for positive definite matrices. Linear Algebra Appl. 52/53, 31–37.

Ando, T., Lib, C.-K., Mathias, R., 2004. Geometric means. Linear Algebra Appl. 385, 305–334.

Beesack, P.R., Pečarić, J., 1985. On Jenssen's inequality for convex functions. J. Math. Anal. Appl. 110, 536–552.

Bhatia, R., Kittaneh, F., 2000. Notes on matrix arithmetic–geometric mean inequalities. Linear Algebra Appl. 308, 203–211.

Bhatia, R., Kosaki, H., 2007. Means matrices and infinite divisibility. Linear Algebra Appl. 424, 36–54.

Bhatia, R., Grover, P., 2012. Norm Inequalities related to the matrix geometric mean, Linear Algebra Appl. 437, 726–733.

Cao, C.-G., Zhang, X., Yang, Z.-P., 2002. Some inequalities for the Khatri–Rao products of matrices. Elec. J. Linear Algebra. 9, 276–281.

Fiedler, M., Ptak, V., 1997. A new positive definite geometric mean of two positive definite matrices. Linear Algebra Appl. 251, 21–57.

Fujii, M., Lee, E.-Y., Seo, Y., 2010. A difference counterpart to a matrix holder inequality. Linear Algebra Appl. 432, 2565–2571.

Furuichi, S., Yanagi, K., Kuriyama, K., 2005. A note on operator inequalities of Tsallis relative operator entropy. Linear Algebra Appl. 407, 19–31.

Furuta, T., 2006. Two reverse inequalities associated with Tsallis relative operator entropy via generalized Kantorovich constant and their applications. Linear Algebra Appl. 421, 526–537.

Hernandez Cifre, M.A., Salinas, G., Gomis, S.S., 2001. Complete systems of inequalities. J. Ineq. Pure Appl. Math. 2 (1), 1–28. Article 10.

Hu, Y., Zhang, X., Yang, Z., 2005. Mixed mean inequalities for several positive definite matrices. Linear Algebra Appl. 395, 247–263.

Ito, M., Seo, K., Yamazaki, T., Yanagida, M., 2011. Geometric properties of positive definite matrices cone with respect to Thompson metric. Linear Algebra Appl. 435, 2054–2064.

Jung, C., Kim, H.-M., Lim, Y., 2009. On the solution of the nonlinear matrix equation $X^n = f(X)$. Linear Algebra Appl. 430, 2042–2052.

Kim, S., Lawson, J., Lim, Y., 2011. The matrix geometric mean of parameterized, weighted arithmetic and Harmonic means. Linear Algebra Appl. 435, 2114–2131.

Kilicman, A., Al-Zhour, Z., 2005. Improvements on geometric means related to the Tracy–Singh products of positive matrices. Matematica 21 (2), 49–65.

Liu, S., 2002. Several inequalities involving Khatri–Rao products of positive semi- definite matrices. Linear Algebra Appl. 354, 175–186.

Lee, H., Lim, Y., Yamazaki, T., 2011. Multi-variable weighted geometric means positive definite matrices. Linear Algebra Appl. 435, 307–322.

Lim, Y., 2008. On Ando–Li–Mathias geometric mean equations. Linear Algebra Appl. 428, 1767–1777.

Lim, Y., Palfia, M., 2012. Matrix power means and the Karcher mean. J. Funct. Anal. 262, 1498–1514.

Lim, Y., Yamazaki, T., 2013. On some inequalities for the matrix power and Karcher means. Linear Algebra Appl. 438, 1293–1304.

Lim, Y., 2012. Factorizations and geometric means of positive definite matrices. Linear Algebra Appl. 437, 2159–2172.

Liu, S., 1999. Matrix results on the Khatri–Rao and Tracy–Singh products. Linear Algebra Appl. 289, 267–277.

Micic, J., Pečarić, J., Seo, Y., 2000. Complementary inequalities to inequalities of Jenssen and Ando based on the Mond–Pečarić method. Linear Algebra Appl. 318, 87–107.

Mond, B., Pečarić, J., 1997. Matrix inequalities for convex functions. J. Math. Anal. Appl. 209 (1), 147–153.

Mond, M., Pečarić, J., Zagreb, 1996. A mixed arithmetic-mean–harmonic-mean matrix inequality. Linear Algebra Appl. (237/238), 449–454.

Qi, F., Guo, B.-N., 2003. An inequality between ratio of the extended logarithmic means and ratio of exponential means. Taiwanese J. Math. 7 (2), 229–237.

Sagae, M., Tanabe, K., 1994. Upper and lower bounds for the arithmetic–geometric–harmonic means of positive definite matrices. Linear Multilinear Alg. 37, 279–282.

Satnoianu, R.A., 2002. Mproved GA-convexity inequalities. J. Ineq. Pure Appl. Math. 3 (5), 1–14. Article 82.

Xia, M., Xu, Z., Bin, Zhu., 2013. Geometric bonferroni means with their applicatios in multi-criteria decision making. Knowledge-Based Sys. 40, 88–100.

Xiao, Z.-G., Zhang, Z.-H., 2003. The inequalities $G \leqslant L \leqslant I \leqslant A$ in n variables, J. Ineq. Pure Appl. Math. 4 (2), 1–13. Article 39.

Zhang, F., 1999. Matrix Theory: Basic Results and Techniques. Springer-Verlag, NewYork.