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Some topological properties on C-α-Normality and C-β-Normality

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A R T I C L E I N F O

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A topological space (Y,τ) is called *C-α-normal (C-β-normal)* if there exist a bijective function g from Y onto α-normal (β-normal) space Z such that the restriction map $g_{|_B}$ from B onto $g(B)$ is a homeomorphism for any compact subspace B of Y. We discuss some relationships between C-α-normal (C-β-normal) and other properties.

1. Introduction

In 2017 we discuss the topological property *"C-normal"* (AlZahrani and Kalantan, 2017). In this paper we introduce a new property called *C-α-Normality and C-β-Normality*. We show any α-normal (β-normal) space is C-α-normal (C-β-normal), but the converse is not true in general. And we show that any C-normal, lower compact, epinormal, *epi*-α-normal and *epi*-β-normal spaces is C-α-normal (C-β-normal). We prove any locally compact is C-α-normal (C-β-normal) but the converse is not true in general. Also observe that a witness function of C-α-normal (C-β-normal) not necessarily to be continuous in general, but it will be continuous under some conditions.

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2. C–-Normality and C--Normality

Recall that a topological space (Y,τ) is called an *α-normal* space (Arhangel'skii and Ludwig, 2001) if for every-two disjoint closed subsets F and E of Y there are two open subsets G and W of Y such that $F\cap G$ is dense in F; $E\cap W$ is dense in E, and $G\cap W=\varnothing$, and a topological space (Y,τ) is called a *β-normal* space (Arhangel'skii and Ludwig, 2001) if for every-two disjoint closed subsets F and E of Y there are two open subsets G and W of Y such that $F\cap G$ is dense in F; $E\cap W$ is dense in E, and $\overline{G}\cap\overline{W}=\varnothing$. A topological space (Y,τ) is called *C-normal* (AlZahrani and Kalantan, 2017) if there exist a bijective function g from Y onto a normal space Z such that the restriction map $g_{|_B}$ from B onto $g(B)$ is a homeomorphism for any compact subspace B of Y.

Definition 1.1. *A topological space (Y,τ) is called C-α-normal (C-β-normal) if there exist a bijective function g from Y onto α-normal (β-normal) space Z such that the restriction map $g_{|_B}$ from B onto $g(B)$ is a homeomorphism for any compact subspace B of Y.*

In these definition, we call the space Z a witness of *C-α-normal* (*C-β-normal*) and the function g is called a witness function.

A topological space (Y,τ) is called *α-regular* (Alzahrani, 2022) if for any $x\in Y$ and a closed subset $A\subset Y$ such that $x\notin A$ there are

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two disjoint open sets G; $H \subset Y$ such that $x \in G$ and $\overline{A \cap H} = A$. And topological space (Y, τ) is called almost *α-regular* (Alzahrani, 2022) if for any $x \in Y$ and a regular closed subset $A \subset Y$ such that $x \notin A$ there are two disjoint open sets G; $H \subset Y$ such that $x \in G$ and $\overline{A \cap H} = A$.

Lemma 1.2. *Any regular space is α-regular.*

Proof. Let (Y, τ) be a regular space. Pick $y \in Y$ and $F \subseteq Y$ be a closed set such that $y \notin F$, then there exist two disjoint open sets W_1 and W_2 subsets of Y where $y \in W_1$ and $F \subseteq W_2$, hence $\overline{F \cap W_2} = F$ (note that $\overline{F} = F$ since F is closed), and $W_1 \cap W_2 = \varnothing$, therefore (Y, τ) is α-regular space.

Lemma 1.3. (Alzahrani, 2022) *Any α-regular space is almost α-regular.*

From Lemma 1.2. and Lemma 1.3. we conclude the following corollary.

Corollary 1.4. *Any regular space is almost α-regular.*

Lemma 1.5. *Any normal space is α-normal.*

Lemma 1.6 ((*Arhangel'skii and Ludwig, 2001*)). *Any normal space is β-normal.*

Proof. Let Y be a normal space. Pick two disjoint closed sets F_1 and F_2 subsets of Y. Since Y is normal, then there exist two disjoint open sets W_1 and W_2 subsets of Y where $F_1 \subseteq W_1$, $F_2 \subseteq W_2$ and $W_1 \cap W_2 = \varnothing$. Hence $\overline{F_1 \cap W_1} = F_1$ and $\overline{F_2 \cap W_2} = F_2$. It remains to prove $\overline{W_1} \cap \overline{W_2} = \varnothing$. For a normal space Y, if F is a closed set, U is an open set and F $\subseteq$ U, then there exist an open set V such that F $\subseteq$ V $\subseteq$ $\bar{V}$ $\subseteq$ U. Now apply this to F_1 and set W_1 = V and W_2 = Y\ $\bar{V}$.

So we have the following theorem.

Theorem 1.7. *Any C-normal space is C-α-normal (C-β-normal).*

The converse is true under some conditions, first we mention some definition.

A Hausdorff space Y is *extremally disconnected* (Engelking, 1977) if the closure of any open set in Y is open. A topological space is called *mildly normal* (Shchepin, 1972) if any two disjoint regular closed subsets can be separated.

Theorem 1.8 ((*Arhangel'skii and Ludwig, 2001*)). *Any α-normal extremally disconnected space is normal.*

Proof. Let Y be a α-normal extremally disconnected space. Pick two disjoint closed sets F_1 and F_2 subsets of Y. Since Y is α-normal, then there exist two disjoint open sets W_1 and W_2 subsets of Y where $\overline{F_1 \cap W_1} = F_1$ and $\overline{F_2 \cap W_2} = F_2$. Hence $F_1 \subseteq \overline{W_1}$ and $F_2 \subseteq \overline{W_2}$. However, $\overline{W_1} \cap W_2 = \varnothing$ since W_2 is open and $W_1 \cap W_2 = \varnothing$. Thus, $\overline{W_1} \cap \overline{W_2} = \varnothing$ as well since $\overline{W_1}$ is open (by extremally disconnectedness) and $\overline{W_1} \cap W_2 = \varnothing$.

Therefore Y is normal space.

From Theorem 1.8, we have the following.

Theorem 1.9. *If Y is C-α-normal (C-β-normal) such that the witness of C-α-normal (C-β-normal) is extremally disconnected, then Y is C-normal.*

Theorem 1.10. *If Y is C-β-normal such that the witness of C-β-normal is mildly normal, then Y is C-normal.*

Proof. Let Y be C-β-normal. Then the codomain Z witness of C-β-normal is β-normal. Let F_1 and F_2 be any disjoint closed subsets of Z. Since Z is β-normal, there exist open subsets W_1 and W_2 of Z where $\overline{W_1} \cap \overline{W_2} = \varnothing$, $\overline{F_1 \cap W_1} = F_1$ and $\overline{F_2 \cap W_2} = F_2$. So $\overline{W_1}$, $\overline{W_2}$ are disjoint regular closed subsets containing F_1 and F_2 respectively. Since Z is mildly normal, there exist disjoint open subsets U_1 and U_2 of Z where $F_1 \subseteq \overline{W_1} \subseteq U_1$ and $F_2 \subseteq \overline{W_2} \subseteq U_2$. Hence Z is normal.

Lemma 1.11. *Any α-normal space satisfying T_1 axiom is Hausdorff.*

Proof. Let Y be any α-normal T_1-space. Let y, z be any two distinct elements in Y. Hence $\{y\}$ and $\{z\}$ are disjoint closed subsets of Y, by α-normality, there exist two disjoint open subsets G_1 and G_2 of Y where $\overline{\{y\} \cap G_1} = \{y\}$ and $\overline{\{z\} \cap G_2} = \{z\}$ which implies $y \in G_1$ and $z \in G_2$. Therefore Y is Hausdorff.

Lemma 1.12 ((*Arhangel'skii and Ludwig, 2001*)). *Any β-normal space satisfying T_1 axiom is regular (hence Hausdorff).*

By Corollary 1.4. we have the following result.

Corollary 1.13. *Any β-normal space satisfying T_1 axiom is almost α-regular.*

Also by Lemma 1.12 and Lemma 1.2 we have the following result.

Lemma 1.14. *Any β-normal space satisfying T_1 axiom is α-regular.*

Corollary 1.15. *Any α-normal space satisfying T_1 axiom is α-regular.*

By Lemma 1.3. we conclude the following corollary.

Corollary 1.16. *Any α-normal space satisfying T_1 axiom is almost α-regular.*

Proposition 1.17. (Murtinová, 2002) *Every first countable α-normal Hausdorff space is regular.*

Recall that a topological space (Y, τ) is called submetrizable (AlZahrani and Kalantan, 2017) if there exists a metric d on Y such that the topology τ d on Y generated by d is coarser than τ.

Theorem 1.18. *Every submetrizable space is C-α-normal (C-β-normal).*

Proof. Let (Y, τ) be a submetrizable space, the there exists a metrizable τ' such that $\tau' \subseteq \tau$. Hence (Y, τ') is α-normal since it is normal, and the identity function id_Y from (Y, τ) onto (Y, τ') is a one-to-one and continuous function. If we take B any compact subspace of (Y, τ), then $id_Y(B)$ is hausdorff, since it is subspace of (Y, τ'), and by (Engelking, 1977);3.1.13]; $id_{Y_{|_B}}$ is a homeomorphism.

Example 1.19. *The Rational Sequence Topology* $(\mathbb{R}, \mathscr{RS})$ (Steen and Seebach, 1995) *is submetrizable being finer than the usual topology* $(\mathbb{R}, \mathscr{U})$, *so* $(\mathbb{R}, \mathscr{RS})$ *is C-α-normal (C-β-normal).*

The converse of Theorem 1.18. is not true in general, for example $\omega_1 + 1$ is C-α-normal (C-β-normal) which is not submetrizable.

Apparently, any α-normal (β-normal) space is C-α-normal (C-β-normal), to prove this, just by considering $Z = Y$ and g is the identity function.

While in general the converse is not true. We provide some examples below.

Example 1.20.

1. *The Half-Disc topological space* (Steen and Seebach, 1995) *is C-α-normal (C-β-normal) because it is submetrizable by* Theorem 1.18**.** *but it is not α-normal nor β-normal because it is first countable and Hausdorff but not regular, so by* Proposition 1.17. *the Half-Disc topological space is not α-normal space, hence not β-normal. In general C-α-normality (C-β-normality) do not imply α-normality (β-normality) even with Hausdorff or first countable properties.*
2. The Deleted Tychonoff Plank (Steen and Seebach, 1995), it is C-α-normal (C-β-normal) since it is locally compact by Theorem 2.7**.** but it is not α-normal nor β-normal see (Arhangel'skii and Ludwig, 2001).
3. The Dieudonn $\hat{\acute{e}}$ Plank (AlZahrani and Kalantan, 2017), in example 1.10 we proved that it is C-normal, hence it is C-α-normal (C-β-normal) by Theorem 1.9. but it is not α-normal nor β-normal see (Arhangel'skii and Ludwig, 2001), also not locally compact, hence this example also shows that the converse of Theorem 2.7. is not true.
4. The Sorgenfrey line square $\mathscr{S} \times \mathscr{S}$ see (Steen and Seebach, 1995) is not normal, but it is submetrizable space being it is finer than the usual topology on $\mathbb{R} \times \mathbb{R}$, so by Theorem 1.18**.** it is C-α-normal (C-β-normal).

Theorem 1.21. *If Y is a compact non–α-normal (non–β-normal) space, then Y can not be C-α-normal (C-β-normal).*

Proof. Assume Y is a compact non–α-normal (non–β-normal) space. Suppose Y is C-α-normal (C-β-normal), then there exists α-normal (β-normal) space Z and a bijective function $g : Y \to Z$ where the restriction map $g_{|_B}$ from B onto $g(B)$ is a homeomorphism for any compact subspace K of Y. As Y is compact, then $Y \cong Z$, and we have a contradiction as Z is α-normal (β-normal) while Y is not. Hence Y can not be C-α-normal (C-β-normal).

Observe that a function $g : Y \to Z$ witnessing of C-α-normal (C-β-normal) of Y not necessarily to be continuous in general, and here is an example.

Example 1.22. *Let $\mathbb{R}$ with the countable complement topology $\mathscr{CC}$* (Steen and Seebach, 1995). *We know $(\mathbb{R}, \mathscr{CC})$ is T_1 and the only compact sets are finite, hence the compact subspaces are discrete. If we let $\mathscr{D}$ be the discrete topology on $\mathbb{R}$, then obviously the identity function from $(\mathbb{R}, \mathscr{CC})$ onto $(\mathbb{R}, D)$ is a witnessing of the C-α-normality (C-β-normality) which is not continuous.*

But it will be continuous under some conditions as the following theorems.

Theorem 1.23. *If (Y, τ) is a C-α-normal (C-β-normal) and Fréchet space, then any function witnessing of C-α-normality (C-β-normality) is continuous.*

Proof. Let Y be a Fréchet C-α-normal (C-β-normal) space and $g : Y \to Z$ be a witness of the C-α-normality (C-β-normality) of Y. Let $A \subseteq Y$ and pick $z \in g\left(\overline{A}\right)$. There is a unique $y \in Y$ where $g(y) = z$, thus $y \in \overline{A}$. since Y is Fréchet, then there exists a sequence $(a_n) \subseteq A$ where $a_n \to y$. As the subspace $K = \{y\} \cup \{a_n : n \in \mathbb{N}\}$ of Y is compact, the induced map $g_{|_K} : K \to g(K)$ is a homeomorphism. Let $U \subseteq Z$ be any open neighborhood of z. Then $U \cap g(K)$ is an open neighborhood of z in the subspace $g(K)$. Since $g_{|_K}$ is a homeomorphism, then $g^{-1}(U \cap g(K)) = g^{-1}(U) \cap K$ is an open neighborhood of y in K, then there exists $m \in \mathbb{N}$ where $a_n \in g^{-1}(U \cap g(K))$ $\forall n \geq m$, hence $g(a_n) \in (U \cap g(K)) \forall n \geq m$, then $U \cap g(A) \neq \emptyset$. Hence $z \in \overline{g(A)}$ and $g\left(\overline{A}\right) \subseteq \overline{g(A)}$. Thus g is continuous.

Since any first countable space is Fréchet, we conclude that, In C-α-normality (C-β-normality) first countable space a function $g : Y \to Z$ is a witness of the C-α-normality (C-β-normality) of Y is continuous. Also, by theorem (Engelking, 1977),3.3.21], we conclude the following.

Corollary 1.24. *If Y is a C-α-normal (C-β-normal) k-space and g is a witness function of the C-α-normality (C-β-normality), then g is continuous.*

For simplicity, let us call a T_1 space which satisfies that the only compact subspaces are the finite subsets *F-compact*. Clearly F-compactness is a topological property.

Theorem 1.25. *If Y is F-compact, then Y is C-α-normal (C-β-normal).*

Proof. Let Y be a *F-compact*. Let $Z = Y$ and let Z with the discrete topology. Hence the identity function from Y onto Z does the job.

Example 1.26. *Consider $(\mathbb{R}, \mathscr{CC})$, where $\mathscr{CC}$ is the countable complement topology* (Steen and Seebach, 1995). *We know $(\mathbb{R}, \mathscr{CC})$ is T_1 and the only compact sets are finite, therefore, by* Theorem 1.25. *$(\mathbb{R}, \mathscr{CC})$ is C-α-normal (C-β-normal). This a fourth example of C-α-normal (C-β-normal) but not α-normal (nor β-normal).*

Notice that any topology finer than a T_1 topological space is T_1. Also any compact sub set of a topological space (Y, τ) is compact in any topology coarser than τ on Y.

Hence any topology finer than F-compact topological space is also F-compact. As an example, $(\mathbb{R}, \tau)$ denotes the Fortissimo topology on $\mathbb{R}$, see [14, Example 25]. We know that $(\mathbb{R}, \tau)$ is finer than $(\mathbb{R}, \mathscr{CC})$ which is F-compact, hence $(\mathbb{R}, \tau)$ F-compact too. Thus, $(\mathbb{R}, \tau)$ is *C-α-normal (C-β-normal).*

Theorem 1.27. *C-α-normality (C-β-normality) is a topological property.*

Proof. Let Y be a C-α-normal (C-β-normal) space and let $Y \cong W$. Let Z be a α-normal (β-normal) space and let $g : Y \to Z$ be a bijective function where the restriction map $g_{|_B}$ from B onto $g(B)$ is a homeomorphism for any compact subspace $B \subseteq Y$. Let $k : W \to Y$ be a homeomorphism. Hence Z and $g \circ k : W \to Z$ satisfy the requirements.

3. C-–Normality (C--Normality) and some other properties

Definition 2.1. *A topological space (Y, τ) is called C-α-regular if there exists a bijective function g from Y onto α-regular space Z such that the restriction map $g_{|_B}$ from B onto $g(B)$ is a homeomorphism for any compact subspace B of Y.*

This definition is new and we will study some of its properties later.

Corollary 2.2. *If Y is C-α-normal (C-β-normal) space and the witness of the C-α-normality (C-β-normality) of Y is T_1, then Y is C-α-regular.*

We prove this corollary by Lemma 1.11, Lemma 1.12. We defined C-regular in (AlZahrani, 2018).

Corollary 2.3. *If Y is C-β-normal space and the codomain witness of the C-β-normality of Y is T_1, then Y is C-regular.*

We prove this corollary by Lemma 1.12**.**

Corollary 2.4. *If Y is a C-α-normal (C-β-normal) Fréchet space and the witness of the C-α-normality (C-β-normality) is T_1, then Y is T_2.*

Proof. Let Y is a C-α-normal (C-β-normal) Fréchet space, then there exist α-normal (β-normal) space Z (witness of the C-α-normality (C-β-normality)) and a bijective function $g : Y \to Z$ such that the restriction map $g_{|_B}$ from B onto $g(B)$ is a homeomorphism for any compact subspace B of Y, then by Theorem 1.23. g is continuous. Let any $a, b \in Y$ be such that $a \neq b$, then $g(a) \neq g(b)$, $g(a), g(b) \in Z$. Since Z is α-normal (β-normal) and T_1, then by Lemma 1.11 (Lemma 1.12) the space Z is T_2 , then there exist W_1 and W_2 are open sets in Z where $g(a) \in W_1, g(b) \in W_2$ and $W_1 \cap W_2 = \varnothing$. Since W_1, W_2 are open sets in Z and g *is continuous*, then $g^{-1}(W_1)$ and $g^{-1}(W_2)$ are open sets in Y, $a \in g^{-1}(W_1)$, $b \in g^{-1}(W_2)$ and $g^{-1}(W_1) \cap g^{-1}(W_2) = g^{-1}(W_1 \cap W_2) = \varnothing$. Hence Y is T_2 .

Theorem 2.5. *Any C-regular Fréchet Lindelof space is C-α-normal (C-β-normal).*

Proof. Let Y be any C-regular Fréchet Lindelof space. Let Z be a regular space and $g : Y \to Z$ be a continuous bijective function see Theorem 1.23**.** By (Engelking, 1977); 3.8.7] Z is Lindelof. Since any regular Lindelof space is normal (Engelking, 1977), 3.8.2]. Hence Y is C-α-normal (C-β-normal).

C-α-normality (C-β-normality) does not imply C-α-regularity nor C-regular, for example.

Example 2.6. *Consider the real numbers set* $\mathbb{R}$ *with its right ray topology* $\mathscr{R}$, *where* $\mathscr{R} = \{\varnothing, \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$. *As any two non-empty closed sets must be intersect in* $(\mathbb{R}, \mathscr{R})$, *then it is normal, and by Lemma in above, it is* α*-normal* (β*-normal*), *hence C-*α*-normal* (*C-*β*-normal*). *Now, suppose that* $(\mathbb{R}, \mathscr{R})$ *is C-*α*-regular. Take* α*-regular space Z and a bijective function g from* $\mathbb{R}$ *onto Z where the restriction map* $g_{|_B}$ *from B onto* $g(B)$ *is a homeomorphism for any compact subspace B of* $\mathbb{R}$. *We know that a subspace B of* $(\mathbb{R}, \mathscr{R})$ *is compact if and only if B has a minimal element. Hence* $[1, \infty)$ *is compact, then* $g_{|_{[1,\infty)}} : [1, \infty) \to g([1, \infty)) \subset Z$ *is a homeomorphism, it means* $[1, \infty)$ *as a subspace of* $(\mathbb{R}, \mathscr{R})$ *is* α*-regular which is a contradiction, since* $[1, 4]$ *is closed in subspace* $[1, \infty)$ *and* $4.5 \notin [1, 4]$, *but any non-empty open sets on* $[1, \infty)$ *must intersect. Then* $(\mathbb{R}, \mathscr{R})$ *cannot be C-*α*-regular* (*C-regular*).

Recall that a topological space (Y, τ) is called *Locally Compact* (AlZahrani and Kalantan, 2017) if (Y, τ) is Hausdorff and for every $y \in Y$ and every open neighborhood V of y there exists an open neighborhood U of y such that $y \in U \subseteq \overline{U} \subseteq V$ and $\overline{U}$ is compact.

Theorem 2.7. *Every locally compact space is C-α-normal (C-β-normal).*

Proof. Let Y be locally compact space. By (Engelking, 1977), 3.3.D], there exists T_2 compact space Z and hence α-normal (β-normal), and a continuous bijective function $g : Y \to Z$. We have $g_{|_K}$ from K onto $g(K)$ is a homeomorphism for any compact subspace K of Y, because continuity ,1–1 and onto are inherited by g, also $g_{|_K}$ is closed since K is compact and g(K) is T_2.

Example 2.8. *Consider* ω_1, *the first uncountable ordinal*, *we consider* ω_1 *as an open subspace of its successor* $(\omega_1 + 1)$, *which is compact and hence is locally compact* [14, *Example* 43]. *Thus*, ω_1 *is locally compact as an open subspace of a locally compact space*, *see* (Engelking, 1977),3.3.8]. *Then by* Theorem 2.7**.** ω_1 *is C-*α*-normal* (*C-*β*-normal*).

The converse of Theorem 2.7**.** is not true in general. We introduce the following example of C-α-normal (C-β-normal) which is not locally compact.

Example 2.9. *Consider the quotient space* $\mathbb{R}/\mathbb{N}$. *Let* $Z = (\mathbb{R}\{\mathbb{N}) \cup \{i\}$, *where* $i = \sqrt{-1}$. *Define* $g : \mathbb{R} \to Z$ *as follows*:

$$g(a) = \begin{cases} a & for \quad a \in \mathbb{R} \backslash \mathbb{N} \\ i & for \quad a \in \mathbb{N} \end{cases}$$

Now consider $\mathbb{R}$ with the usual topology $\mathscr{U}$. Define the topology $\tau = \{V \subseteq Z : g^{-1}(V) \in \mathscr{U}\}$ on Z. Then $g : (\mathbb{R}, \mathscr{U}) \to (Z, \tau)$ is a closed quotient mapping. We explain the open neighborhoods of any element in Z as follows: The open neighborhoods of each $a \in \mathbb{R}\{\mathbb{N}$ are $(a - \varepsilon, a + \varepsilon)\{\mathbb{N}$ where ε is a natural number. The open neighborhoods of $i \in Z$ are $(G\{\mathbb{N}) \cup \{i\}$, where G is an open set in $(\mathbb{R}, \mathscr{U})$ such that $\mathbb{N} \subseteq G$. It is clear that (Z, τ) is T_3, but it is not locally compact . (Z, τ) is a continuous image of $\mathbb{R}$ with its usual topology, so it is Lindelof and T_3, then (Z, τ) is T_4. Hence it is C-α-normal (C-β-normal).

A topological space (Y, τ) is called *Epi-α-normal* (Gheith and AlZahrani, 2021) if there is a coarse topology τ' on Y such that (Y, τ') is α-normal and T_1. A topological space (Y, τ) is called *Epi-β-normal* (Gheith and AlZahrani, 2021) if there is a coarse topology τ' on Y such that (Y, τ') is β-normal and T_1. We defined Epinormal in (AlZahrani and Kalantan, 2016). By the same argument of Theorem 1.18. we can prove the following corollary.

Corollary 2.10. *Every epinormal space is C-α-normal (C-β-normal).*

Corollary 2.11. *Every epi-α-normal (epi-β-normal)space is C-α-normal(C-β-normal).*

Any indiscrete space which has more than one point is an example of a C-α-normal (C-β-normal) space which is not *epi*-α-normal (*epi*-β-normal).

The converse of Corollary 2.9 is true with Fréchet property.

Theorem 2.12. *Any C-α-normal (C-β-normal) Fréchet space is epi-α-normal (epi-β-normal).*

Proof. Let (Y, τ) be any C-α-normal (C-β-normal) Fréchet space. Let $(Z, \tau\prime)$ be α-normal (β-normal) and $g : (Y, \tau) \to (Z, \tau\prime)$ be a bijective function. Since Y is Fréchet, g is continuous (see Theorem 1.23). Define $\tau^* = \{g^{-1}(V) : V \in \tau\prime\}$. Obviously, τ^* is a topology on Y coarser than τ such that $g : (Y, \tau^*) \to (Z, \tau\prime)$ is continuous. Also g is open, since if we take $U \in \tau^*$, then $U = g^{-1}(V)$ where $V \in \tau\prime$. Thus $g(U) = g(g^{-1}(V)) = V$ which gives that g is open. Therefore g is a homeomorphism. Thus (Y, τ^*) is α-normal (β-normal). Hence (Y, τ) is *epi*-α-normal (*epi*-β-normal).

A topological space (Y, τ) is called *lower compact* (Kalantan et al., 2019) if there exists a coarser topology τ' on Y such that (Y, τ') is T_2-compact.

Theorem 2.13. *Any lower compact space is C-α-normal (C-β-normal).*

Proof. Let (Y,τ) is lower compact, then (Y,τ') is T_2-compact, hence normal and the identity function $id_Y:(Y,\tau)\to(Y,\tau')$ is a continuous and bijective. If we take B any compact subspace of (Y,τ), then $id_{Y_{|_B}}$ is a homeomorphism by (Engelking, 1977);3.1.13].

In general, the converse of Theorem 2.13. is not true, for example consider a countable complement topology on an uncountable set, it is C-α-normal (C-β-normal) since it is F-compact, but it is not lower compact because it is not T_2.

Theorem 2.14. *If (Y,τ) is C-α-normal compact Fréchet space and the witness of the C-α-normality is T_1, then (Y,τ) is lower compact.*

Proof. Pick α-normal space (Z,τ^*) and a bijective function $g:(Y,\tau)\to(Z,\tau^*)$ such that $g_{|_B}:B\to g(B)$ is a homeomorphism for any compact subspace $B\subseteq Y$. Since Y is Fréchet, then g is continuous. Hence (Z,τ^*) is compact. Since (Z,τ^*) is T_1 α-normal space, then by Lemma 1.11. it is Hausdorff. Hence (Z,τ^*) is T_2 compact. Define a topology τ' on Y as follows $\tau'=\{g^{-1}(V):V\in\tau^*\}$. Then τ' is coarser than τ and $g:(Y,\tau')\to(Z,\tau^*)$ is a bijection continuous function. Let any $U\in\tau'$, then U is of the form $g^{-1}(V)$ for some $V\in\tau^*$. Hence $g(U)=g(g^{-1}(V))=V$. Thus g is open. Hence g is a homeomorphism. So (Y,τ') is T_2 compact. Therefore (Y,τ) is lower compact.

Theorem 2.15. *If (Y,τ) is C-β-normal compact Fréchet space and the witness of the C-β-normality is T_1, then (Y,τ) is lower compact.*

4. Conclusion

The aim of this paper is to introduce a new weaker version of normality called C-α-normal and C-β-normal. We show that some relationships between this a new topological property and some other topological properties, and there are still many topological properties that the researcher can study in this topic.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Supplementary data

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