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ORIGINAL ARTICLE

Dependence of a class of non-integer power functions

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Abstract This short article exhibits that there exists critical point of the power for the generalized function t^{-a} for $a > 0$. The present results show that it is long-range dependent if $0 < a < 1$ and short-range dependent when $a > 1$. My motivation of studying that dependence issue comes from the power-law type functions in fractal time series. The present results may yet be useful to investigate fractal behavior of fractal time series from a new point of view.

1. Introduction

Dependence analysis of functions is an interesting topic. That is particularly true in time series with long-range dependence (LRD), (see e.g., Arzano and Calcagni, 2013; Asgari et al., 2011; Cattani, 2010a,b; Cattani et al., 2012; Lévy Véhel, 2013; Mandelbrot, 2001; Stanley et al., 1993; Yang and Baleanu, 2013; Yang et al., 2013; Zhao and Ye, 2013), simply mentioning a few. The particularity in time series with LRD or in fractal time series in general is power-laws in probability density function (PDF), power spectrum density (PSD), and autocorrelation function (ACF) (Li, 2010; Stanley, 1995). By power-laws, we mean that things one concern about are described by power functions, for instance, $f(t) = At^{\lambda}$ $(t > 0)$ where A is a constant and $\lambda \in \mathbf{R}$ (the set of real numbers).

Functions in the class of t^{λ} play a role in the domain of generalized functions (Rennie, 1982; Kanwal, 2004). Its Fourier transform has been well studied (Gelfand and Vilenkin, 1964; Lighthill, 1958). However, its correlation dependence from a view of statistical analysis is rarely seen. This article aims at providing my analysis of its correlation dependence. For facilitating the consistence with those in fractal time series, we are also interested in t^{-a} for $a > 0$, which are decayed power functions. This article will show that there exists a critical point for t^{-a}. When $a > 1$, t^{-a} is of short-range dependence (SRD). If $0 < a < 1$, t^{-a} is of LRD.

2. Analysis and results for $t^{-a}H(t)$

Denote by $y(t)$ a time series. Its ACF is denoted by $r_{yy}(\tau) = \mathrm{E}[y(t)y(t+\tau)]$. By LRD (Mandelbrot, 2001), one implies

$$\int_0^{\infty} r_{yy}(\tau)d\tau = \infty. \tag{1}$$

A typical case of $r_{yy}(\tau)$, which satisfies the above, is a decayed power function expressed by

$$r_{yy}(\tau) \sim c\tau^{-b} \quad (0 < b < 1), \tag{2}$$

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where c is a constant. If

$$\int_0^\infty r_{yy}(\tau)\, d\tau < \infty, \tag{3}$$

$y(t)$ is of SRD. Note that exponentially decayed ACFs are trivial in the field of fractal time series for the meaning of SRD.

Denote by $S_{yy}(\omega)$ the PSD of $y(t)$. Then,

$$S_{yy}(\omega) = \mathrm{F}[r_{yy}(\tau)] = \int_{-\infty}^{\infty} r_{yy}(\tau)e^{-i\omega\tau}d\tau. \tag{4}$$

The case of LRD expressed by (1) implies $S_{yy}(0) = \infty$, meaning $1/f$ noise. On the other hand, the SRD case expressed by (3) means that $S_{yy}(\omega)$ is convergent at $\omega = 0$. Both reflect the statistical dependence of LRD and SRD in the frequency domain.

Let $f(t) = t^{-a}H(t)$, where $a > 0$ and $H(t)$ is the Heaviside unit step function. To discuss its correlation dependence, the following lemma is needed.

Lemma 1. *The Fourier transform of $t^{\lambda}H(t)$ is given by*

$$\begin{aligned}\mathrm{F}[t^{\lambda}H(t)] &= \frac{1}{i\omega}\exp\left[-\frac{i\pi\lambda}{2}\mathrm{sgn}\left(\frac{\omega}{2\pi}\right)\right]\lambda!|\omega|^{-\lambda}\\ &= -i\exp\left[-\frac{i\pi\lambda}{2}\mathrm{sgn}\left(\frac{\omega}{2\pi}\right)\right](\lambda)!|\omega|^{-\lambda-1}\\ &= \exp\left[-\frac{i\pi(1+\lambda)}{2}\mathrm{sgn}\left(\frac{\omega}{2\pi}\right)\right](\lambda)!|\omega|^{-\lambda-1},\end{aligned} \tag{5}$$

referring to (Lighthill, 1958) *for the proof, also see* (Li, 2013).

From Lemma 1, we have the following corollary.

Corollary 1. *The Fourier transform of $f(t)$ is given by*

$$\begin{aligned}F[t^{-a}H(t)] &= \frac{1}{i\omega}\exp\left[\frac{i\pi a}{2}\mathrm{sgn}\left(\frac{\omega}{2\pi}\right)\right]\lambda!|\omega|^{a}\\ &= -i\exp\left[\frac{i\pi a}{2}\mathrm{sgn}\left(\frac{\omega}{2\pi}\right)\right](-a)!|\omega|^{a-1}\\ &= \exp\left[-\frac{i\pi(1-a)}{2}\mathrm{sgn}\left(\frac{\omega}{2\pi}\right)\right](-a)!|\omega|^{a-1}.\end{aligned} \tag{6}$$

Lemma 2. *Denote the ACF of $f(t)$ by $r_{ff}(\tau)$. Representing $r_{ff}(\tau)$ by using the convolution* (Papoulis, 1977) *produces*

$$r_{ff}(\tau) = f(\tau) * f(-\tau), \tag{7}$$

where * stands for the convolution operation.

Corollary 2. *The Fourier transform of $f(-t)$ is given by*

$$\mathrm{F}[f(-t)] = \exp\left[\frac{i\pi(1-a)}{2}\mathrm{sgn}\left(\frac{\omega}{2\pi}\right)\right](-a)!(-1)^{a-1}|\omega|^{a-1}. \tag{8}$$

Proof. Denote by $\mathrm{F}(\omega)$ the Fourier transform of $f(t)$. Then, the Fourier transform of $f(-t)$ is $\mathrm{F}(-\omega)$. Replacing ω in (6) by $-\omega$ produces (8). Hence, Corollary 2 holds.

Let $S_{ff}(\omega)$ be the PSD of $f(t)$. Then, we present the following theorem.

Theorem 1. *The PSD of $f(t)$ is expressed by*

$$S_{ff}(\omega) = (-1)^{a-1}[(-a)!]^2|\omega|^{2(a-1)}. \tag{9}$$

Proof. According to the convolution theorem, one has $S_{ff}(\omega) = \mathrm{F}[f(t)]\,\mathrm{F}[f(-t)]$. Therefore, with Corollaries 1 and 2, we have

$$\begin{aligned}\mathrm{F}[f(t)]\,\mathrm{F}[f(-t)] &= \exp\left[-\frac{i\pi(1-a)}{2}\mathrm{sgn}\left(\frac{\omega}{2\pi}\right)\right](-a)!|\omega|^{a-1}\\ &\quad\times\exp\left[\frac{i\pi(1-a)}{2}\mathrm{sgn}\left(\frac{\omega}{2\pi}\right)\right](-a)!(-1)^{a-1}|\omega|^{a-1}\\ &= (-1)^{a-1}[(-a)]!^2|\omega|^{2(a-1)}.\end{aligned}$$

Thus, Theorem 1 holds.

Theorem 2. *$f(t)$ is SRD if $a > 1$ and LRD if $0 < a < 1$.*

Proof. From (9) in Theorem 1, we see that $S_{ff}(\omega)$ is convergent at $\omega = 0$ for $a > 1$, meaning $f(t)$ is SRD. On the other side, it is divergent $\omega = 0$ if $0 < a < 1$, implying $f(t)$ is LRD. This completes the proof.

The ACF of $f(t)$, $r_{ff}(\tau)$, gives the quantitative description of how $f(t)$ at time t correlates to the one at $t + \tau$. Thus, suppose $f(t)$ is a PDF or ACF or PSD of a specific time series. Theorem 2 may provide a tool to deeply investigate or describe dynamics of a fractal random function from another point of view. I shall work at this issue in future.

3. Analysis and results for $|t|^{-a}$

Lemma 3. *The Fourier transform of $|t|^{\lambda}$ is given by*

$$\mathrm{F}\left(|t|^{\lambda}\right) = -2\sin\left(\frac{\lambda\pi}{2}\right)\lambda!|\omega|^{-\lambda-1}, \tag{10}$$

where $\lambda \neq -1, -3, \ldots$ (Lighthill, 1958; Li, 2013).

Corollary 3. *The Fourier transform of $|t|^{-a}$ is given by*

$$\mathrm{F}(|t|^{-a}) = 2\sin\left(\frac{a\pi}{2}\right)(-a)!|\omega|^{a-1}. \tag{11}$$

Proof. Replacing λ in (10) by $-a$ yields this corollary.

Theorem 3. *Let $S_{gg}(\omega)$ be the PSD of $g(t) = |t|^{-a}$. Then,*

$$S_{gg}(\omega) = [\mathrm{F}(|t|^{-a})]^2 = 4\sin^2\left(\frac{a\pi}{2}\right)[(-a)!]^2|\omega|^{2(a-1)}. \tag{12}$$

Proof. According to the convolution theorem, we have $S_{gg}(\omega) = \mathrm{F}[g(t)]\,\mathrm{F}[g(-t)] = \{\mathrm{F}[g(t)]\}^2$. Using (11), we have $\{\mathrm{F}[g(t)]\}^2 = 4\sin^2(\frac{a\pi}{2})[(-a)!]^2|\omega|^{2(a-1)}$. Therefore, Theorem 3 holds.

Theorem 4. *$g(t)$ is LRD if $0 < a < 1$ and it is SRD if $a > 1$.*

Proof. Omitted as it is similar to that in Theorem 2.

4. Concluding remarks

I have explained that there exists a critical point of power for the class of generalized power functions $|t|^{-a}$ and $|t|^{-a}H(t)$ to classify its dependence (LRD or SRD). For $0 < a < 1$, they are of LRD and SRD if $a > 1$. The potential utility of the present results may be in the aspect of deeply investigating the fractal dynamics of a fractal time series from the point of view of the dependence behavior of its stochastic models in power laws, such as PSD, PDF or ACF. Indeed, in addition to fractal time series, the generalized functions discussed in this research are also essential in other mathematics branches, such as fractional calculus, where the Mittag–Leffler function, denoted by $E_\gamma(-t^\gamma)$, plays a role (Jumarie, 2009; Klafter et al., 2012).

$$E_\gamma(-t^\gamma)- = \sum_{k=0}^{\infty}(-1)^k \frac{t^{\gamma k}}{\Gamma(1+\gamma k)}, \tag{13}$$

see (Gorenflo et al., 2014; Turner, 2013); and references therein for the Mittag–Leffler function. That is essential in fractional calculus (Kiryakova, 2010). More about it is given in the Addendum.

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Appendix A.

In addition to (13) mentioned previously, I would like to show more cases about the functions of t^{-a} type from the point of view of fractional calculus.

A.1. Function t^{-a} in Riemann–Liouville integral

Denote by ${}_0D_t^{-v}$ the Riemann–Liouville integral operator of order v (Miller and Ross, 1993, p. 45). When $v > 0$ and $f(t)$ is a piecewise continuous on $(0, \infty)$ and integrable on any finite subinterval of $[0, \infty)$, one has the differential of order v of $f(t)$, for $t > 0$, in the form

$${}_0D_t^{-v}f(t) = \frac{1}{\Gamma(v)}\int_0^t (t-u)^{v-1}f(u)du, \tag{A1}$$

where Γ is the Gamma function.

Taking into account the definition of the convolution stated by Mikusinski, 1959, one has

$${}_0D_t^{-v}f(t) = \frac{t^{v-1}}{\Gamma(v)} * f(t) \tag{A2}$$

The above gives a case of t^{-a} type functions to the Riemann–Liouville integral.

A.2. Application of $\mathbf{t^{-a}}$ to fractional Brownian motion of the Riemann–Liouville integral type

Denote by $B(t)$, $t \in (0, \infty)$, the standard Brownian motion, see e.g., Hida, 1980. Then, replacing v with $H + 0.5$ in (A1) for $0 < H < 1$, where H is the Hurst parameter, the fractional Brownian motion (fBm) of the Riemann–Liouville type, $B_H(t)$, is given by

$$B_H(t) = {}_0D_t^{-(H+1/2)}B'(t) = \frac{1}{\Gamma(H+1/2)}\int_0^t (t-u)^{H-1/2}dB(u). \tag{A3}$$

The above may be written by

$$B_H(t) = \frac{dB(t)}{dt} * \frac{t^{H-1/2}}{\Gamma(H+1/2)}, \tag{A4}$$

implying a case of application of t^{-a} type functions to the fBm of the Riemann–Liouville type.

A.3. Application of $\mathbf{t^{-a}}$ to fractional vibrations

An oscillator system with the damping zero may be expressed by

$$\left(\frac{d^2}{dt^2} + \omega_0^2\right)x(t) = e(t), \tag{A5}$$

where $x(t)$ is the system response, $e(t)$ is the excitation, and $\omega_0 > 0$ the angular natural frequency. Generalizing (A5) using the fractional calculus to the form given by

$$\left(\frac{d^2}{dt^2} + \omega_0^2\right)^\beta x(t) = e(t), \tag{A6}$$

where $\beta > 0$ is a fractional index. When $x(0) = x'(0) = 0$, $e(t) = \delta(t)$ (Diract-δ function), following Li et al., 2011, we have the impulse response, denoted by $h(t)$, in the form

$$h(t) = \frac{\sqrt{\pi}}{\Gamma(\beta)(2\omega_0)^{\beta-1/2}} t^{\beta-1/2} J_{\beta-1/2}(\omega_0 t), \quad \beta > 0,\ t \geqslant 0, \tag{A7}$$

where $J_{\beta-1/2}(\omega_0 t)$ is the Bessel function of the first kind of order 1/2. As can be seen from (A7), the factor $t^{\beta-1/2}$ exhibits a case of application of t^{-a} type functions to a class of fractal vibration phenomena described in Li et al., 2011. One may find such an application to other fractional vibration phenomenon, e.g., that explained by Achar et al. (2002, 2004).

The above describes the relationship between the type of functions of t^{-a} and fractional calculus from three points. To be precise, the definition of fractional integral, fBm, and fractional oscillating.

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