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On strict common fixed points of hybrid mappings in 2-metric spaces

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Abstract In this paper, we introduce an implicit relation with a view to cover several contractive conditions in one go and utilize the same to prove a general common fixed point theorem for two hybrid pairs of occasionally weakly compatible mappings defined on 2-metric spaces. Our results extend, generalize and unify several known common fixed point theorems of the existing literature.

1. Introduction

The concept of 2-metric spaces was introduced and investigated by Gähler in his papers (Gähler, 1963; Gähler, 1965) which were later developed by many other mathematicians including Gähler himself. Like various other aspects of the theory, a number of authors also studied a multitude of results of metric fixed point theory in the setting of 2-metric spaces. In doing so, the authors are indeed motivated by various concepts already known in respect of metric spaces which enable them to introduce analogous concepts in the frame work of 2-metric spaces. For this kind of work, we refer to Cho et al. (1988), Murthy et al. (1992), Tan et al. (2003), Naidu and Prasad (1986), Abu-Donia and Atia (2007), Pathak et al. (1995) wherein the weak conditions of commutativity such as: compatible mappings, compatible mappings of type (A) and type (P), weakly compatible mappings of type (A) and weakly compatible mappings were lifted to the setting of 2-metric spaces which were subsequently utilized to prove results on common fixed points in 2-metric spaces.

On the other hand, Al-Thagafi and Shahzad (2008) introduced the notion of occasional weak compatibility (in short OWC) as a generalization of weak compatibility. Jungck and Rhoades (2006) utilized this notion of OWC to prove common fixed point theorems in symmetric spaces. In fact, OWC is not a proper generalization of weak compatibility for those pairs of mappings whose set of coincidence points is empty. Imdad et al. (2011) pointed out that OWC is pertinent in respect of nontrivial weak compatible pairs (i.e., pairs with at least one coincidence point). In the same spirit, Pant and Pant (2010) redefined OWC and termed it as conditional commutativity wherein authors assumed that the set of coincidence points is nonempty. Most recently, Doric et al. (2011) proved that

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OWC and weak compatibility are identical notions in respect of single-valued pairs of mappings whenever point of coincidence is unique. But, the same is not true for pairs of hybrid mappings, i.e., OWC property is weaker than weak compatibility in respect of hybrid pairs of mappings.

2. Preliminaries

A 2-metric space is a set X equipped with a real-valued function d on X^3 which satisfies the following conditions:

(M_1) to each pair of distinct points x, y in X, there exists a point $z \in X$ such that $d(x,y,z) \neq 0$,
(M_2) $d(x,y,z) = 0$ when at least two of x,y,z are equal,
(M_3) $d(x,y,z) = d(x,z,y) = d(y,z,x)$,
(M_4) $d(x,y,z) \leqslant d(x,y,u) + d(x,u,z) + d(u,y,z)$ for all $x,y,z,u \in X$.

The function d is called a 2-metric on the set X whereas the pair (X,d) stands for 2-metric space. Geometrically, in respect of a 2-metric d, $d(x,y,z)$ represents the area of a triangle with vertices x, y and z.

It is known (cf. Gähler, 1965; Naidu and Prasad, 1986) that a 2-metric d is a non-negative continuous function in any one of its three arguments but the same need not be continuous in two arguments. A 2-metric d is said to be continuous if it is continuous in all of its arguments. Throughout this paper d stands for a continuous 2-metric.

Definition 2.1. A sequence $\{x_n\}$ in a 2-metric space (X,d) is said to be convergent to a point $x \in X$ (denoted by $\lim_{n\to\infty}x_n = x$) if $\lim_{n\to\infty}d(x_n,x,z) = 0$ for all $z \in X$.

Definition 2.2. A sequence $\{x_n\}$ in a 2-metric space (X,d) is said to be Cauchy sequence if $\lim_{n,m\to\infty}d(x_n,x_m,z) = 0$ for all $z \in X$.

Definition 2.3. A 2-metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

Remark 2.1 (Naidu and Prasad, 1986). In general, a convergent sequence in a 2-metric space (X,d) need not be Cauchy, but every convergent sequence is a Cauchy sequence whenever 2-metric d is continuous on X.

Definition 2.4 (Murthy et al., 1992). A pair of self mappings (S,T) of a 2-metric space (X,d) is said to be compatible if $\lim_{n\to\infty} d(STx_n,TSx_n,z) = 0$ for all $z \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty}Tx_n = t$ for some $t \in X$.

Definition 2.5. A pair of self mappings (S,T) of a nonempty set X is said to be weakly compatible if $Sx = Tx$ (for some $x \in X$) implies $STx = TSx$.

Let (X,d) be a 2-metric space. We denote by $B(X)$, the family of bounded subsets of (X,d). For all A, B and C in $B(X)$, let $D(A,B,C)$ and $\delta(A,B,C)$ be the functions defined by

$$D(A,B,C) = \inf\{d(a,b,c) : a \in A, b \in B, c \in C\},$$
$$\delta(A,B,C) = \sup\{d(a,b,c) : a \in A, b \in B, c \in C\}.$$

If A consists of a single point 'a', we write $\delta(A,B,C) = \delta(a,B,C)$. Further, if B and C also consist of single points 'b' and 'c', respectively, then we write $\delta(A,B,C) = D(a,b,c) = d(a,b,c)$.

It follows from the definition that

$\delta(A,B,C) = 0$ if at least two A, B, C are identically equal and singleton,

$$\delta(A,B,C) = \delta(A,C,B) = \delta(B,A,C) = \delta(B,C,A) = \delta(C,B,A) = \delta(C,A,B) \geqslant 0,$$
$$\delta(A,B,C) \leqslant \delta(A,B,E) + \delta(A,E,C) + \delta(E,B,C) \text{ for all } A,B,C,E \text{ in } B(X).$$

Definition 2.6. A sequence $\{A_n\}$ of subsets of a 2-metric space (X,d) is said to be convergent to a subset A of X if:

(i) given $a \in A$, there exists $\{a_n\}$ in X such that $a_n \in A_n$ for $n = 1,2,3,\ldots$ and $\lim_{n\to\infty}d(a_n,a,z) = 0$ for each $z \in X$, and
(ii) given $\epsilon > 0$, there exists a positive integer N such that $A_n \subset A_\epsilon$ for $n > N$ where A_ϵ is the union of all open balls with centers in A and radius ϵ.

Definition 2.7. The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be weakly commuting at x if $IFx \in B(X)$ and

$$\delta(FIx,IFx,z) \leqslant \max\{\delta(Ix,Fx,z),\delta(IFx,IFx,z)\}. \tag{2.1}$$

Remark 2.2. If F is a single-valued mapping, then the set IFx becomes singleton. Therefore, $\delta(IFx,IFx,z) = 0$ and condition (2.1) reduces to the condition given by Khan (1984), that is $D(FIx,IFx,z) \leqslant D(Ix,Fx,z)$.

Definition 2.8. The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be compatible if $\lim_{n\to\infty}D(FIx_n,IFx_n,z) = 0$ for all $z \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}Ix_n = t \in A = \lim_{n\to\infty} Fx_n$ for some $t \in X$ and $A \in B(X)$.

Definition 2.9. The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be δ-compatible if $\lim_{n\to\infty}\delta(FIx_n,IFx_n,z) = 0$ for all $z \in X$, whenever $\{x_n\}$ is a sequence in X such that $IFx_n \in B(X)$, $Fx_n \to \{t\}$ and $Ix_n \to t$ for some $t \in X$.

Definition 2.10. Let $I : X \to X$ and $F : X \to B(X)$. A point $x \in X$ is said to be a fixed point (strict fixed point) of F if $x \in Fx$ ($Fx = \{x\}$). Also, a point $x \in X$ is said to be a coincidence point (strict coincidence point) of (I,F) if $Ix \in Fx$ ($Fx = \{Ix\}$).

Definition 2.11 (Jungck and Rhoades, 1998). The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be weakly compatible if they commute at all strict coincidence points, i.e., for each x in X such that $Fx = \{Ix\}$, we have $FIx = IFx$.

Remark 2.3 (Jungck and Rhoades, 1998). Any δ-compatible pair (I,F) is weakly compatible but not conversely.

Definition 2.12. The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be strict occasionally weakly compatible if the pair commutes at some of it's strict coincidence points.

Quite recently, Abd El-Monsef et al. (2009) proved the following common fixed point theorem in 2-metric spaces.

Theorem 2.1. *If $I, J : X \to X$ and $F, G : X \to B(X)$ are mappings which satisfy*

(i) $\cup G(X) \subseteq I(X)$ *and* $\cup F(X) \subseteq J(X)$,
(ii)

$$\delta(Fx,Gy,C) \leqslant \alpha \max\{\delta(Ix,Jy,C),\delta(Ix,Fx,C),\delta(Jy,Gy,C)\} + (1-\alpha)[aD(Ix,Gy,C)+bD(Jy,Fx,C)]$$

for all $x, y \in X$ and $C \in B(X)$, where $0 \leqslant \alpha < 1, a + b < 1$, $a, b \geqslant 0$ and $\alpha|a - b| < 1 - (a + b)$,
(iii) *$I(X)$ (or $J(X)$) is complete subspace of (X, d),*
(vi) *both the pairs (F, I) and (G, J) are weakly compatible, then F, G, I and J have a unique common fixed point in X.*

There exists considerable literature on hybrid fixed point theorem involving diametral distances in metric spaces (e.g., Abd El-Monsef et al., 2007; Jungck and Rhoades, 1998; Sessa et al., 1986). The purpose of this paper is to prove a general common fixed point theorem for two pairs of OWC hybrid pair of mappings satisfying a newly defined implicit relation. Our results generalize and extend several previously known results of the existing literature.

3. Implicit relations

The study of common fixed point theorems in metric spaces for class of mappings satisfying implicit relations was initiated in Popa (1997, 1999). Following the lines of Imdad et al. (2002), Popa et al. (2010), employed this idea to prove common fixed point theorems in 2-metric spaces. Now, we define the following class of implicit relations.

Definition 3.1. Let Φ be the set of all continuous functions $\phi : \mathfrak{R}_+^6 \to \mathfrak{R}$ satisfying the following conditions:

(ϕ_1) ϕ is nondecreasing in variable t_1 and nonincreasing in variables $t_2 \ldots, t_6$.
(ϕ_2) there exists $h, k > 0$ with $hk < 1$ such that for $u, v \geqslant 0$
(ϕ_a): $\phi(u, v, v, u, u + v, 0) \leqslant 0$ implies $u \leqslant hv$,
(ϕ_b): $\phi(u, v, u, v, 0, u + v) \leqslant 0$ implies $u \leqslant kv$.

(ϕ_3) $\phi(t, t, 0, 0, t, t) > 0\ \forall t > 0$.

Example 3.1. Define $\phi(t_1, t_2, \ldots, t_6) : \mathfrak{R}_+^6 \to \mathfrak{R}$ as

$\phi(t_1, t_2, \ldots, t_6) = t_1 - \alpha \max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}$, where $\alpha \in (0, 1)$.

Setting $h = k = \alpha < 1$, the requirements of Definition 3.1 are met out.

Example 3.2. Define $\phi(t_1, t_2, \ldots, t_6) : \mathfrak{R}_+^6 \to \mathfrak{R}$ as

$$\phi(t_1, t_2, \ldots, t_6) = t_1^2 - c_1 \max\{t_2^2, t_3^2, t_4^2\} - c_2 \max\{t_3 t_5, t_4 t_6\} - c_3 t_5 t_6,$$

where $c_1 > 0$, $c_2, c_3 \geqslant 0$, $c_1 + 2c_2 < 1$ and $c_1 + c_3 < 1$.
Choosing $h = k = \sqrt{c_1 + 2c_2} < 1$, one can easily verify the requirements of Definition 3.1.

Example 3.3. Define $\phi(t_1, t_2, \ldots, t_6) : \mathfrak{R}_+^6 \to \mathfrak{R}$ as

$$\phi(t_1, t_2, \ldots, t_6) = t_1 - \alpha t_2 - \beta \min\{t_3, t_5\} - \eta \min\{t_4, t_6\},$$

where $\alpha, \beta, \eta > 0$, $\alpha + \beta < 1$, $\alpha + \eta < 1$ and $(\alpha + \beta)(\alpha + \eta) < 1$.
Setting $h = \alpha + \beta < 1$, $k = \alpha + \eta < 1$ with $hk < 1$, one can easily check the requirements of Definition 3.1.

Example 3.4. Define $\phi(t_1, t_2, \ldots, t_6) : \mathfrak{R}_+^6 \to \mathfrak{R}$ as

$$\phi(t_1, t_2, \ldots, t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1 - \alpha)(\beta t_5 + \eta t_6),$$

where $0 \leqslant \alpha < 1$, $\beta + \eta < 1$, $\beta, \eta \geqslant 0$ and $\alpha|\beta - \eta| < 1 - (\beta + \eta)$. Choosing $h = \max\left\{\frac{\alpha+(1-\alpha)\beta}{1-(1-\alpha)\beta}, \frac{\beta}{1-\beta}\right\}$, $k = \max\left\{\frac{\alpha+(1-\alpha)\eta}{1-(1-\alpha)\eta}, \frac{\eta}{1-\eta}\right\}$ with $hk < 1$ (see Abd El-Monsef et al., 2009, p. 1438), one can easily verify the requirements of Definition 3.1.

Example 3.5. Define $\phi(t_1, t_2, \ldots, t_6) : \mathfrak{R}_+^6 \to \mathfrak{R}$ as

$$\phi(t_1, t_2, \ldots, t_6) = t_1 - \psi\left(\max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\right\}\right)$$

where $\psi : \mathfrak{R}^+ \to \mathfrak{R}^+$ is an upper semi-continuous function such that $\psi(t) < t$ for all $t > 0$.

Example 3.6. Define $\phi(t_1, t_2, \ldots, t_6) : \mathfrak{R}_+^6 \to \mathfrak{R}$ as

$$\phi(t_1, t_2, \ldots, t_6) = t_1 - \psi(t_2, t_3, t_4, t_5, t_6)$$

where $\psi : \mathfrak{R}_+^5 \to \mathfrak{R}_+$ is an upper semi-continuous and increasing function in $t_2, \ldots, t_6$ such that $\psi(t, t, t, \alpha t, \beta t) < t$ for all $t > 0$ and $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 2$.

Example 3.7. Define $\phi(t_1, t_2, \ldots, t_6) : \mathfrak{R}_+^6 \to \mathfrak{R}$ as

$$\phi(t_1, t_2, \ldots, t_6) = \int_0^{t_1} \psi(t)\,dt - \alpha \int_0^{\max\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\}} \psi(t)\,dt$$

where $\alpha \in (0,1)$ and $\psi : \mathfrak{R}^+ \to \mathfrak{R}^+$ is a Lebesgue integrable function which is summable and $\int_0^{\epsilon} \psi(t)\,dt > 0$ for all $\epsilon > 0$.

Example 3.8. Define $\phi(t_1, t_2, \ldots, t_6) : \mathfrak{R}_+^6 \to \mathfrak{R}$ as

$$\phi(t_1, t_2, \ldots, t_6) = \int_0^{t_1} \psi(t)\,dt - \alpha \max\left\{\int_0^{t_2} \psi(t)\,dt, \int_0^{t_3} \psi(t)\,dt, \int_0^{t_4} \psi(t)\,dt, \frac{1}{2}\left(\int_0^{t_5} + \int_0^{t_6}\right)\psi(t)\,dt\right\}$$

where $\alpha \in (0,1)$ and $\psi : \mathfrak{R}^+ \to \mathfrak{R}^+$ is a Lebesgue integrable function which is summable and satisfies $\int_0^{\epsilon} \psi(t)\,dt > 0$ for all $\epsilon > 0$.

4. Main results

We begin with the following observation.

Theorem 4.1. *Let (X,d) be a 2-metric space wherein the mappings $I: X \to X$ and $F: X \to B(X)$ are strict OWC pair. If I and F have a unique point of strict coincidence $\{z\} = \{Ix\} = Fx$, then z is the unique common fixed point of I and F which also remains a strict fixed point of F.*

Proof. Since the mappings I and F are strict OWC, there exists a point $x \in X$ with $\{z\} = \{Ix\} = Fx$ implies that $FIx = IFx$. Therefore $\{Iz\} = \{IIx\} = IFx = FIx = Fz = \{u\}$ which shows that u is a point of strict coincidence of I and F. Now, in view of the uniqueness of point of coincidence, one infers $z = u$ and henceforth $\{z\} = \{Iz\} = Fz$ which shows that z is a common fixed point of I and F. Suppose that $v \neq z$ is another common fixed point of I and F which is also a strict fixed point for F, then $\{v\} = \{Iv\} = Fv$ implies that v is a point of strict coincidence of I and F. Now, due to the uniqueness of point of strict coincidence one gets $v = z$. This concludes the proof. □

Theorem 4.2. *Let (X,d) be a 2-metric space wherein $I,J: X \to X$ and $F,G: X \to B(X)$ are the mappings which satisfy the inequality*

$$\phi(\delta(Fx,Gy,C),\delta(Ix,Jy,C),\delta(Ix,Fx,C),\delta(Jy,Gy,C),$$
$$D(Ix,Gy,C),D(Jy,Fx,C)) \leqslant 0 \tag{4.1}$$

for all $x,y \in X$, every $C \in B(X)$ and $\phi \in \Phi$. Suppose that there exist $x,y \in X$ such that $u = \{Ix\} = Fx$ and $v = \{Jy\} = Gy$.

Then u is the unique point of strict coincidence of I and F whereas v is the unique point of strict coincidence of J and G.

Proof. Firstly, we show that $Ix = Jy$. Let on contrary that $Ix \neq Jy$, then using (4.1) and (ϕ_1), we obtain

$$\phi(\delta(Ix,Jy,C),\delta(Ix,Jy,C),0,0,\delta(Ix,Jy,C),\delta(Ix,Jy,C)) \leqslant 0$$

a contradiction to (ϕ_3). Hence $Ix = Jy$. Thus $u = \{Ix\} = Fx = \{Jy\} = Gy$. Suppose that there is some $z \in X$, $z \neq x$ with $\{w\} = \{Iz\} = Fz$. Then using (4.1) and (ϕ_1), we obtain

$$\phi(\delta(Iz,Jy,C),\delta(Iz,Jy,C),0,0,\delta(Iz,Jy,C),\delta(Iz,Jy,C)) \leqslant 0$$

a contradiction to (ϕ_3) provided $\delta(Iz,Jy,C) = 0$. Hence $\{w\} = \{Iz\} = Fz = \{Jy\} = Gy$, $u = \{Ix\} = Fx$, and u is the unique point of strict coincidence of I and F. Similarly, one can show that v is the unique point of strict coincidence of J and G. This completes the proof. □

Let $I,J: X \to X$ and $F,G: X \to B(X)$ be mappings such that inequality (4.1) holds for all $x,y \in X$ and $C \in B(X)$ and

$$F(X) \subset J(X) \quad \text{and} \quad G(X) \subset I(X). \tag{4.2}$$

Since $F(X) \subset J(X)$ for an arbitrary $x_0 \in X$ there exists a point $x_1 \in X$ such that $Jx_1 \in Fx_0 = Y_0$. Since $G(X) \subset I(X)$ for this point x_1, there exists a point $x_2 \in X$ such that $Ix_2 \in Gx_1 = Y_1$ and so on. Consequently, we can inductively define a sequence $\{x_n\}$ as follows:

$$Jx_{2n+1} \in Fx_{2n} = Y_{2n} \quad \text{and} \quad Ix_{2n+2} \in Gx_{2n+1}$$
$$= Y_{2n+1}, \text{for all } n = 0,1,2,\ldots \tag{4.3}$$

Lemma 4.1. *If $I,J: X \to X$ and $F,G: X \to B(X)$ are mappings which satisfy (4.1) and (4.2), then (for every $n \in N$), $\delta(Y_n, Y_{n+1}, Y_{n+2}) = 0$.*

Proof. By using (4.1) and (ϕ_1), we can have

$$\phi(\delta(Fx_{2n+2},Gx_{2n+1},Y_{2n}),\delta(Ix_{2n+2},Jx_{2n+1},Y_{2n}),\delta(Ix_{2n+2},Fx_{2n+2},Y_{2n}),$$
$$\delta(Jx_{2n+1},Gx_{2n+1},Y_{2n}),D(Ix_{2n+2},Gx_{2n+1},Y_{2n}),D(Fx_{2n+2},Jx_{2n+1},Y_{2n})) \leqslant 0$$
$$\text{or} \quad \phi(\delta(Y_{2n+2},Y_{2n+1},Y_{2n}),\delta(Y_{2n+1},Y_{2n},Y_{2n}),\delta(Y_{2n+1},Y_{2n+2},Y_{2n}),$$
$$\delta(Y_{2n},Y_{2n+1},Y_{2n}),D(Y_{2n+1},Y_{2n+1},Y_{2n}),D(Y_{2n},Y_{2n+2},Y_{2n})) \leqslant 0$$
$$\text{or} \quad \phi(\delta(Y_{2n+2},Y_{2n+1},Y_{2n}),0,\delta(Y_{2n+1},Y_{2n+2},Y_{2n}),0,0,D(Y_{2n},Y_{2n+2},Y_{2n})) \leqslant 0$$
$$\text{or} \quad \phi(\delta(Y_{2n+2},Y_{2n+1},Y_{2n}),0,\delta(Y_{2n+1},Y_{2n+2},Y_{2n}),0,0,\delta(Y_{2n+1},Y_{2n+2},Y_{2n})) \leqslant 0$$

which implies (due to (ϕ_b)) $\delta(Y_{2n},Y_{2n+1},Y_{2n+2}) = 0$. Similarly, using (ϕ_a), we can also show that $\delta(Y_{2n+1}, Y_{2n+2}, Y_{2n+3}) = 0$. Thus, in all, $\delta(Y_n, Y_{n+1}, Y_{n+2}) = 0$. □

Lemma 4.2 (Abd El-Monsef et al., 2007). *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B respectively, then $\delta(A_n,B_n,C)$ converges to $\delta(A,B,C)$ for every $C \in B(X)$.*

Theorem 4.3. *Let $I,J: X \to X$ and $F,G: X \to B(X)$ be the mappings such that* (4.1) and (4.2) *hold (for all $x,y \in X$ and for all $C \in B(X)$). If $I(X)$ (or $J(X)$) is a complete subspace of X, then*

(i) *I and F have a strict coincidence point,*
(ii) *J and G have a strict coincidence point.*

Moreover, if the pairs (I,F) and (J,G) are strict OWC, then I, J, F and G have a unique common fixed point which also remains a strict fixed point of F and G.

Proof. Owing to (4.1), (4.2), (4.3) and (ϕ_1), we can write

$$\phi(\delta(Fx_{2n},Gx_{2n+1},C),\delta(Ix_{2n},Jx_{2n+1},C),\delta(Ix_{2n},Fx_{2n},C),$$
$$\delta(Jx_{2n+1},Gx_{2n+1},C),D(Ix_{2n},Gx_{2n+1},C),D(Fx_{2n},Jx_{2n+1},C)) \leqslant 0$$
$$\text{or} \quad \phi(\delta(Y_{2n},Y_{2n+1},C),\delta(Y_{2n-1},Y_{2n},C),\delta(Y_{2n-1},Y_{2n},C),$$
$$\delta(Y_{2n},Y_{2n+1},C),\delta(Y_{2n-1},Y_{2n+1},C),\delta(Y_{2n},Y_{2n},C)) \leqslant 0.$$

Since $\delta(Y_{2n-1},Y_{2n+1},C) \leqslant \delta(Y_{2n-1},Y_{2n}, C) + \delta(Y_{2n},Y_{2n+1}, C) + \delta(Y_{2n-1},Y_{2n+1},Y_{2n})$ and $\delta(Y_{2n-1},Y_{2n+1},Y_{2n}) = 0$ (due to Lemma 4.1), therefore

$$\phi(\delta(Y_{2n},Y_{2n+1},C),\delta(Y_{2n-1},Y_{2n},C),\delta(Y_{2n-1},Y_{2n},C),$$
$$\delta(Y_{2n},Y_{2n+1},C),\delta(Y_{2n-1},Y_{2n},C)+\delta(Y_{2n},Y_{2n+1},C),0) \leqslant 0.$$

(due to (ϕ_a)) gives rise

$$\delta(Y_{2n},Y_{2n+1},C) \leqslant h\delta(Y_{2n-1},Y_{2n},C). \tag{4.6}$$

Similarly, using (ϕ_b), we obtain

$$\delta(Y_{2n+1},Y_{2n+2},C) \leqslant k\delta(Y_{2n},Y_{2n+1},C). \tag{4.7}$$

Therefore, inductively

$$\delta(Y_{2n},Y_{2n+1},C) \leqslant (hk)^n\delta(Fx_0,Gx_1,C), \tag{4.8}$$

and

$$\delta(Y_{2n+1}, Y_{2n+2}, C) \leqslant (hk)^n \delta(Gx_1, Fx_2, C), \tag{4.9}$$

which, in all, gives rise $\lim \delta(Y_n, Y_{n+1}, C) = 0$.

For all $C \in B(X)$ and $m > n$, we have (by Lemma 4.1)

$$\delta(Y_n, Y_m, C) \leqslant \delta(Y_n, Y_{n+1}, Y_{n+2}) + \delta(Y_{n+1}, Y_{n+2}, Y_{n+3}) + \cdots + \delta(Y_{m-2}, Y_{m-1}, Y_m) + \delta(Y_{m-1}, Y_m, C),$$

which on letting $n, m \to \infty$ gives rise that $\lim \delta(Y_n, Y_m, C) = 0$.

Suppose that $J(X)$ is complete and $Jx_{2n+1} \in Fx_{2n} = Y_{2n}$, for $n = 0, 1, 2, \ldots$, we can have

$$d(Jx_{2m+1}, Jx_{2n+1}, C) \leqslant \delta(Y_{2m}, Y_{2n}, C)$$

which implies that $\lim d(Jx_{2m+1}, Jx_{2n+1}, C) = 0$. Hence $\{Jx_{2n+1}\}$ is a Cauchy sequence and is also convergent to a limit $p \in J(X)$, hence $p = Jv$ for some $v \in X$. But $Ix_{2n} \in Gx_{2n-1} = Y_{2n-1}$, so that we obtain

$$\lim \delta(Ix_{2n}, Jx_{2n+1}, C) \leqslant \lim \delta(Y_{2n-1}, Y_{2n}, C) = 0.$$

Consequently, $\lim Ix_{2n} = p$. Moreover, we obtain

$$\delta(Fx_{2n}, p, C) \leqslant \delta(Fx_{2n}, Ix_{2n}, C) + \delta(Ix_{2n}, p, C) + \delta(Fx_{2n}, p, Ix_{2n}).$$

Since $\delta(Fx_{2n}, Ix_{2n}, C) \leqslant \delta(Y_{2n}, Y_{2n-1}, C)$ implies $\lim \delta(Fx_{2n}, Ix_{2n}, C) = 0$, therefore $\lim \delta(Fx_{2n}, p, C) = 0$. Similarly, we can have $\lim \delta(Gx_{2n-1}, p, C) = 0$. Using the inequality (4.1), we obtain

$$\phi(\delta(Fx_{2n}, Gv, C), \delta(Ix_{2n}, Jv, C), \delta(Ix_{2n}, Fx_{2n}, C), \delta(Jv, Gv, C), D(Ix_{2n}, Gv, C), D(Jv, Fx_{2n}, C)) \leqslant 0.$$

Since $\delta(Jx_{2n+1}, Gv, C) \leqslant \delta(Fx_{2n}, Gv, C)$, then by (ϕ_1), we have

$$\phi(\delta(Jx_{2n+1}, Gv, C), \delta(Ix_{2n}, Jv, C), \delta(Ix_{2n}, Fx_{2n}, C), \delta(Jv, Gv, C), \delta(Ix_{2n}, Gv, C), \delta(Jv, Fx_{2n}, C)) \leqslant 0.$$

Letting $n \to \infty$, we obtain

$$\phi(\delta(p, Gv, C), 0, 0, \delta(p, Gv, C), \delta(p, Gv, C), 0) \leqslant 0$$

which implies by (ϕ_a) that $\delta(p, Gv, C) = 0$, i.e., $Gv = \{p\}$. Therefore, $Gv = \{Jv\} = \{p\}$ and v is a strict coincidence point of J and G.

Since $G(X) \subset I(X)$, there exists $u \in X$ such that $\{Iu\} = Gv = \{Jv\}$. By (4.1) and (ϕ_1) we obtain

$$\phi(\delta(Fu, Gv, C), \delta(Iu, Jv, C), \delta(Iu, Fu, C), \delta(Jv, Gv, C), D(Iu, Gv, C), D(Fu, Jv, C)) \leqslant 0$$
$$\phi(\delta(Fu, p, C), 0, \delta(p, Fu, C), 0, 0, \delta(Fu, p, C)) \leqslant 0.$$

By (ϕ_b), we obtain $(Fu, p, C) = 0$ which implies $Fu = \{p\}$. Hence u is a strict coincidence point of I and F. Therefore, $\{p\} = \{Iu\} = Fu = \{Jv\} = Gv$.

In view of Theorem 4.2, $\{p\} = \{Iu\} = Fu$ is the unique point of strict coincidence of I and F. Similarly, $\{p\} = \{Jv\} = Gv$ is the unique point of strict coincidence of J and G. Since (I, F) and (J, G) are strict OWC and p is a unique point of coincidence, then by Theorem 4.1, p is the unique common fixed point of I, J, F and G which is a strict common fixed point for F and G. In case $I(X)$ is complete, the proof is similar. This completes the proof. □

Corollary 4.1. *The conclusions of Theorem* 4.3 *remain valid if inequality* (4.1) *is replaced by any one of the following contraction conditions:*

(a_1) $$\delta(Fx, Gy, C) \leqslant \alpha \max\left\{\delta(Ix, Jy, C), \delta(Ix, Fx, C), \delta(Jy, Gy, C), \frac{1}{2}[D(Ix, Gy, C) + D(Jy, Fx, C)]\right\}$$

where $\alpha \in (0, 1)$.

(a_2) $$\delta^2(Fx, Gy, C) \leqslant c_1 \max\{\delta^2(Ix, Jy, C), \delta^2(Ix, Fx, C), \delta^2(Jy, Gy, C)\} + c_2 \max\{\delta(Ix, Fx, C)D(Ix, Gy, C), \delta(Jy, Gy, C)D(Jy, Fx, C)\} + c_3 D(Ix, Gy, C)D(Jy, Fx, C)$$

where $c_1 > 0$, c_2, $c_3 \geqslant 0$, $c_1 + 2c_2 < 1$ and $c_1 + c_3 < 1$.

(a_3) $$\delta(Fx, Gy, C) \leqslant \alpha\delta(Ix, Jy, C) + \beta \min\{\delta(Ix, Fx, C), D(Ix, Gy, C)\} + \eta \min\{\delta(Jy, Gy, C), D(Jy, Fx, C)\}$$

where $\alpha, \beta, \eta > 0$, $\alpha + \beta < 1$, $\alpha + \eta < 1$ and $(\alpha + \beta)(\alpha + \eta) < 1$.

(a_4) $$\delta(Fx, Gy, C) \leqslant \alpha \max\{\delta(Ix, Jy, C), \delta(Ix, Fx, C), \delta(Jy, Gy, C)\} + (1 - \alpha)(\beta D(Ix, Gy, C) + \eta D(Jy, Fx, C))$$

where $0 \leqslant \alpha < 1$, $\beta, \eta \geqslant 0$, $\beta + \eta < 1$ and $\alpha|\beta - \eta| < 1 - (\beta + \eta)$.

(a_5) $$\delta(Fx, Gy, C) \leqslant \psi(\max\{\delta(Ix, Jy, C), \delta(Ix, Fx, C), \delta(Jy, Gy, C), \tfrac{1}{2}[D(Ix, Gy, C) + D(Jy, Fx, C)]\})$$

where $\psi : \Re^+ \to \Re^+$ is an upper semi-continuous function such that $\psi(t) < t$ for all $t > 0$.

(a_6) $$\delta(Fx, Gy, C) \leqslant \psi(\delta(Ix, Jy, C), \delta(Ix, Fx, C), \delta(Jy, Gy, C), D(Ix, Gy, C), D(Jy, Fx, C))$$

where $\psi : \Re^5_+ \to \Re_+$ is an upper semi-continuous function such that $\psi(t, t, t, \alpha t, \beta t) < t$ for all $t > 0$ and $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 2$.

(a_7) $$\int_0^{\delta(Fx,Gy,C)} \psi(t)\,\mathrm{d}t \leqslant \alpha \int_0^{\max\{\delta(Ix,Jy,C),\delta(Ix,Fx,C),\delta(Jy,Gy,C),\frac{1}{2}[D(Ix,Gy,C)+D(Jy,Fx,C)]\}} \psi(t)\,\mathrm{d}t$$

where $\alpha \in (0,1)$ and $\psi : \Re^+ \to \Re^+$ is a Lebesgue integrable function which is summable and satisfies $\int_0^\epsilon \psi(t)\,\mathrm{d}t > 0$ for all $\epsilon > 0$.

(a_8) $$\int_0^{\delta(Fx,Gy,C)} \psi(t)\,\mathrm{d}t \leqslant \alpha \max\left\{\int_0^{\delta(Ix,Jy,C)} \psi(t)\,\mathrm{d}t, \int_0^{\delta(Ix,Fx,C)} \psi(t)\,\mathrm{d}t, \int_0^{\delta(Jy,Gy,C)} \psi(t)\,\mathrm{d}t, \frac{1}{2}\left[\int_0^{D(Ix,Gy,C)} \psi(t)\,\mathrm{d}t + \int_0^{D(Jy,Fx,C)} \psi(t)\,\mathrm{d}t\right]\right\}$$

where $\alpha \in (0,1)$ and $\psi : \Re^+ \to \Re^+$ is a Lebesgue integrable function which is summable and satisfies $\int_0^\epsilon \psi(t)\,\mathrm{d}t > 0$ for all $\epsilon > 0$.

Remark 4.1. In view of Theorem 4.3 with inequality (a_4), we obtain a generalization of Theorem 2.1 besides some relevant results contained in Abd El-Monsef et al. (2007). Using inequalities $(a_1$–$a_3)$ and $(a_5$–$a_8)$ together with Theorem 4.3, we obtain generalization and extension of relevant results from Jungck and Rhoades (1998), Khan (1984), Naidu and Prasad (1986), Popa et al. (2010), Sessa et al. (1986) and also obtain some new results.

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