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Existence and uniqueness of a class of uncertain Liouville-Caputo fractional difference equations

Hari Mohan Srivastava [a,b,c], Pshtiwan Othman Mohammed [d,*], Cheon Seoung Ryoo [e], Y.S. Hamed [f]

[a] Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada
[b] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
[c] Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, Baku AZ1007, Azerbaijan
[d] Department of Mathematics, College of Education, University of Sulaimani, Sulaimani, Kurdistan Region, Iraq
[e] Department of Mathematics, Hannam University, Daejeon 34430, KOREA
[f] Department of Mathematics and Statistics, College of Science, Taif University, P. O. Box 11099, Taif 21944, Saudi Arabia

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ABSTRACT

We consider a class of uncertain fractional difference equation of the Liouville-Caputo type (UFLCDE). An equivalent uncertain fractional sum equation is found to the UFLCDE by using the basic properties. The successive Picard iteration method for finding a solution to the UFLCDE is introduced. Using the theory of Banach contraction under the Lipschitz constant condition, we investigate the structure of algebras of existence and uniqueness of the UFLCDE. The article finally exhibits three examples to show the effectiveness of the proposed investigation.

1. Introduction

In recent years, many experiments and theories have shown that a large number of abnormal phenomena that occurs in the engineering and applied sciences can be well described by using discrete fractional calculus. Especially, fractional difference equations have been found to be powerful tools in the modeling of various phenomena in many different fields of engineering and science, for example, in physics, fluid mechanics and heat conduction (see, for example, Bohner and Peterson, 2003; Srivastava et al., 2019; Liu, 2010; Kilbas et al., 2006; Srivastava, 2020; Srivastava, 2020; Goodrich and Peterson, 2015; Atici and Eloe, 2007; Atici and Eloe, 2009; Goodrich, 2011; Wu and Baleanu, 2015; Wu et al., 2017; Suwan et al., 2018; Mohammed and Abdeljawad, 2020; Zhu, 2015; Zhu, 2015; Lu and Zhu, 2019 and the references which are cited therein).

In the last few years, considerable attention has been given to the subject of fractional difference equations on the finite time scales. There are a few papers which investigate the existence and uniqueness of fractional difference equations in the sense of the Riemann–Liouville (RL) fractional calculus (see, for example, He et al., 2018 (2018),; Mohammed, 2019; Lu and Zhu, 2020; Srivastava and Mohammed, 2020; Mohammed et al., 2020; Lu et al., 2019; see also several recent developments Srivastava and Saad, 2020; Khader et al., 2020; Srivastava et al., 2020; Izadi and Srivastava, 2020; Srivastava and Saad, 2020; Singh et al., 2021 on the theory and applications of fractional-order ODEs and PDEs modelling various real-world situations). In particular, Lu et al.

* Corresponding author.
E-mail addresses: pshtiwansangawi@gmail.com (P.O. Mohammed), yasersalah@tu.edu.sa (Y.S. Hamed).

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(2019) investigated the existence and uniqueness of the following uncertain fractional forward difference equation (UFFDE):

$$\left({}^{RL}{}_{\phi-1}\Delta^{\phi}\wp\right)(z) = \mathscr{H}_1(z+\phi, \wp(z+\phi)) + \mathscr{H}_2(z+\phi, \wp(z+\phi))\varepsilon_{z+phi}, \left({}^{RL}{}_{\phi-1}\Delta^{-(1-\phi)}\wp\right)(z)|_{z=0}c := a_0, \quad (1.2)$$

where ${}^{RL}{}_{\phi-1}\Delta^{\phi}$ denotes fractional RL forward difference with $0<\phi\leqq 1$, and $\mathscr{H}_1$ and $\mathscr{H}_2$ are two real-valued functions defined on $[1,\infty]\times\mathscr{R}, z\in\mathscr{N}_0\cap[0,\mathscr{T}], a_0\in\mathscr{R}$ is a crisp number, and $\varepsilon_{\phi},\varepsilon_{\phi+1},\cdots,\varepsilon_{\mathscr{T}+\phi}$ are $(\mathscr{T}+1)$ IID uncertain variables with symmetrical uncertainty distribution $\mathbf{L}(\ell_1,\ell_2)$. The above work was generalized by Mohammed (2019) and, subsequently, by Srivastava and Mohammed (2020). In addition, Mohammed et al. (2020) obtained the existence and uniqueness of the nabla case (backward) of the Eq. (1.2).

To the best of our knowledge, there are few studies that consider the existence and uniqueness of the RL fractional difference equations. Therefore, in the sense of the Liouville-Caputo fractional calculus, it is generally important to study this kind of difference equations by using the uncertainty theory, which extends and enriches the existing body of literature. Motivated by the above-cited investigations, in this article, we study the existence and uniqueness of the following uncertain fractional Liouville-Caputo like difference equation (UFLCDE):

$$\left({}^{C}{}_{\phi-1}\Delta^{\phi}\wp\right)(z) = \mathscr{H}_1(z+\phi-1, \wp(z+\phi-1)) + \mathscr{H}_2(z+\phi-1, \wp(z+\phi-1))\varepsilon_{z+\phi-1}, \quad (1.3)$$

with the initial condition given by

$$\left({}^{C}{}_{\phi-1}\Delta^{-(1-\phi)}\wp\right)(z)|_{z=0} = \wp(\phi-1) := c_0, \quad (1.4)$$

where ${}^{C}{}_{\phi-1}\Delta^{\phi}$ denotes a fractional Liouville-Caputo forward difference with $0<\phi\leqq 1$, and $\mathscr{H}_1 and \mathscr{H}_2$ are two real-valued functions defined on $[1,\infty]\times\mathscr{R}, z\in\mathscr{N}_0\cap[0,\mathscr{T}], c_0\in\mathscr{R}$ is a crisp number, and $\varepsilon_{\phi-1},\varepsilon_{\phi},\cdots,\varepsilon_{\mathscr{T}+\phi-1}$ are the $(\mathscr{T}+1)$-IID uncertain variables with symmetrical uncertainty distribution $\mathbf{L}(a,b)$.

The rest of this article is organized as follows. In Sections 2.1 and 2.2, we revisit some necessary definitions, lemmas and axioms in the context of discrete fractional calculus and the uncertainty theory, respectively. In Section 3, we state the main result. Finally, we give some examples of applications in Section 4.

2. Preliminaries

In this section, we revisit notations, definitions, and preliminary facts associated with the discrete fractional calculus and the uncertainty theory, which are used throughout this article.

2.1. Discrete fractional calculus

Here, in this subsection, we recall some basics from discrete fractional calculus for later use in the following sections (see, for details, Goodrich and Peterson, 2015; Abdeljawad, 2013; Abdeljawad et al., 2017; Abdeljawad, 2018). The functions we consider are always defined on the isolated time scale $\mathscr{N}_{\ell} := \{\ell,\ell+1,\ell+2,\cdots\}$ for a fixed $\ell\in\mathscr{R}$. The operators given by

$$\rho(\kappa) := \kappa-1 \quad \text{and} \quad \sigma(\kappa) := \kappa+1$$

are, respectively, the backward and forward jump operators for $\kappa\in\mathscr{N}_{\ell}$. Moreover, the following operators:

$$\nabla\wp(\kappa) = \wp(\kappa)-\wp(\rho(\kappa)) \quad \text{and} \quad \Delta\wp(\kappa) = \wp(\sigma(\kappa))-\wp(\kappa)$$

are, respectively, the backward and forward difference operators for $\kappa\in\mathscr{N}_{\ell}$.

Definition 1 (*see* Goodrich and Peterson, 2015). Suppose that $\wp:\mathscr{N}_{\ell}\to\mathscr{R}$ and $\beta>0$. Then the Δ-RL fractional sum of $\wp$ is defined by

$$\left({}_{\ell}\Delta^{-\beta}\wp\right)(z) := \frac{1}{\Gamma(\beta)}\sum_{r=\ell}^{z-\beta}[z-\sigma(r)]^{(\beta-1)}\wp(r) \quad (\forall z\in\mathscr{N}_{\ell+\beta}). \quad (2.1)$$

Here $z^{(\beta)}$ represents the falling factorial function, which is defined by

$$z^{(\beta)} = \frac{\Gamma(z+1)}{\Gamma(z+1-\beta)}, \quad (2.2)$$

for each $z,\beta\in\mathscr{R}$.

Lemma 1 (*see* Goodrich and Peterson, 2015). *Suppose that $\ell\in\mathscr{R},\beta>0$ and $\phi\geqslant q0$. Then*

$$_{\ell}+\phi\Delta^{-\beta}(z-a)^{(\phi)} = \frac{\Gamma(\phi+1)}{\Gamma(\phi+\beta+1)}(z-a)^{(\phi+\beta)} \quad (\forall z\in\mathscr{N}_{\ell+\phi+\beta}).$$

Lemma 2 (*see* Atici and Eloe, 2007; Atici and Eloe, 2009; Abdeljawad, 2013; Abdeljawad et al., 2017; Abdeljawad, 2018). *For any function $\wp$ defined on $\mathscr{N}_{\ell}$ and any $\beta,\phi>0$, it is asserted that*

1. (i) $\left({}_{\ell}+\phi\Delta^{-\beta}{}_{\ell}\Delta^{-\phi}\wp\right)(z) = \left({}_{\ell}\Delta^{-(\phi+\beta)}\wp\right)(z) = \left({}_{\ell}+\beta\Delta^{-\phi}{}_{\ell}\Delta^{-\beta}\wp\right)(z)$ for $z\in\mathscr{N}_{\ell+\phi+\beta}$.
2. (ii) $\left({}_{\ell}\Delta^{\beta}\wp\right)(z) = \left(\Delta{}_{\ell}\Delta^{-(1-\beta)}\wp\right)(z)$.
3. (iii) $\left({}_{\ell}\Delta^{-\beta}\Delta\wp\right)(z) = \left(\Delta{}_{\ell}\Delta^{-\beta}\wp\right)(z) - \frac{(z-a)^{\overline{\beta-1}}}{\Gamma(\beta)}\wp(\ell)$.
4. (iv) $\left({}_{\ell}\Delta^{-\beta}\Delta^{\beta}\wp\right)(z) = \left(\Delta^{\beta}{}_{\ell}\Delta^{-\beta}\wp\right)(z) = \wp(z)$ for $\beta\notin\mathscr{N}$ and $\left({}_{\ell}\Delta^{-\beta}\Delta^{\beta}\wp\right)(z) = \wp(z)-\sum_{k=0}^{\beta-1}\frac{(z-a)^{\overline{k}}}{k!}\Delta^k\wp(\ell) \quad (\beta\in\mathscr{N})$.

Lemma 3 (*see* Abdeljawad, 2011). *Suppose that $\ell\in\mathscr{R},\beta>0,n-1<\beta\leqq n$ and $\wp$ is defined on $\mathscr{N}_{\ell}$. Then*

$$\left({}_{\ell}+n-\beta\Delta^{-\beta C}{}_{\ell}\Delta^{\beta}\wp\right)(z) = \wp(z)-\sum_{k=0}^{n-1}\frac{(z-a)^{(k)}}{k!}\Delta^k\wp(\ell). \quad (1.3)$$

Particularly, for $0<\beta\leqq 1$, it is asserted that

$$\left({}_{\ell}+1-\beta\Delta^{-\beta C}{}_{\ell}\Delta^{\beta}\wp\right)(z) = \wp(z)-\wp(\ell). \quad (1.4)$$

Definition 2 (*see* Abdeljawad, 2018). Let $\wp$ be defined on $\mathscr{N}_{\ell}$ and $n-1<\beta\leqq n$ for $n\in\mathscr{N}_1$ and $\beta>0$. Then the delta Liouville-Caputo fractional differences of order ϕ are defined by

$$\left({}^{C}{}_{\ell}\Delta^{-\beta}\wp\right)(z) := \left({}_{\ell}\Delta^{-(n-\beta)}\Delta^n\wp\right)(z) = \frac{1}{\Gamma(n-\beta)}\sum_{r=\ell}^{z-(n-\beta)}[z-\sigma(r)]^{(\beta-1)}\Delta^n\wp(r) \quad (\forall z\in\mathscr{N}_{\ell+n-\beta}). \quad (1.5)$$

Recently, Lu et al. (2019) introduced the βth order Riemann–Liouville fractional sum for uncertain sequence ε_z.

Definition 3 (*see* Lu et al., 2019). For any $\beta>0$, $\ell\in\mathscr{R}$ and uncertain sequence ε_z indexed by $z\in\mathscr{N}_{\ell}$, we define the βth order RL fractional sum of ε_z as follows:

$${}_{\ell}\Delta^{-\beta}\varepsilon_z = \frac{1}{\Gamma(\beta)}\sum_{r=\ell}^{z-\beta}[z-\sigma(r)]^{\overline{\beta-1}}\varepsilon_r.$$

We now define the βth order Liouville-Caputo fractional sum for uncertain sequence ε_z in the following definition.

Definition 4. For any $\beta > 0, \ell \in \mathscr{R}$ and the uncertain sequence ε_z indexed by $z \in \mathscr{N}_{\ell}$, we define the βth order Liouville-Caputo fractional sum of ε_z as follows:

$${}^{C}{}_{\ell}\Delta^{-\beta}\varepsilon_z = \frac{1}{\Gamma(n-\beta)}\sum_{r=\ell}^{z-(n-\beta)}[z-\sigma(r)]^{(\beta-1)}\Delta^n\varepsilon_r.$$

Definition 5 (*see Lu et al., 2019*). For any $\beta > 0$, the fractional RL backward difference for the uncertain sequence ε_z is defined by

$$\Delta_{\ell}^{\beta}\varepsilon_z = \Delta_{\ell}\left(\Delta_{\ell}^{-(1-\beta)}\varepsilon_z\right)$$

for $n-1 < \phi \leqq n$ for $n \in \mathscr{N}_1$

Next, we recall the definition of delta discrete Mittag–Leffler (delta-ML) functions.

Definition 6 (*see Haider et al., 2020*). Assume that $\phi, \mu, \zeta \in \mathscr{C}$ with $Re(\phi) > 0$ and $\lambda \in \mathscr{R}$ with $|\lambda| < 1$. Then the discrete delta-ML functions are given by

$$\mathrm{E}_{(\phi,\mu)}(\lambda,\zeta) := \sum_{e=0}^{\infty}\frac{[\zeta + e(\phi-1)]^{(e\phi+\mu-1)}}{\Gamma(e\phi+\mu)}\lambda^e. \tag{1.6}$$

Particularly, if $\mu = 1$, we obtain

$$\mathrm{E}_{(\phi)}(\lambda,\zeta) := \sum_{e=0}^{\infty}\frac{[\zeta + e(\phi-1)]^{(e\phi)}}{\Gamma(e\phi+1)}\lambda^e. \tag{1.7}$$

2.2. Uncertainty theory

In this subsection, we focus on the uncertainty theory concepts (see Liu, 2010). Let χ be a non-empty set and $\mathbf{L}$ be a σ-algebra over the set χ. Each element δ in $\mathbf{L}$ is called an event. A set function $\mathscr{M}$ defined on the σ-algebra $\mathbf{L}$ is called an uncertain measure if it satisfies the following axioms:

[Normality axiom:] $\mathscr{M}\{\chi\} = 1$ for the universal set χ.
[Duality axiom:] $\mathscr{M}\{\delta\} + \mathscr{M}\{\delta^c\} = 1$ for each event δ.
[Subadditivity axiom:] $\mathscr{M}\{\bigcup_{k=1}^{\infty}\delta_k\} \leqq \sum_{k=1}^{\infty}\mathscr{M}\{\delta_k\}$ for each countable sequence of events $\delta_2, \delta_2, \cdots$.
[Product axiom:] In view the above three axioms, it is clear that uncertain measure is a monotone increasing set function. The triplet $(\chi, \mathbf{L}, \mathscr{M})$ is called an uncertainty space.
We now suppose that $\left(\chi_j, \mathbf{L}_j, \mathscr{M}_j\right)$ are the uncertainty spaces and δ_j are any arbitrarily chosen events for $j = 1, 2, \cdots$. Then the product uncertain measure $\mathscr{M}$ is an uncertain measure satisfying the following condition:

$$\mathscr{M}\left\{\prod_{j=1}^{\infty}\delta_j\right\} = \bigwedge_{j=1}^{\infty}\mathscr{M}_j\{\delta_j\},$$

where $\bigwedge$ is the minimum operator.

Consequently, we have the following definitions.

Definition 7 (*see Liu, 2010*). A function ε from an uncertainty space $(\chi, \mathbf{L}, \mathscr{M})$ to $\mathscr{R}$ (the set of real numbers) is called an uncertain variable such that the set $\{\varepsilon \in \mathscr{B}\} := \{\gamma \in \chi; \varepsilon(\gamma) \in \mathscr{B}$ is an event for any Borel set $\mathscr{B}$ of real numbers. The uncertainty distribution $\Psi(x)$ of an uncertain variable ε is defined as $\Psi(x) := \mathscr{M}\{\varepsilon \leqq x\}$.

Definition 8 (*see Liu, 2010*). An uncertainty distribution Ψ is called regular if it is a continuous and strictly increasing function with respect to x for which $0 < \Psi(x) < 1$ and it satisfies the following condition:

$$\lim_{x\to-\infty}\Psi(x) = 0 \quad \text{and} \quad \lim_{x\to+\infty}\Psi(x) = 1.$$

Definition 9 (*see Liu, 2010*). Let ε be an uncertain variable with a regular uncertainty distribution Ψ. Then the inverse function Ψ^{-1} is called the inverse uncertainty distribution (IUD) of ε.

Example 1. In the light of Definition 9, one can observe that

1. (i) The IUD of linear uncertain variable $\mathbf{L}(\ell_1, \ell_2)$ is given by
$$\Psi^{-1}(\phi) = (1-\phi)\ell_1 + \phi\ell_2.$$
2. (ii) The IUD of a normal uncertain variable $\mathscr{N}(\ell_1, \ell_2)$ is given by
$$\Psi^{-1}(\phi) = \ell_1 + \frac{\sqrt{3}\ell_2}{\pi}\ln\left(\frac{\phi}{1-\phi}\right).$$
3. (iii) The IUD of a normal uncertain variable $\mathscr{LOGN}(\ell_1, \ell_2)$ is given by
$$\Psi^{-1}(\phi) = \exp(\ell_1) + \left(\frac{\phi}{1-\phi}\right)^{\frac{\sqrt{3}\ell_2}{\pi}}.$$

Definition 10 (*see Liu, 2010*). Let $\Psi(x)$ be a regular uncertainty distribution of ε. Then we say that an uncertain variable ε is symmetrical if

$$\Psi(x) + \Psi(-x) = 1.$$

Remark 1. From Definition (10, we can deduce that the symmetrical uncertain variable has the inverse uncertainty distribution $\Psi^{-1}(\phi)$ that satisfies the following condition:

$$\Psi^{-1}(\phi) + \Psi^{-1}(1-\phi) = 0.$$

Example 2. From Definition 10, we deduce that

1. The linear uncertain variable $\mathbf{L}(-\ell, \ell)$ is symmetrical for any positive real number a.
2. The normal uncertain variable $\mathbf{N}(0, 1)$ is symmetrical.

Definition 11 (*see Liu, 2010*). The uncertain variables $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ are said to be independent if, for any Borel sets $\mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_n$ of real numbers, we have

$$\mathscr{M}\left\{\bigcap_{j=1}^{n}\{\varepsilon_j \in \mathscr{B}_j\}\right\} = \bigwedge_{j=1}^{n}\mathscr{M}\{\varepsilon_j \in \mathscr{B}_j\}.$$

Definition 12 (*see Liu, 2010*). (The IID) The uncertain variables $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ are said to be independent identical distribution (or, briefly, IID) if they are independent and have the same uncertainty distribution.

3. UFLCDE and the associated existence and uniqueness theorem

In view of the earlier works Lu et al., 2019 and Mohammed et al., 2020, we can state the definition of the UFLCDE as follows.

Definition 13. An uncertain fractional difference equation is a fractional difference equation which is driven by an uncertain sequence. Moreover, an uncertain fractional forward difference equation in the Liouville-Caputo sense (UFLCDE) is the uncertain fractional difference equation with the Liouville-Caputo forward difference.

Lemma 4. *The initial-value problem (1.3) with the initial conditions (1.4) is equivalent to the following uncertain fractional sum equation:*

$$\wp(z) = c_0 + \frac{1}{\Gamma(\phi)}\sum_{r=0}^{z-\phi}[z-\sigma(r)]^{(\phi-1)}\left[\mathscr{H}_1(r+\phi-1, \wp(r+\phi-1)) + \mathscr{H}_2(r+\phi-1, \wp(r+\phi-1))\varepsilon_{r+\phi-1}\right] \qquad (z \in \mathscr{N}_\phi \cap [0, \mathscr{T}]). \tag{3.1}$$

Proof 1. By applying ${}_0\Delta^{-\phi}$ on IVP (1.3) and, by using (1.4), we can directly obtain the desired result. Moreover, the readers can see Abdeljawad, 2011, Example 17 and Baleanu et al., 2020, Lemma 2.10 for more details.

In this investigation, we focus now on the following special linear UFLCDE:

$$\left({}^{C}_{\phi-1}\Delta^{\phi}\wp\right)(z) = \lambda\,\wp(z+\phi-1) + \lambda\,\varepsilon_{z+\phi-1}, \tag{3.2}$$

$${}^{C}_{\phi-1}\Delta^{\phi-1}\wp(z)|_{z=0} = c_0, \tag{3.3}$$

for $z \in \mathscr{N}_0 \cap [0, \mathscr{T}]$ and $\lambda \in (0, 1)$.

Theorem 1. *For any $z \in \mathscr{N}_\phi \cap [0, \mathscr{T}]$ and $|\lambda| < 1$, the linear UFLCDE (3.2) with the initial condition (3.3) has a solution given by*

$$\wp(z) = \mathrm{E}_{(\phi)}(\lambda, z)c_0 + \varepsilon_z, \tag{3.4}$$

where ε_z is an uncertain sequence with the uncertainty distribution as follows:

$$\mathbf{L}\left(\ell \cdot \lambda \mathrm{E}_{(\phi,\phi+1)}(\lambda, z), b \cdot \lambda \mathrm{E}_{(\phi,\phi+1)}(\lambda, z)\right).$$

Proof 2. Applying ${}_0\Delta^{-\phi}$ on the Eq. (3.2), we get

$${}_0\Delta_\phi^{-\phi}\left[{}^{C}_{\phi-1}\Delta^{\phi}\wp(z)\right] = \lambda_0\Delta^{-\phi}\wp(z+\phi-1) + \lambda_0\Delta^{-\phi}\varepsilon_{z+\phi-1} \quad (z \in \mathscr{N}_\phi \cap [0, \mathscr{T}]). \tag{3.5}$$

Thus, by making use of Lemma 3, it follows that

$$\wp(z) = c_0 + \lambda_0\Delta^{-\phi}\wp(z+\phi-1) + \lambda_0\Delta^{-\phi}\varepsilon_{z+\phi-1} \quad (z \in \mathscr{N}_\phi \cap [0, \mathscr{T}]), \tag{3.6}$$

which is the solution of the given UFLCDE (3.4).

To obtain an explicit solution, we use the method of the Picard approximation with a starting point $\wp_0(z) = c_0$. In addition, we can obtain the other components by using the following recurrence relation:

$$\wp_q(z) = c_0 + \lambda_0\Delta^{-\phi}\wp_q(z+\phi-1) + \lambda_0\Delta^{-\phi}\varepsilon_{z+\phi-1} \quad (z \in \mathscr{N}_\phi \cap [0, \mathscr{T}];\ q = 0, 1, \cdots). \tag{3.7}$$

We can now write $\varepsilon_{z+\phi-1} = \varepsilon$ in the distribution, since $\varepsilon_{\phi-1}, \varepsilon_\phi, \cdots, \varepsilon_{\mathscr{T}+\phi-1}$ are the IID uncertain variables. By making use of Lemma 1 and the fact that the linear combination of finite independent uncertain variables is also an uncertain variable with positive linear combination coefficient (see, for details, Liu, 2010, Theorems 1.21 to 1.24), we can obtain

$$\begin{aligned}\wp_1(z) &= c_0 + \lambda_0\Delta^{-\phi}\wp_0(z) + \lambda_0\Delta^{-\phi}\varepsilon \\ &= \left[1 + \lambda\frac{z^{(\phi)}}{\Gamma(\phi+1)}\right]c_0 + \lambda\frac{z^{(\phi)}}{\Gamma(\phi+1)}\,\varepsilon\end{aligned}$$

$$\begin{aligned}\wp_2(z) &= \tfrac{(z-\phi+1)^{\overline{\phi-1}}}{\Gamma(\phi)}c_0 + \lambda_0\Delta^{-\phi}\wp_1(z) + \lambda_0\Delta^{-\phi}\varepsilon, \\ &= \left[1 + \lambda\tfrac{z^{(\phi)}}{\Gamma(\phi+1)} + \lambda^2\tfrac{z^{(2\phi)}}{\Gamma(2\phi+1)}\right]c_0 + \left[\lambda\tfrac{z^{(\phi)}}{\Gamma(\phi+1)} + \lambda^2\tfrac{z^{(2\phi)}}{\Gamma(2\phi+1)}\right]\varepsilon \\ &\ \vdots\end{aligned}$$

and so on. Continuing the process up to the qth term, we find that

$$\wp_q(z) = c_0\sum_{k=0}^{q}\lambda^k\frac{[z+k(\phi-1)]^{(k\phi)}}{\Gamma(k\phi+1)} + \sum_{k=1}^{q}\lambda^k\frac{[z+(k-1)(\phi-1)]^{(k\phi)}}{\Gamma(k\phi+1)}\,\varepsilon,$$

for $z \in \mathscr{N}_\phi \cap [0, \mathscr{T}]$. Now, by letting $q \to \infty$, we get the following solution:

$$\begin{aligned}\hat{\wp}(z) := \lim_{q\to\infty}\wp_q(z) &= c_0\sum_{k=0}^{\infty}\lambda^k\tfrac{[z+k(\phi-1)]^{(k\phi)}}{\Gamma(k\phi+1)} + \sum_{k=1}^{\infty}\lambda^k\tfrac{[z+(k-1)(\phi-1)]^{(k\phi)}}{\Gamma(k\phi+1)}\,\varepsilon \\ &= c_0\mathrm{E}_{(\phi)}(\lambda, z) + \lambda\,\mathrm{E}_{(\phi,\phi+1)}(\lambda, z)\,\varepsilon, \quad z \in \mathscr{N}_\phi \cap [0, \mathscr{T}].\end{aligned} \tag{3.8}$$

The solution $\hat{\wp}(z)$ exists, since the Mittag-Liffler functions $\mathrm{E}_{(\phi)}(\lambda, z)$ and $\mathrm{E}_{(\phi,\phi+1)}(\lambda, t+\phi-1)$ are absolutely convergent for $|\lambda| < 1$.

In addition, if we take $lim_{q\to\infty}$ on both sides of (3.7), we get

$$\hat{\wp}(z) = c_0 + \lambda_0\Delta^{-\phi}\hat{\wp}(z+\phi-1) + \lambda_0\Delta^{-\phi}\varepsilon_{z+\phi-1} \quad (z \in \mathscr{N}_\phi \cap [0, \mathscr{T}]).$$

This means that $\hat{\wp}(z)$ satisfies the Eq. (3.6). Hence, clearly, $\hat{\wp}(z)$ is a solution of the Eqs. 3.2,3.3. Our proof of Theorem 1 is thus completed.

We now state the existence and uniqueness of the solution of UFLCDEs.

Theorem 2 (*Existence and Uniqueness*). *Let $\mathscr{H}_1(z, \wp_1)$ and $\mathscr{H}_2(z, \wp_1)$ be two real-valued functions in (1.3) and satisfy the following Lipschitz condition*:

$$|\mathscr{H}_1(z, \wp_1) - \mathscr{H}_1(z, \wp_2)| + |\mathscr{H}_2(z, \wp_1) - \mathscr{H}_2(z, \wp_2)| \leq L|\wp_1 - \wp_2|, \tag{3.9}$$

with the Lipschitz constant L that satisfies the inequality given by

$$L < \frac{\Gamma(\phi+1)\Gamma(\mathscr{T}+1-\phi)}{\Gamma(\mathscr{T}+1)(\vartheta+1)}, \tag{3.10}$$

where $\vartheta = |\ell_1| \vee |\ell_2|$. Then the UFLCDE (1.3)–(1.4) have a unique solution $\wp(z)$ for all $z \in \mathscr{N}_\phi \cap [0, \mathscr{T}]$ almost surely.

Proof 3. Let us define l_ϕ^k (the set of all finite real sequences $\{\wp(z)\}_\phi^k$) with the norm $\|\wp\|$ as follows:

$$l_\phi^k := \left\{\wp;\ \wp = \{\wp(z)\}_\phi^k,\ k \in \mathscr{N}_1\right\}$$

and

$$\|\wp\| := \max_{z\in\mathscr{N}_\phi\cap[0,\mathscr{T}]} |\wp(z)|,$$

which has k terms. It is easy to see that $\left(l_\phi^k, \|\cdot\|\right)$ is a Banach space (see, for details, Sacks, 2017, Chapter 4).

We now define the operator $\mathscr{D}$ for $\wp_z \in l_\phi^k$ as follows:

$$\mathscr{D}\wp_z = c_0 + \frac{1}{\Gamma(\phi)} \sum_{r=0}^{z-\phi} [z-\sigma(r)]^{(\phi-1)} \Big[\mathscr{H}_1\Big(r+\phi-1, \wp_{r+\phi-1}\Big) + \mathscr{H}_2\Big(r+\phi-1, \wp_{r+\phi-1}\Big)\varepsilon_{r+\phi-1}\Big].$$

We also assume that χ represents the universal set on the uncertainty space. Clearly, $\mathscr{M}\{(\varepsilon_z < a)\cup(\varepsilon_z > b)\} = 0$, since $\varepsilon_z(z\in\mathscr{N}_\phi\cap[0,\mathscr{T}])$ is an uncertain variable at each time t with the linear uncertainty distribution $\mathbf{L}(\ell_1,\ell_2)$. In addition, the inequality $\varepsilon_z(\gamma) \leqq \vartheta$, where $\vartheta = |\ell_1|\vee|\ell_2|$, holds true almost surely for each γ given by

$$\gamma \in \chi\big\{(\varepsilon_z < \ell_1)\cup(\varepsilon_z > \ell_2);\ z\in\mathscr{N}_\phi\cap[0,\mathscr{T}]\big\}.$$

For any $x_z, \wp_z \in l_\phi^k$, we then obtain

$$\begin{aligned}\left\|\mathscr{D}x_z(\gamma) - \mathscr{D}\wp_z(\gamma)\right\| &= \max_{z\in\mathscr{N}_\phi\cap[0,\mathscr{T}]} \left|\mathscr{D}x_z(\gamma) - \mathscr{D}\wp_z(\gamma)\right| \\ &\leqq \tfrac{1}{\Gamma(\phi)} \max_{z\in\mathscr{N}_\phi\cap[0,\mathscr{T}]} \sum_{r=0}^{z-\phi}[z-\sigma(r)]^{\overline{\phi-1}}\Big(\Big|\mathscr{H}_1(r+\phi-1, x_{r+\phi-1}(\gamma)) - \mathscr{H}_1\Big(r+\phi-1, \wp_{r+\phi-1}(\gamma)\Big)\Big| + \Big|\Big[\mathscr{H}_2(r+\phi-1, x_{r+\phi-1}(\gamma)) - \mathscr{H}_2\Big(r+\phi-1, \wp_{r+\phi-1}(\gamma)\Big)\Big]\varepsilon_r\Big|\Big) \\ &\leqq \tfrac{1}{\Gamma(\phi)} \max_{z\in\mathscr{N}_\phi\cap[0,\mathscr{T}]} \sum_{r=0}^{z-\phi}[z-\sigma(r)]^{\overline{\phi-1}}\Big(\Big|\mathscr{H}_1(r+\phi-1, x_{r+\phi-1}(\gamma)) - \mathscr{H}_1\Big(r+\phi-1, \wp_{r+\phi-1}(\gamma)\Big)\Big| + \vartheta\Big|\mathscr{H}_2(r+\phi-1, x_{r+\phi-1}(\gamma)) - \mathscr{H}_2\Big(r+\phi-1, \wp_{r+\phi-1}(\gamma)\Big)\Big|\Big)\end{aligned}$$

Thus, by the help of the assumptions and Lemma 1, we get

$$\begin{aligned}\left\|\mathscr{D}x_z(\gamma) - \mathscr{D}\wp_z(\gamma)\right\| &\leqq L(1+\vartheta)\tfrac{1}{\Gamma(\phi)} \max_{z\in\mathscr{N}_\phi\cap[0,\mathscr{T}]} \sum_{r=0}^{z-\phi}(z-\sigma(r))^{\overline{\phi-1}}|x_r(\gamma) - \wp_r(\gamma)| \\ &\leqq L(1+\vartheta)\|x_z(\gamma) - \wp_z(\gamma)\| \max_{z\in\mathscr{N}_\phi\cap[0,\mathscr{T}]}\left({}_0\Delta^{-\phi} z^{(0)}\right) \\ &= L(1+\vartheta)\|x_z(\gamma) - \wp_z(\gamma)\| \max_{z\in\mathscr{N}_\phi\cap[0,\mathscr{T}]}\left(\tfrac{1}{\Gamma(\phi+1)} z^{(\phi)}\right) \\ &\leqq \tfrac{L(1+\vartheta)\mathscr{T}^{(\phi)}}{\Gamma(\phi+1)}\|x_z(\gamma) - \wp_z(\gamma)\| \\ &= \tfrac{L(1+\vartheta)\Gamma(\mathscr{T}+1)}{\Gamma(\phi+1)\Gamma(\mathscr{T}-\phi+1)}\|x_z(\gamma) - \wp_z(\gamma)\|.\end{aligned}$$

The mapping $\mathscr{D}$ is a contraction in l_ϕ^k almost surely such that (see Sacks, 2017, Chapter 4)

$$0 < L < \frac{\Gamma(\phi+1)\Gamma(\mathscr{T}-\phi+1)}{(1+\vartheta)\Gamma(\mathscr{T}+1)}.$$

Therefore, by using the Banach contraction mapping theorem (see Sacks, 2017, Chapter 4), we obtain a unique fixed point $\wp_z(\gamma)$ of $\mathscr{D}$ in l_ϕ^k almost surely. Furthermore, we have

$$\wp_z(\gamma) = \lim_{j\to\infty}\wp_z^j(\gamma),$$

where

$$\wp_z^j(\gamma) = \mathscr{D}\left(\wp_z^{j-1}(\gamma)\right)$$

with

$$\wp_z^0(\gamma) = \frac{(z-\phi)^{\overline{\phi-1}}}{\Gamma(\phi)} c_0.$$

On the other hand, the operator $\mathscr{D}$ is measurable for any $z\in\mathscr{N}_\phi\cap[0,\mathscr{T}]$, since $\mathscr{H}_1$ and $\mathscr{H}_2$ are Lipschitz continuous functions. Since

$$\wp_z^1(\gamma), \wp_z^2(\gamma), \cdots, \wp_z^j(\gamma), \cdots$$

are uncertain variables and since $\wp_z^0(\gamma)$ is a real-valued measurable function of uncertain variables, $\wp_z^0(\gamma)$ is seen to be an uncertain variable by using Liu, 2010, Theorem 1.10. Hence, clearly, $\wp_z = \lim_{j\to\infty}\wp_z^j$ is an uncertain variable by using Zhu, 2015, Theorem 3. Therefore, the UFLCDE (3.2) subject to the initial condition (3.3) has a unique solution $\wp_z$ for $z\in\mathscr{N}_\phi\cap[0,\mathscr{T}]$ almost surely. We thus have completed the existence and uniqueness asserted by Theorem 2.

4. Illustrative examples

In this section, we deal with some UFLCDE applications to confirm the validity our Theorem 2.

Example 3. Consider the following UFLCDE:

$$^{C}{}_{-0.5}\Delta^{0.5}\wp(z) = \begin{cases}\frac{\ln\left(|\wp(z-0.5)|+1\right)}{64(z-0.5)^3} + 0.5\varepsilon_{z-0.5} & (z\in\mathscr{N}_0\cap[0,3]), \\ \wp(\frac{1}{2}) = 1,\end{cases} \tag{4.1}$$

where $\varepsilon_{-0.5}, \varepsilon_{0.5}, \varepsilon_{1.5}, \varepsilon_{2.5}$ are four IID uncertain variables with the uncertainty distribution $\mathbf{L}(-1,2)$.

According to Lemma 4 with $\phi = \frac{1}{2}$, the inverse uncertainty distribution of the solution for the UFLCDE (4.1) is the solution of the following sum equation:

$$\wp(z) = c_0 + \frac{1}{\Gamma(0.5)}\sum_{r=0}^{z-0.5}[z-\sigma(r)]^{(-0.5)} \left(\frac{\ln\left(|\wp(r-0.5)|+1\right)}{4(r-0.5)^3} + 0.25\varepsilon_{r-0.5}\right) \quad (z\in\mathscr{N}_0\cap[0,3]).$$

Thus, for $z\in\mathscr{N}_0\cap[0,3]$, we have

$$\begin{aligned}
&|\mathscr{H}_1(z,\wp_1)-\mathscr{H}_1(z,\wp_2)|+|\mathscr{H}_2(z,\wp_1)-\mathscr{H}_2(z,\wp_2)|\\
&\quad=\left|\frac{\ln(|\wp_1|+1)}{64(z-0.5)^3}-\frac{\ln(|\wp_2|+1)}{64(z-0.5)^3}\right|\\
&\quad=\frac{1}{64(z-0.5)^3}|\ln(|\wp_1|+1)-\ln(|\wp_2|+1)|\\
&\quad\leqq\frac{1}{64(\frac{1}{2})^3}||\wp_1|-|\wp_2||\\
&\quad\leqq\frac{|\wp_1-\wp_2|}{8}
\end{aligned}$$

and

$$\frac{\Gamma(0.5+1)\Gamma(3+1-0.5)}{3\Gamma(3+1)}\approx 0.1636>\frac{1}{8}=0.125.$$

Therefore, in view of Theorem 2, this confirms that the UFLCDE (4.1) has a unique solution almost surely.

Example 4. We consider the following UFLCDE:

$$ {}^{C}_{-0.75}\Delta^{0.25}\wp(z)=\frac{\wp^2(z-0.75)}{40}+\varepsilon_{z-0.75}\quad(z\in\mathscr{N}_0\cap[0,3]), \tag{4.2}$$

where $\varepsilon_{-0.75},\varepsilon_{0.25},\varepsilon_{1.25},\varepsilon_{2.25}$ are four IID linear uncertain variables with the linear uncertainty distribution $\mathbf{L}(-3,3)$.

According to Lemma 4 with $\phi=\frac{1}{4}$, the inverse uncertainty distribution of the solution for the UFLCDE (4.2) is the solution of the following sum equation:

$$\wp(z)=c_0+\frac{1}{\Gamma(0.25)}\sum_{r=0}^{z-0.25}[z-\sigma(r)]^{(-\frac{3}{4})}\left(\frac{\wp^2(r-0.75)}{40}+\varepsilon_{r-0.75}\right).$$

We observe that

$$|\mathscr{H}_1(z,\wp_1)-\mathscr{H}_1(z,\wp_2)|+|\mathscr{H}_2(z,\wp_1)-\mathscr{H}_2(z,\wp_2)|$$

is Lipschitz continuous in $[-20,20]$ with Lipschitz constant 0.1 as follows:

$$\begin{aligned}
&|\mathscr{H}_1(z,\wp_1)-\mathscr{H}_1(z,\wp_2)|+|\mathscr{H}_2(z,\wp_1)-\mathscr{H}_2(z,\wp_2)|\leqq\frac{1}{40}|\wp_1\\
&\quad+\wp_2||\wp_1-\wp_2|=0.1|\wp_1-\wp_2|.
\end{aligned}$$

We also have

$$\frac{\Gamma(0.25+1)\Gamma(3+1-0.25)}{4\Gamma(3+1)}\approx 0.167>0.1.$$

Consequently, in the light of Theorem 2. this confirms that the UFLCDE (4.3) has a unique solution almost surely.

Example 5. Consider the following UFLCDE:

$$ {}^{C}_{-0.5}\Delta^{0.5}\wp(z)=\frac{\sin\left(\wp(z-\frac{1}{2})\right)}{10+(z-\frac{1}{2})^2}+0.1\varepsilon_{z-\frac{1}{2}}\quad(z\in\mathscr{N}_0\cap[0,3]), \tag{4.3}$$

where $\varepsilon_{-\frac{1}{2}},\varepsilon_{\frac{1}{2}},\varepsilon_{\frac{3}{2}},\varepsilon_{\frac{5}{2}}$ are four IID linear uncertain variables with the linear uncertainty distribution $\mathbf{L}(-1,1)$.

According to Lemma 4 with $\phi=\frac{1}{2}$, the inverse uncertainty distribution of the solution for the UFLCDE (4.3) is the solution of the following sum equation:

$$\wp(z)=c_0+\frac{1}{\sqrt{\pi}}\sum_{r=0}^{z-\frac{1}{2}}[z-\sigma(r)]^{(-0.5)}\left(\frac{\sin(\wp(r-\frac{1}{2}))}{10+(r-\frac{1}{2})^2}+0.1\varepsilon_{r-\frac{1}{2}}\right).$$

We can thus verify directly that

$$|\mathscr{H}_1(z,\wp_1)-\mathscr{H}_1(z,\wp_2)|+|\mathscr{H}_2(z,\wp_1)-\mathscr{H}_2(z,\wp_2)|\leqq\frac{1}{10}|\wp_1-\wp_2|$$

and

$$\frac{\Gamma(\frac{1}{2}+1)\Gamma(3+1-\frac{1}{2})}{2\Gamma(3+1)}\approx 0.2454>\frac{1}{10}=0.1.$$

Therefore, in view of Theorem 2, this confirms that the UFLCDE (4.3) has a unique solution almost surely.

5. Conclusion

Our investigation in this article can be summarized as follows:

- The basic concepts of the discrete fractional calculus and the uncertainty theory have been recalled and applied.
- A certain UFLCDE (uncertain fractional forward difference equation in the Liouville-Caputo sense) has been introduced and investigated systematically.
- An uncertain fractional sum equation, corresponding to the UFLCDE considered here, has been found.
- The successive Picard iteration method has been successfully used for finding a solution to the UFLCDE investigated here.
- The theory of Banach contraction under the Lipschitz constant condition has been used in order to investigate the existence and uniqueness of the solution of the UFLCDE studied here.
- Three illustrative examples are presented to exhibit and verify the validity of the proposed investigations.

In concluding this investigation, we remark that several recent developments (see, for example, Srivastava and Saad, 2020; Khader et al., 2020; Srivastava et al., 2020; Izadi and Srivastava, 2020; Srivastava and Saad, 2020; Singh et al., 2021) on the theory and applications of fractional-order ODEs and PDEs modelling various real-world situations will provide motivation for interesting further directions for researches related to the subject-matter which we have presented here.

Data Availability

No data were used to support this study.

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Authors' contributions

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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