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Sequential fractional differential equations and inclusions with semi-periodic and nonlocal integro-multipoint boundary conditions

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ABSTRACT

This paper is concerned with the existence of solutions for Caputo type sequential fractional differential equations and inclusions supplemented with semi-periodic and nonlocal integro-multipoint boundary conditions involving Riemann-Liouville integral. We make use of standard fixed point theorems for single-valued and multivalued maps to obtain the desired results. Examples are constructed for the illustration of the main results.

1. Introduction

The topic of nonlocal nonlinear boundary value problems of integro-differential equations constitutes an important area of research and has attracted considerable attention over the decades in view of its numerous applications. Integro-differential equations, regarded as approximation to partial differential equations, are employed to model much of the continuum phenomena and appear in a variety of disciplines such as population models, ecology, fluid dynamics, aerodynamics, etc. Lakshmikantham and Rao (1995) and Kot (2001). The failure of classical boundary conditions to describe some peculiar processes taking place inside the given domain, led to the birth of nonlocal boundary conditions (Bitsadze and Samarskii, 1969) which relate the boundary values of the unknown function to its values at some interior positions of the domain. Integral boundary conditions find useful applications in computational fluid dynamics (CFD) studies of blood flow problems and provide the means to assume an arbitrary shaped cross-section of blood vessels in CFD of blood flow problems. Integral boundary conditions are also used in the regularization of the ill-posed backward problems in time partial differential equations. For further details on integral boundary conditions, see Ahmad et al. (2008) and Ciegis and Bugajev (2012). During the last few decades, fractional differential equations have been studied by many authors and the literature on the topic is now much enriched. In fact, fractional-order differential and integral operators are found to be great interest in the mathematical modeling of real world problems occuring in engineering and scientific disciplines. The importance of such operators can be understood in the sense that they can describe memory and hereditary properties of various materials and processes and provide more degree of freedom than their integer-order counterparts. For theoretical development and applications, for instance, see Kilbas et al. (2006), Magin (2006), Sabatier et al. (2007), Konjik et al. (2011), Zhou

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(2014) and Wang and Schiavone (2015) and the references cited therein. Fractional-order boundary value problems involving a variety of conditions such as classical, nonlocal, multipoint, periodic/anti-periodic, fractional-order, and integral boundary conditions have recently been studied by many researchers. For some recent works on boundary value problems involving non-sequential and sequential fractional differential, integro-differential equations and inclusions, we refer the reader to works (Agarwal et al., 2011; Ahmad and Ntouyas, 2013; O'Regan and Stanek, 2013; Ahmad and Nieto, 2013; Graef et al., 2014; Wang et al., 9162; Ahmad and Ntouyas, 2015; Ahmad, 2017; Zhou and Peng, 2017; Ahmad et al., 2017) and the references cited therein. In this paper, we discuss the existence of solutions for sequential fractional differential equations and inclusions:

$$({}^cD^q_{0+} + k\,{}^cD^{q-1}_{0+})x(t) = f(t, x(t), {}^cD^{\delta}_{0+}x(t), I^{\gamma}x(t)),\ \ t \in [0,1], \tag{1.1}$$

$$({}^cD^q_{0+} + k\,{}^cD^{q-1}_{0+})x(t) \in F(t, x(t), {}^cD^{\delta}_{0+}x(t), I^{\gamma}x(t)),\ \ t \in [0,1], \tag{1.2}$$

supplemented with semi-periodic and nonlocal integro-multipoint boundary conditions involving Riemann-Liouville integral given by

$$x(0) = x(1),\ \ x'(0) = 0,\ \ \sum_{i=1}^{m} a_i x(\zeta_i) = \lambda \int_0^{\eta} \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} x(s)ds, \tag{1.3}$$

where ${}^cD^{(\cdot)}_{0+}$ denotes the left Caputo derivatives of fractional order $(\cdot), 2 < q \leqslant 3, 0 < \delta, \gamma < 1, k > 0, \beta > 0, 0 < \eta < \zeta_1 < \cdots < \zeta_m < 1, I^{(\cdot)}$ denotes the left Riemann-Liouville integral of fractional order $(\cdot)$ (see Definition 2.1), $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is given continuous function, $F : [0,1] \times \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $\lambda, a_i, i = 1, \ldots, m$ are real constants. Here we remark that the word "sequential" is used in the sense that the operator ${}^cD^q_{0+} + k\,{}^cD^{q-1}_{0+}$ can be written as the composition of operators ${}^cD^{q-1}_{0+}(D + k)$. Further, it is imperative to note that the last condition in (1.3) connecting the nonlocal multi-point and Riemann-Liouville type strip conditions can be interpreted as the linear combination of the values of the unknown function at nonlocal points $\zeta_i \in (0,1)$ is proportional to the strip contribution of the unknown function on an arbitrary segment $(0,\eta) \subset [0,1]$. The present work is motivated by a recent paper (Ahmad et al., 2016) in which the authors considered the problem (1.1) and (1.3) with the first condition $x(0) = 0$ instead of the semi-periodic condition $x(0) = x(1)$ in (1.3). It means that the initial-nolocal type conditions were considered in Ahmad et al. (2016). On the other hand, one can notice that the semi-periodic type condition $x(0) = x(1)$ assumed in (1.3) implies that the difference of the values of the unknown function at $t = 0$ and $t = 1$ is zero, that is, $x(0) - x(1) = 0$. In other words, we can say that the solutions of the problems (1.1) and (1.3) and (1.2) and(1.3) experience the effect from the nonlocal multipoint-strip condition with zero flux at $t = 0$. Thus the present work is more interesting and practical as the right end point $t = 1$ of the interval under consideration is introduced via semi-periodic boundary conditions. Moreover, the scope of the present study can be extended to the cases of Riemann-Liouville and Hadamard type fractional differential and integral operators. For some works involving Riemann-Liouville fractional differential and integral operators, for instance, see Li et al. (2012) and Alsaedi et al. (2016), while the text (Ahmad et al., 2017) contains many interesting results on Hadamard type fractional differential equations and inclusions. The rest of the paper is arranged as follows. In Section 2, we prove a basic result that plays a key role in the forthcoming analysis. Section 3 contains the existence and uniqueness results for the single-valued problem (1.1) and (1.3), which rely on fixed point theorems due to Banach and Krasnoselskii. In Section 4, we prove the existence results for convex and Lipschitz type multivalued maps involved in the problem (1.2) and (1.3) by applying nonlinear alternative for contractive maps and Covitz and Nadler fixed point theorem respectively. In Section 5, we discuss illustrative examples for the obtained results.

2. Background material

This section is devoted to some fundamental concepts of fractional calculus (Kilbas et al., 2006) and a basic lemma related to the linear variant of the given problem.

Definition 2.1. The Riemann-Liouville fractional integral of order r with the lower limit zero for a function $f : [0,\infty) \to \mathbb{R}$ is defined as

$$I^r f(t) = \frac{1}{\Gamma(r)} \int_0^t \frac{f(s)}{(t-s)^{1-r}} ds, \quad t > 0, \quad r > 0,$$

provided the right hand-side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(r) = \int_0^{\infty} t^{r-1}e^{-t}dt$. Note that the above integral exists on $[0,\infty)$ when $f \in C([0,\infty),\mathbb{R})$ (Zhou, 2014).

Definition 2.2. The Riemann-Liouville fractional derivative of order $r > 0,\ n-1 < r < n,\ n \in \mathbb{N}$ for a function $f : [0,\infty) \to \mathbb{R}$ is defined as

$$D^r_{0+} f(t) = \frac{1}{\Gamma(n-r)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-r-1} f(s)ds.$$

Notice that the Riemann-Liouville fractional derivative of order $r \in [n-1, n)$ exists almost everywhere on $[0,\infty)$ if $f \in AC^n([0,\infty),\mathbb{R})$, for details, see Lemma 2.2 in Kilbas et al. (2006).

The Caputo fractional derivative is defined via above Riemann-Liouville fractional derivatives as follows.

Definition 2.3. The Caputo derivative of order $r \in [n-1, n)$ for a function $f : [0,\infty) \to \mathbb{R}$ can be written as

$${}^cD^r_{0+} f(t) = D^r_{0+}\left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)\right), \quad t > 0, \quad n-1 < r < n.$$

Note that the Caputo fractional derivative of order $r \in [n-1, n)$ exists almost everywhere on $[0,\infty)$ if $f \in AC^n([0,\infty),\mathbb{R})$.

Remark 2.4. If $f \in C^n[0,\infty)$, then

$${}^cD^r_{0+} f(t) = \frac{1}{\Gamma(n-r)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{r+1-n}} ds = I^{n-r} f^{(n)}(t),\ t > 0,\ n-1 < r < n.$$

(see Theorem 2.2 in Kilbas et al., 2006).

To define the solution for problem (1.1)–(1.3), we consider the following lemma dealing with the linear variant of (1.1)–(1.3).

Lemma 2.1. *For any $y \in C([0,1],\mathbb{R})$, a function $x \in C^3([0,1],\mathbb{R})$ is a solution of the linear sequential fractional differential equation:*

$$({}^cD^q_{0+} + k\,{}^cD^{q-1}_{0+})x(t) = y(t), \tag{2.1}$$

supplemented with the boundary conditions (1.3) if and only if it satisfies the following integral equation

$$\begin{aligned} x(t) = \frac{1}{\Delta_1}\Bigg[&-\sum_{i=1}^{m} a_i \left\{ \chi(\zeta_i) \int_0^1 e^{-k(1-s)} I^{q-1} y(s)ds + \int_0^{\zeta_i} e^{-k(\zeta_i - s)} I^{q-1} y(s)ds \right\} \\ &+ \lambda \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \left(\chi(s) \int_0^1 e^{-k(1-u)} I^{q-1} y(u)du + \int_0^s e^{-k(s-u)} I^{q-1} y(u)du \right) ds \Bigg] \\ &+ \chi(t) \int_0^1 e^{-k(1-s)} I^{q-1} y(s)ds + \int_0^t e^{-k(t-s)} I^{q-1} y(s)ds, \end{aligned} \tag{2.2}$$

where

$$\chi(w)=\frac{kw-1+e^{-kw}}{1-e^{-k}-k},\ w=t,s,\zeta_i;\ I^{q-1}y(s)=\int_0^s\frac{(s-\tau)^{q-2}}{\Gamma(q-1)}y(\tau)d\tau,\ \Delta_1=\sum_{i=1}^m a_i-\frac{\lambda\eta^\beta}{\Gamma(\beta+1)}\neq 0. \quad (2.3)$$

Proof. As argued in Ahmad and Nieto (2013), the general solution of the system (2.1) can be written as

$$x(t)=b_0e^{-kt}+\frac{b_1}{k}(1-e^{-kt})+\frac{b_2}{k^2}(kt-1+e^{-kt})+\int_0^t e^{-k(t-s)}I^{q-1}y(s)ds. \quad (2.4)$$

Using the conditions $x(0)=x(1)$ and $x'(0)=0$ in (2.4), we find that

$$b_0=\frac{b_1}{k}+\frac{b_2}{k^2}\left(\frac{k-(1-e^{-k})}{1-e^{-k}}\right)+\frac{1}{1-e^{-k}}\int_0^1 e^{-k(1-s)}I^{q-1}y(s)ds,\ b_1=kb_0,$$

which imply that

$$b_2=\frac{k^2}{1-e^{-k}-k}\int_0^1 e^{-k(1-s)}I^{q-1}y(s)ds.$$

Thus (2.4) take the form

$$x(t)=\frac{b_1}{k}+\frac{kt-1+e^{-kt}}{1-e^{-k}-k}\int_0^1 e^{-k(1-s)}I^{q-1}y(s)ds+\int_0^t e^{-k(t-s)}I^{q-1}y(s)ds. \quad (2.5)$$

Using the integro-multipoint condition: $\sum_{i=1}^m a_i x(\zeta_i)=\lambda\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\delta)}x(s)ds$ in (2.5), we get

$$\frac{b_1}{k}=\frac{1}{\Delta_1}\Bigg[-\sum_{i=1}^m a_i\left\{\frac{k\zeta_i-1+e^{-k\zeta_i}}{1-e^{-k}-k}\int_0^1 e^{-k(1-s)}I^{q-1}y(s)ds+\int_0^{\zeta_i}e^{-k(\zeta_i-s)}I^{q-1}y(s)ds\right\}+\frac{\lambda}{1-k-e^{-k}}\int_0^\eta\frac{(\eta-s)^{\beta-1}(ks-1+e^{-ks})}{\Gamma(\beta)}\int_0^1 e^{-k(1-u)}I^{q-1}y(u)du+\lambda\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_0^s e^{-k(s-u)}I^{q-1}y(u)duds\Bigg].$$

Substituting the value of b_1/k in (2.5) together with (2.3) yields the solution (2.2). The converse of the lemma follows by direct computation. This completes the proof. □

3. Main results for the problem (1.1) and (1.3)

This section is devoted to the main results concerning the existence and uniqueness of solutions for the problem (1.1)–(1.3). First of all, we fix our terminology.

Let $X=\{x: x\in C([0,1],\mathbb{R})\text{and}^cD^\delta_{0+}x\in C([0,1],\mathbb{R})\}$ denotes the space equipped with the norm $\|x\|_X=\|x\|+\|^cD^\delta_{0+}x\|=\sup_{t\in[0,1]}|x(t)|+\sup_{t\in[0,1]}|^cD^\delta_{0+}x(t)|$. It has been shown in Su (2009) that $(X,\|\cdot\|_X)$ is a Banach space.

Using Lemma 2.1, we introduce an operator $F: X\to X$as follows:

$$F(x)(t)=\frac{1}{\Delta_1}\Bigg[-\sum_{i=1}^m a_i\left\{\chi(\zeta_i)\int_0^1 e^{-k(1-s)}I^{q-1}\hat f_x(s)ds+\int_0^{\zeta_i}e^{-k(\zeta_i-s)}I^{q-1}\hat f_x(s)ds\right\}+\lambda\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\chi(s)\int_0^1 e^{-k(1-u)}I^{q-1}\hat f_x(u)du+\int_0^s e^{-k(s-u)}I^{q-1}\hat f_x(u)du\right)ds\Bigg]+\chi(t)\int_0^1 e^{-k(1-s)}I^{q-1}\hat f_x(s)ds+\int_0^t e^{-k(t-s)}I^{q-1}\hat f_x(s)ds, \quad (3.1)$$

where

$$\hat f_x(t)=f(t,x(t),{}^cD^\delta_{0+}x(t),I^\gamma x(t)),\quad t\in[0,1]. \quad (3.2)$$

Observe that problem (1.1)–(1.3) has solutions if the operator (3.1) has fixed points.

For the sake of convenience, we set

$$\Lambda=\frac{1}{|\Delta_1|}\left\{\sum_{i=1}^m\frac{|a_i|}{k\Gamma(q)}\left(|\chi(\zeta_i)|(1-e^{-k})+\zeta_i^{q-1}(1-e^{-k\zeta_i})\right)+\frac{|\lambda|\eta^\beta}{k\Gamma(\beta+1)\Gamma(q)}\left((1-e^{-k})|\chi(\eta)|+\eta^{q-1}(1-e^{-k\eta})\right)\right\}+\frac{2(1-e^{-k})}{k\Gamma(q)}. \quad (3.3)$$

$$\Lambda_1=\frac{(1-e^{-k})^2}{|1-e^{-k}-k|\Gamma(q)}+\frac{2-e^{-k}}{\Gamma(q)}. \quad (3.4)$$

$$L_1=1+\frac{1}{\Gamma(\gamma+1)}. \quad (3.5)$$

Theorem 3.5. *Let $f:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ be a continuous function satisfying the condition*

(H_1) $|f(t,x,y,z)-f(t,x_1,y_1,z_1)|\leqslant L[\|x-x_1\|+\|y-y_1\|+\|z-z_1\|]$,

for all $t\in[0,1], x,y,z,x_1,y_1,z_1\in\mathbb{R}$, where L is the Lipschitz constant. Then the boundary value problem (1.1) and (1.3) has a unique solution on $[0,1]$ if $LL_1(\Lambda+\Lambda_2)<1$, where Λ,Λ_1,L_1 are respectively given by 3.3, 3.4, 3.5 and $\Lambda_2=\Lambda_1/\Gamma(2-\delta)$.

Proof. Let us fix $r\geqslant M_0(\Lambda+\Lambda_2)/(1-LL_1(\Lambda+\Lambda_2))$, and $M_0=\sup_{t\in[0,1]}|f(t,0,0,0)|$. Then we show that $FB_r\subset B_r$where $B_r=\{x\in X:\|x\|_X\leqslant r\}$.For $x\in B_r$, notice that

$$\begin{aligned}|\hat f_x(t)|&=|f(t,x(t),{}^cD^\delta_{0+}x(t),I^\gamma x(t))|\\&\leqslant|f(t,x(t),{}^cD^\delta_{0+}x(t),I^\gamma x(t))-f(t,0,0,0)|+|f(t,0,0,0)|\\&\leqslant L[|x(t)|+|^cD^\delta_{0+}x(t)|+|I^\gamma x(t)|]+M_0\leqslant L\left[\|x\|_X+\frac{1}{\Gamma(\gamma+1)}\|x\|\right]+M_0\\&\leqslant L\left(1+\frac{1}{\Gamma(\gamma+1)}\right)\|x\|_X+M_0=LL_1\|x\|_X+M_0\leqslant LL_1r+M_0.\end{aligned}$$

Then, for $x\in X$, we have

$$\begin{aligned}\|F(x)\|\leqslant&\sup_{t\in[0,1]}\Bigg\{\frac{1}{|\Delta_1|}\Bigg[\sum_{i=1}^m|a_i|\left\{|\chi(\zeta_i)|\int_0^1 e^{-k(1-s)}I^{q-1}|\hat f_x|(s)ds+\int_0^{\zeta_i}e^{-k(\zeta_i-s)}I^{q-1}|\hat f_x|(s)ds\right\}\\&+|\lambda|\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\chi(s)\int_0^1 e^{-k(1-u)}I^{q-1}|\hat f_x|(u)du+\int_0^s e^{-k(s-u)}I^{q-1}|\hat f_x|(u)du\right)ds\Bigg]\\&+|\chi(t)|\int_0^1 e^{-k(1-s)}I^{q-1}|\hat f_x|(s)ds+\int_0^t e^{-k(t-s)}I^{q-1}|\hat f_x|(s)ds\Bigg\}\\\leqslant&(LL_1r+M_0)\Bigg[\frac{1}{|\Delta_1|}\left\{\sum_{i=1}^m\frac{|a_i|}{k\Gamma(q)}\left(|\chi(\zeta_i)|(1-e^{-k})+\zeta_i^{q-1}(1-e^{-k\zeta_i})\right)\right.\\&\left.+\frac{|\lambda|\eta^\beta}{k\Gamma(\beta+1)\Gamma(q)}\left((1-e^{-k})|\chi(\eta)|+\eta^{q-1}(1-e^{-k\eta})\right)\right\}+\frac{2(1-e^{-k})}{k\Gamma(q)}\Bigg]\\=&(LL_1r+M_0)\Lambda.\end{aligned}$$

Also we have

$$\begin{aligned}|F'(x)(t)|\leqslant&\left|\frac{k-ke^{-kt}}{1-e^{-k}-k}\right|\int_0^1 e^{-k(1-s)}\left(\int_0^s\frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|\hat f(\tau)|d\tau\right)ds\\&+k\int_0^t e^{-k(t-s)}\left(\int_0^s\frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|\hat f(\tau)|d\tau\right)ds+\int_0^t\frac{(t-s)^{q-2}}{\Gamma(q-1)}|\hat f(s)|ds\\\leqslant&(LL_1r+M_0)\left\{\frac{(1-e^{-k})^2}{|1-e^{-k}-k|\Gamma(q)}+\frac{2-e^{-k}}{\Gamma(q)}\right\}\\\leqslant&(LL_1r+M_0)\Lambda_1.\end{aligned}$$

By definition of Caputo fractional derivative with $0 < \delta < 1$, we get

$$\begin{aligned} |\,^cD^{\delta}_{0+}(Fx)(t)\,| &\leqslant \int_0^t \frac{(t-s)^{-\delta}}{\Gamma(1-\delta)}\,|\,F'(x)(s)\,|\,ds \\ &\leqslant (LL_1r+M_0)\Lambda_1\int_0^t \frac{(t-s)^{-\delta}}{\Gamma(1-\delta)}ds \\ &\leqslant \frac{1}{\Gamma(2-\delta)}(LL_1r+M_0)\Lambda_1. \end{aligned}$$

Hence

$$\begin{aligned} \|F(x)\|_X &= \|F(x)\| + \|^cD^{\delta}_{0+}F(x)\| \\ &\leqslant (LL_1r+M_0)\Lambda + \frac{1}{\Gamma(2-\delta)}(LL_1r+M_0)\Lambda_1 < r. \end{aligned} \quad (3.6)$$

This shows that F maps B_r into itself. Now, for $x, y \in B_r$ and for each $t \in [0,1]$, we obtain

$$\begin{aligned} |(Fx)(t)-(Fy)(t)| &\leqslant \frac{1}{|\Delta_1|}\Bigg[\sum_{i=1}^m |a_i|\Bigg\{|\chi(\zeta_i)|\int_0^1 e^{-k(1-s)}I^{q-1}|\hat f_x - \hat f_y|(s)ds \\ &\quad + \int_0^{\zeta_i} e^{-k(\zeta_i-s)}I^{q-1}|\hat f_x - \hat f_y|(s)ds\Bigg\} \\ &\quad + \lambda\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\Bigg(\chi(s)\int_0^1 e^{-k(1-u)}I^{q-1}|\hat f_x - \hat f_y|(u)du \\ &\quad + \int_0^s e^{-k(s-u)}I^{q-1}|\hat f_x - \hat f_y|(u)du\Bigg)ds\Bigg] \\ &\quad + \chi(t)\int_0^1 e^{-k(1-s)}I^{q-1}|\hat f_x - \hat f_y|(s)ds \\ &\quad + \int_0^t e^{-k(t-s)}I^{q-1}|\hat f_x - \hat f_y|(s)ds \\ &\leqslant L\Lambda\left[\|x-y\| + \|^cD^{\delta}_{0+}x - {}^cD^{\delta}_{0+}y\| + \frac{1}{\Gamma(\gamma+1)}\|x-y\|\right] \\ &\leqslant LL_1\Lambda\|x-y\|_X. \end{aligned}$$

Also we have $|(Fx)'(t)-(Fy)'(t)| \leqslant LL_1\Lambda_1\|x-y\|_X$, which implies that

$$\begin{aligned} |^cD^{\delta}_{0+}F(x)(t) - {}^cD^{\delta}_{0+}F(y)(t)| &\leqslant \int_0^t \frac{(t-s)^{-\delta}}{\Gamma(1-\delta)}|F'(x)(s)-F'(y)(s)|ds \\ &\leqslant \frac{LL_1\Lambda_1}{\Gamma(2-\delta)}\|x-y\|_X. \end{aligned}$$

From the above inequalities, we get

$$\begin{aligned} \|F(x)-F(y)\|_X &= \|F(x)-F(y)\| + \|^cD^{\delta}_{0+}F(x) - {}^cD^{\delta}_{0+}F(y)\| \\ &\leqslant LL_1\left(\Lambda + \frac{\Lambda_1}{\Gamma(2-\delta)}\right)\|x-y\|_X. \end{aligned} \quad (3.7)$$

As $LL_1\left(\Lambda + \frac{\Lambda_1}{\Gamma(2-\delta)}\right) < 1$, F is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. This completes the proof. □

Now, we state a known result due to Krasnoselskii (1955) which is needed to prove the existence of at least one solution of (1.1)–(1.3).

Theorem 3.6. *Let M be a closed, convex, bounded and nonempty subset of a Banach space X. Let $\mathcal{G}_1, \mathcal{G}_2$ be the operators such that: (i) $\mathcal{G}_1x + \mathcal{G}_2y \in M$ whenever $x, y \in M$; (ii) $\mathcal{G}_1$ is compact and continuous; (iii) $\mathcal{G}_2$ is a contraction mapping. Then there exists $z \in M$ such that $z = \mathcal{G}_1z + \mathcal{G}_2z$.*

Theorem 3.7. *Assume that $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function satisfying (H_1). In addition we suppose that the following assumption holds:*

$(H_2)\, |f(t,x_1,x_2,x_3)| \leqslant \mu(t), \forall (t,x_1,x_2,x_3) \in [0,1]\times\mathbb{R}^3$ with $\mu \in C([0,1],\mathbb{R})$.

Then the boundary value problem 1.1,1.2,1.3 has at least one solution on $[0,1]$ if

$$LL_1\left[\Lambda - \frac{(1-e^{-k})}{k\Gamma(q)} + \frac{(1-e^{-k})^2}{|1-e^{-k}-k|\Gamma(2-\delta)}\right] < 1, \quad (3.8)$$

where Λ is given by (3.3).

Proof. Letting $\sup_{t\in[0,1]}|\mu(t)| = \|\mu\|$, we fix

$$r \geqslant \left(\Lambda + \frac{\Lambda_1}{\Gamma(2-\delta)}\right)\|\mu\|, \quad (3.9)$$

where Λ, Λ_1 are given by (3.3), (3.4) and consider $\mathcal{B}_r = \{x \in \mathcal{C} : \|x\|_X \leqslant r\}$. Define the operators F_1 and F_2 on $\mathcal{B}_r$ as

$$(F_1x)(t) = \int_0^t e^{-k(t-s)}I^{q-1}\hat f_x(s)ds,$$

$$\begin{aligned} (F_2x)(t) &= \frac{1}{\Delta_1}\Bigg[-\sum_{i=1}^m a_i\Bigg\{\chi(\zeta_i)\int_0^1 e^{-k(1-s)}I^{q-1}\hat f_x(s)ds + \int_0^{\zeta_i} e^{-k(\zeta_i-s)}I^{q-1}\hat f_x(s)ds\Bigg\} \\ &\quad + \lambda\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\Bigg(\chi(s)\int_0^1 e^{-k(1-u)}I^{q-1}\hat f_x(u)du + \int_0^s e^{-k(s-u)}I^{q-1}\hat f_x(u)du\Bigg)ds\Bigg] \\ &\quad + \chi(t)\int_0^1 e^{-k(1-s)}I^{q-1}\hat f_x(s)ds. \end{aligned}$$

For $x, y \in \mathcal{B}_r$, following the earlier arguments, we can have

$$\|F_1x + F_2y\| \leqslant \Lambda\|\mu\|, \; \|F_1'x + F_2'y\| \leqslant \Lambda_1\|\mu\|, \; \|^cD^{\delta}_{0+}(F_1x+F_2y)\| \leqslant \frac{\Lambda_1}{\Gamma(2-\delta)}\|\mu\|.$$

From the above inequalities, we get

$$\begin{aligned} \|F_1x+F_2y\|_X &= \|F_1x+F_2y\| + \|^cD^{\delta}_{0+}(F_1x+F_2y)\| \\ &\leqslant \left(\Lambda + \frac{\Lambda_1}{\Gamma(2-\delta)}\right)\|\mu\| < r. \end{aligned} \quad (3.10)$$

Thus, $F_1x + F_2y \in \mathcal{B}_r$. In view of the condition (3.8), it can easily be shown that F_2 is a contraction mapping. The continuity of f implies that the operator F_1 is continuous. Also, F_1 is uniformly bounded on $\mathcal{B}_r$ as

$$\|F_1x\| \leqslant \frac{(1-e^{-k})\|\mu\|}{k\Gamma(q)}, \quad \|F_1'x\| \leqslant \frac{(2-e^{-k})\|\mu\|}{\Gamma(q)},$$

$$\|^cD^{\delta}_{0+}F_1x\| \leqslant \frac{1}{\Gamma(2-\delta)}\frac{(2-e^{-k})\|\mu\|}{\Gamma(q)},$$

and

$$\|F_1x\|_X \leqslant \frac{\|\mu\|}{k\Gamma(q)}\left((1-e^{-k}) + \frac{k(2-e^{-k})}{\Gamma(2-\delta)}\right).$$

Now we prove the compactness of the operator F_1. Setting $\Omega = [0,1]\times\mathcal{B}_r\times\mathcal{B}_r\times\mathcal{B}_r$, we define $\sup_{(t,\cdot,\cdot,\cdot)\in\Omega}|f(t,\cdot,\cdot,\cdot)| = M_r$, and consequently we get

$$\begin{aligned} |(F_1x)(t_2)-(F_1x)(t_1)| &= \Bigg|\int_0^{t_2} e^{-k(t_2-s)}\left(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)}\hat f_x(u)du\right)ds \\ &\quad - \int_0^{t_1} e^{-k(t_1-s)}\left(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)}\hat f_x(u)du\right)ds\Bigg| \\ &\leqslant \frac{M_r}{k\Gamma(q)}\left(|t_2^{q-1} - t_1^{q-1}| + |t_2^{q-1}e^{-kt_2} - t_1^{q-1}e^{-kt_1}|\right), \end{aligned}$$

and

$$\begin{aligned} & \mid {}^cD^{\delta}_{0+}F_1(x)(t_2) - {}^cD^{\delta}_{0+}F_1(x)(t_1) \mid \\ \leqslant & \left| \int_0^{t_2} \frac{(t_2-s)^{-\delta}}{\Gamma(1-\delta)} F_1'(x)(s)ds - \int_0^{t_1} \frac{(t_1-s)^{-\delta}}{\Gamma(1-\delta)} F_1'(x)(s)ds \right| \\ \leqslant & \frac{M_r(2-e^{-k})}{\Gamma(2-\delta)\Gamma(q)} \left(2(t_2-t_1)^{1-\delta} + |t_2^{1-\delta} - t_1^{1-\delta}| \right). \end{aligned}$$

Clearly, $\mid F_1(x)(t_2) - F_1(x)(t_1) \mid \to 0$and $\mid {}^cD^{\delta}_{0+}F_1(x)(t_2) - {}^cD^{\delta}_{0+}F_1(x)(t_1) \mid \to 0$as $t_2 \to t_1$. Thus, F_1 is relatively compact on $\mathcal{B}_r$. Hence, by the Arzel'a-Ascoli Theorem, F_1 is compact on $\mathcal{B}_r$. Thus all the assumptions of Theorem 3.6 are satisfied and the conclusion of Theorem 3.6 implies that the boundary value problem (1.1)–(1.3) has at least one solution on $[0,1]$. This completes the proof. □

Remark 3.8. In the above theorem we can interchange the roles of the operators F_1 and F_2 to obtain a second result replacing (3.8) by the following condition:

$$\frac{(1-e^{-k})}{k\Gamma(q)} < 1.$$

4. Main results for the problem (1.2) and (1.3)

Before presenting the existence results for the problem (1.2) and (1.3), we outline the necessary concepts on multi-valued maps (Deimling, 1992; Hu and Papageorgiou, 1997).

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{isclosed}\}, \mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{isbounded}\}, \mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{iscompact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by *FixG*. A multivalued map $G : [0,1] \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable.

For each $y \in C([0,1],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,1],\mathbb{R}) : v(t) \in F(t, y(t), {}^cD^{\delta}y(t), I^{\gamma}y(t)) \text{fora.e.} t \in [0,1]\}.$$

Definition 4.9. A multivalued map $F : [0,1] \times \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \mapsto F(t,x,y,z)$ is measurable for each $x,y,z \in \mathbb{R}$;
(ii) $(x,y,z) \mapsto F(t,x,y,z)$ is upper semicontinuous for almost all $t \in [0,1]$; Further a Carathéodory function F is called L^1-Carathéodory if
(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1([0,1],\mathbb{R}^+)$ such that

$$\|F(t,x,y,z)\| = \sup\{|v| : v \in F(t,x,y,z)\} \leqslant \varphi_\rho(t)$$

for all $\|x\|, \|y\|, \|z\| \leqslant \rho$ and for a.e. $t \in [0,1]$.

We define the graph of G to be the set $Gr(G) = \{(x,y) \in X \times Y : y \in G(x)\}$ and recall two results for closed graphs and upper semicontinuity.

Lemma 4.2. [Deimling, 1992, Proposition 1.2] *If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty, x_n \to x_*, y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 4.3. [Lasota and Opial, 1965] *Let X be a Banach space. Let $F : [0,1] \times X^3 \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$. Then the operator*

$$\Theta \circ S_{F,x} : C([0,1],X) \to \mathcal{P}_{cp,c}(C([0,1],X)),\ x \mapsto (\Theta \circ S_{F,x})(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,1],X) \times C([0,1],X)$.

For the forthcoming analysis, we need the following lemma.

Lemma 4.4. *(Nonlinear alternative for Kakutani maps) (Granas and Dugundji, 2005). Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp,c}(C)$ is a upper semicontinuous compact map. Then either*

(i) F has a fixed point in $\overline{U}$, or
(ii) there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.

Let (X,d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ defined by $H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}$, where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see Kisielewicz, 1991).

Definition 4.10. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called (**a**) $\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that $H_d(N(x), N(y)) \leqslant \gamma d(x,y)$ foreach$x, y \in X$ and (**b**) a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.

Lemma 4.5. [Covitz and Nadler, 1970] *Let (X,d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Definition 4.11. A function $x \in C^3([0,1],\mathbb{R})$ is said to be a solution of the boundary value problem (1.2) and (1.3) if $x(0) = x(1)$, $x'(0) = 0, \sum_{i=1}^m a_i x(\zeta_i) = \lambda \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} x(s)ds$, and there exists a function $v \in S_{F,x}$ such that $v(t) \in F(t, x(t), {}^cD^{\delta}_{0+}x(t), I^{\gamma}x(t))$ and

$$\begin{aligned} x(t) = & \frac{1}{\Delta_1}\left[-\sum_{i=1}^m a_i \left\{ \chi(\zeta_i) \int_0^1 e^{-k(1-s)} I^{q-1} v(s)ds + \int_0^{\zeta_i} e^{-k(\zeta_i-s)} I^{q-1} v(s)ds \right\} \right. \\ & \left. + \lambda \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \left(\chi(s) \int_0^1 e^{-k(1-u)} I^{q-1} v(u)du + \int_0^s e^{-k(s-u)} I^{q-1} v(u)du \right) ds \right] \\ & + \chi(t) \int_0^1 e^{-k(1-s)} I^{q-1} v(s)ds + \int_0^t e^{-k(t-s)} I^{q-1} v(s)ds. \end{aligned}$$

4.1. The upper semicontinuous case

In the case when F has convex values we prove an existence result based on nonlinear alternative of Leray-Schauder type.

Theorem 4.12. *Assume that:*

(C_1) *$F : [0,1] \times \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$ is L^1-Carathéodory and has nonempty compact and convex values;*

(C_2) *there exist a function $\phi \in C([0,1],\mathbb{R}^+)$, and a nondecreasing, subhomogeneous (that is, $\Omega(\mu x) \leqslant \mu\Omega(x)$ for all $\mu \geqslant 1$ and $x \in \mathbb{R}^+$) function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|w| : w \in F(t,x,y,z)\} \leqslant \phi(t)\Omega(\|x\| + \|y\| + \|z\|)$$

for each $(t,x,y,z) \in [0,1] \times \mathbb{R}^3$;

(C_3) *there exists a constant $M > 0$ such that*

$$\frac{M}{L_1\left(\Lambda + \frac{\Lambda_1}{\Gamma(2-\delta)}\right)\|\phi\|\Omega(M)} > 1,$$

where Λ, Λ_1 and L_1 are defined by 3.3,3.4,3.5.

Then the boundary value problem (1.2) and (1.3) has at least one solution on $[0,1]$.

Proof. Define an operator $\Omega_F : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ by

$$\Omega_F(x) = \{h \in C([0,1],\mathbb{R}) : h(t) = N(x)(t)\}$$

where

$$N(x)(t) = \begin{cases} \frac{1}{\Delta_1}\Big[-\sum_{i=1}^{m} a_i\Big\{\chi(\zeta_i)\int_0^1 e^{-k(1-s)}I^{q-1}v(s)ds + \int_0^{\zeta_i} e^{-k(\zeta_i-s)}I^{q-1}v(s)ds\Big\} \\ +\lambda\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\Big(\chi(s)\int_0^1 e^{-k(1-u)}I^{q-1}v(u)du + \int_0^s e^{-k(s-u)}I^{q-1}v(u)du\Big)ds\Big] \\ +\chi(t)\int_0^1 e^{-k(1-s)}I^{q-1}v(s)ds + \int_0^t e^{-k(t-s)}I^{q-1}v(s)ds,\ v \in S_{F,x}. \end{cases}$$

We will show that Ω_F satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that Ω_F *is convex for each* $x \in C([0,1],\mathbb{R})$. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

In the second step, we show that Ω_F *maps bounded sets (balls) into bounded sets in* $C([0,1],\mathbb{R})$. For a positive number ρ, let $B_\rho = \{x \in C([0,1],\mathbb{R}) : \|x\| \leqslant \rho\}$ be a bounded ball in $C([0,1],\mathbb{R})$. Then, for each $h \in \Omega_F(x), x \in B_\rho$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) = &\frac{1}{\Delta_1}\Big[-\sum_{i=1}^{m} a_i\Big\{\chi(\zeta_i)\int_0^1 e^{-k(1-s)}I^{q-1}v(s)ds + \int_0^{\zeta_i} e^{-k(\zeta_i-s)}I^{q-1}v(s)ds\Big\} \\ &+\lambda\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\Big(\chi(s)\int_0^1 e^{-k(1-u)}I^{q-1}v(u)du + \int_0^s e^{-k(s-u)}I^{q-1}v(u)du\Big)ds\Big] \\ &+\chi(t)\int_0^1 e^{-k(1-s)}I^{q-1}v(s)ds + \int_0^t e^{-k(t-s)}I^{q-1}v(s)ds. \end{aligned}$$

Then, for $t \in [0,1]$ we have

$$\begin{aligned} |h(t)| \leqslant &\Bigg[\frac{1}{|\Delta_1|}\Bigg\{\sum_{i=1}^{m}\frac{|a_i|}{k\Gamma(q)}\left(|\chi(\zeta_i)|(1-e^{-k}) + \zeta_i^{q-1}(1-e^{-k\zeta_i})\right) \\ &+\frac{|\lambda|\eta^{\beta}}{k\Gamma(\beta+1)\Gamma(q)}\left((1-e^{-k})|\chi(\eta)| + \eta^{q-1}(1-e^{-k\eta})\right)\Bigg\} \\ &+\frac{2(1-e^{-k})}{k\Gamma(q)}\Bigg]L_1\Lambda\|\phi\|\Omega(\|x\|_X), \end{aligned}$$

which, on taking the norm for $t \in [0,1]$ yields

$$\|h\| \leqslant \Lambda\|\phi\|L_1\Omega(\|x\|_X) \leqslant \Lambda\|\phi\|L_1\Omega(r).$$

Also we have

$$\begin{aligned} |h'(t)| \leqslant &\left|\frac{k - ke^{-kt}}{1-e^{-k}-k}\right|\int_0^1 e^{-k(1-s)}\left(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|\hat{f}(\tau)|d\tau\right)ds \\ &+k\int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|\hat{f}(\tau)|d\tau\right)ds + \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)}|\hat{f}(s)|ds \\ \leqslant &\left\{\frac{(1-e^{-k})^2}{(1-e^{-k}-k)\Gamma(q)} + \frac{2-e^{-k}}{\Gamma(q)}\right\}\|\phi\|\Omega(L_1\|x\|_X) \leqslant \Lambda_1\|\phi\|L_1\Omega(\|x\|_X). \end{aligned}$$

By definition of Caputo fractional derivative with $0 < \beta < 1$,we get

$$|{}^cD_{0+}^{\delta}h(t)| \leqslant \int_0^t \frac{(t-s)^{-\delta}}{\Gamma(1-\delta)}|h'(s)|ds \leqslant \frac{1}{\Gamma(2-\delta)}\Lambda_1\|\phi\|L_1\Omega(\|x\|_X).$$

As $h \in \Omega_F(x), x \in B_\rho$ is an arbitrary element, therefore we have

$$\|\Omega_F(x)\|_X = \|\Omega_F(x)\| + \|{}^cD_{0+}^{\delta}\Omega_F(x)\| \leqslant \left(\Lambda + \frac{\Lambda_1}{\Gamma(2-\delta)}\right)\|\phi\|L_1\Omega(r). \tag{4.1}$$

Now we show that Ω_F *maps bounded sets into equicontinuous sets of* $C([0,1],\mathbb{R})$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_\rho$. For each $h \in \Omega_F(x)$, we obtain

$$\begin{aligned} |h(t_2) - h(t_1)| \leqslant &\frac{|k(t_2-t_1) + e^{-kt_2} - e^{-kt_1}|}{|1-e^{-k}-k|}\int_0^1 e^{-k(1-s)}\left(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|v(\tau)|d\tau\right)ds \\ &+|\int_0^{t_1}\left(e^{-k(t_2-s)} - e^{-k(t_1-s)}\right)\left(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|v(\tau)|d\tau\right)ds \\ &+\int_{t_1}^{t_2} e^{-k(t_2-s)}\left(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|v(\tau)|d\tau\right)ds| \\ \leqslant &\Bigg[\frac{|k(t_2-t_1) + e^{-kt_2} - e^{-kt_1}|(1-e^{-k})}{|1-e^{-k}-k|} \\ &+t_1^{q-1}\left(e^{-k(t_2-t_1)} - (e^{-kt_2} - e^{-kt_1}) - 1\right) \\ &+t_2^{q-1}\left(1 - e^{-k(t_2-t_1)}\right)\Bigg]\frac{\|\phi\|L_1\Omega(r)}{k\Gamma(q)}. \end{aligned}$$

Also

$$\begin{aligned} |{}^cD_{0+}^{\delta}h(t_2) - {}^cD_{0+}^{\delta}h(t_1)| \leqslant &\left|\int_0^{t_2}(t_2-s)^{-\delta}h'(s)ds - \int_0^{t_1}(t_1-s)^{-\delta}h'(s)ds\right| \\ \leqslant &\frac{\Lambda_1}{\Gamma(2-\delta)}\left(2(t_2-t_1)^{1-\delta} + |t_2^{1-\delta} - t_1^{1-\delta}|\right)\|\phi\|L_1\Omega(r). \end{aligned}$$

Obviously the right hand side of the above inequalities tends to zero independently of $x \in B_\rho$ as $t_2 - t_1 \to 0$. As Ω_F satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\Omega_F : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous.

In our next step, we show that Ω_F *is upper semicontinuous.* To this end it is sufficient to show that Ω_F *has a closed graph,* by Lemma 4.2. Let $x_n \to x_*, h_n \in \Omega_F(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \Omega_F(x_*)$. Associated with $h_n \in \Omega_F(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$\begin{aligned} h_n(t) = &\frac{1}{\Delta_1}\Big[-\sum_{i=1}^{m} a_i\Big\{\chi(\zeta_i)\int_0^1 e^{-k(1-s)}I^{q-1}v_n(s)ds + \int_0^{\zeta_i} e^{-k(\zeta_i-s)}I^{q-1}v_n(s)ds\Big\} \\ &+\lambda\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\Big(\chi(s)\int_0^1 e^{-k(1-u)}I^{q-1}v_n(u)du + \int_0^s e^{-k(s-u)}I^{q-1}v_n(u)du\Big)ds\Big] \\ &+\chi(t)\int_0^1 e^{-k(1-s)}I^{q-1}v_n(s)ds + \int_0^t e^{-k(t-s)}I^{q-1}v_n(s)ds. \end{aligned}$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$\begin{aligned} h_*(t) = &\frac{1}{\Delta_1}\Big[-\sum_{i=1}^{m} a_i\Big\{\chi(\zeta_i)\int_0^1 e^{-k(1-s)}I^{q-1}v_*(s)ds + \int_0^{\zeta_i} e^{-k(\zeta_i-s)}I^{q-1}v_*(s)ds\Big\} \\ &+\lambda\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\Big(\chi(s)\int_0^1 e^{-k(1-u)}I^{q-1}v_*(u)du + \int_0^s e^{-k(s-u)}I^{q-1}v_*(u)du\Big)ds\Big] \\ &+\chi(t)\int_0^1 e^{-k(1-s)}I^{q-1}v_*(s)ds + \int_0^t e^{-k(t-s)}I^{q-1}v_*(s)ds. \end{aligned}$$

Let us consider the linear operator $\Theta : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$v\mapsto\Theta(v)(t)=\frac{1}{\Delta_1}\left[-\sum_{i=1}^{m}a_i\left\{\chi(\zeta_i)\int_0^1 e^{-k(1-s)}I^{q-1}v(s)ds+\int_0^{\zeta_i}e^{-k(\zeta_i-s)}I^{q-1}v(s)ds\right\}\right.$$
$$\left.+\lambda\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\chi(s)\int_0^1 e^{-k(1-u)}I^{q-1}v(u)du+\int_0^s e^{-k(s-u)}I^{q-1}v(u)du\right)ds\right]$$
$$+\chi(t)\int_0^1 e^{-k(1-s)}I^{q-1}v(s)ds+\int_0^t e^{-k(t-s)}I^{q-1}v(s)ds.$$

Observe that

$$\|h_n(t)-h_*(t)\|=\left\|\frac{1}{\Delta_1}\left[-\sum_{i=1}^{m}a_i\left\{\chi(\zeta_i)\int_0^1 e^{-k(1-s)}I^{q-1}(v_n-v_*)(s)ds\right.\right.\right.$$
$$\left.+\int_0^{\zeta_i}e^{-k(\zeta_i-s)}I^{q-1}(v_n-v_*)(s)ds\right\}$$
$$+\lambda\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\chi(s)\int_0^1 e^{-k(1-u)}I^{q-1}(v_n-v_*)(u)du\right.$$
$$\left.\left.+\int_0^s e^{-k(s-u)}I^{q-1}(v_n-v_*)(u)du\right)ds\right]$$
$$+\chi(t)\int_0^1 e^{-k(1-s)}I^{q-1}(v_n-v_*)(s)ds$$
$$\left.+\int_0^t e^{-k(t-s)}I^{q-1}(v_n-v_*)(s)ds\right\|\to 0\,\text{as}\,n\to\infty.$$

Thus, it follows by Lemma 4.3 that $\Theta\circ S_F$ is a closed graph operator. Further, we have $h_n(t)\in\Theta(S_{F,x_n})$. Since $x_n\to x_*$, therefore, we have

$$h_*(t)=\frac{1}{\Delta_1}\left[-\sum_{i=1}^{m}a_i\left\{\chi(\zeta_i)\int_0^1 e^{-k(1-s)}I^{q-1}v_*(s)ds+\int_0^{\zeta_i}e^{-k(\zeta_i-s)}I^{q-1}v_*(s)ds\right\}\right.$$
$$\left.+\lambda\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\chi(s)\int_0^1 e^{-k(1-u)}I^{q-1}v_*(u)du+\int_0^s e^{-k(s-u)}I^{q-1}v_*(u)du\right)ds\right]$$
$$+\chi(t)\int_0^1 e^{-k(1-s)}I^{q-1}v_*(s)ds+\int_0^t e^{-k(t-s)}I^{q-1}v_*(s)ds,\ \text{for some}\ v_*\in S_{F,x_*}.$$

Finally, we show there exists an open set $U\subseteq C([0,1],\mathbb{R})$ with $x\notin\Omega_F(x)$ for any $\theta\in(0,1)$ and all $x\in\partial U$. Let $\theta\in(0,1)$ and $x\in\theta\Omega_F(x)$. Then there exists $v\in L^1([0,1],\mathbb{R})$ with $v\in S_{F,x}$ such that, for $t\in[0,1]$, we can obtain

$$\|x\|_X=\|x\|+\|^cD^{\delta}_{0+}x\|\leqslant\left(\Lambda+\frac{\Lambda_1}{\Gamma(2-\delta)}\right)\|\phi\|L_1\Omega(\|x\|_X),\tag{4.2}$$

which implies that

$$\frac{\|x\|_X}{\left(\Lambda+\frac{\Lambda_1}{\Gamma(2-\delta)}\right)\|\phi\|L_1\Omega(\|x\|_X)}\leqslant 1.$$

In view of (C_3), there exists M such that $\|x\|\neq M$. Let us set

$U=\{x\in C([0,1],\mathbb{R}):\|x\|<M\}.$

Note that the operator $\Omega_F:\overline{U}\to\mathcal{P}(C([0,1],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x\in\partial U$ such that $x\in\theta\Omega_F(x)$ for some $\theta\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 4.4), we deduce that Ω_F has a fixed point $x\in\overline{U}$ which is a solution of the problem (1.2) and (1.3). This completes the proof. □

4.2. The Lipschitz case

We prove in this subsection the existence of solutions for the problem (1.2) and (1.3) with a nonconvex valued right-hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler (1970).

Theorem 4.13. *Assume that:*

- (A_1) *$F:[0,1]\times\mathbb{R}^3\to\mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot,x(t),{}^cD^{\delta}_{0+}x(t),I^{\gamma}x(t)):[0,1]\to\mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x\in\mathbb{R}$;*
- (A_2) *$H_d(F(t,x,y,z),F(t,\bar{x},\bar{y},\bar{z}))\leqslant p(t)[|x-\bar{x}|+|y-\bar{y}|+|z-\bar{z}|]$ for almost all $t\in[0,1]$ and $x,y,z,\bar{x},\bar{y},\bar{z}\in\mathbb{R}$ with $p\in C([0,1],\mathbb{R}^+)$ and $d(0,F(t,0,0,0))\leqslant p(t)$ for almost all $t\in[0,1]$.*

Then the problem (1.2) and (1.3) has at least one solution on $[0,1]$ if

$$\|p\|L_1\left(\Lambda+\frac{\Lambda_1}{\Gamma(2-\delta)}\right)<1.\tag{4.3}$$

Proof. Consider the operator $\Omega_F:C([0,1],\mathbb{R})\to\mathcal{P}(C([0,1],\mathbb{R}))$ defined in the beginning of the proof of Theorem 4.12. Observe that the set $S_{F,x}$ is nonempty for each $x\in C([0,1],\mathbb{R})$ by the assumption (A_1), so F has a measurable selection (see Theorem III.6 Castaing and Valadier, 1977). Now we show that the operator Ω_F satisfies the assumptions of Lemma 4.5. To show that $\Omega_F(x)\in\mathcal{P}_{cl}((C[0,1],\mathbb{R}))$ for each $x\in C([0,1],\mathbb{R})$, let $\{u_n\}_{n\geqslant 0}\in\Omega_F(x)$ be such that $u_n\to u(n\to\infty)$ in $C([0,1],\mathbb{R})$. Then $u\in C([0,1],\mathbb{R})$ and there exists $v_n\in S_{F,x}$ such that, for each $t\in[0,1]$,

$$u_n(t)=\frac{1}{\Delta_1}\left[-\sum_{i=1}^{m}a_i\left\{\chi(\zeta_i)\int_0^1 e^{-k(1-s)}I^{q-1}v_n(s)ds+\int_0^{\zeta_i}e^{-k(\zeta_i-s)}I^{q-1}v_n(s)ds\right\}\right.$$
$$\left.+\lambda\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\chi(s)\int_0^1 e^{-k(1-u)}I^{q-1}v_n(u)du+\int_0^s e^{-k(s-u)}I^{q-1}v_n(u)du\right)ds\right]$$
$$+\chi(t)\int_0^1 e^{-k(1-s)}I^{q-1}v_n(s)ds+\int_0^t e^{-k(t-s)}I^{q-1}v_n(s)ds.$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0,1],\mathbb{R})$. Thus, $v\in S_{F,x}$ and for each $t\in[0,1]$, we have

$$u_n(t)\to u(t)=\frac{1}{\Delta_1}\left[-\sum_{i=1}^{m}a_i\left\{\chi(\zeta_i)\int_0^1 e^{-k(1-s)}I^{q-1}v(s)ds+\int_0^{\zeta_i}e^{-k(\zeta_i-s)}I^{q-1}v(s)ds\right\}\right.$$
$$\left.+\lambda\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\chi(s)\int_0^1 e^{-k(1-u)}I^{q-1}v(u)du+\int_0^s e^{-k(s-u)}I^{q-1}v(u)du\right)ds\right]$$
$$+\chi(t)\int_0^1 e^{-k(1-s)}I^{q-1}v(s)ds+\int_0^t e^{-k(t-s)}I^{q-1}v(s)ds.$$

Hence, $u\in\Omega_F(x)$.

Next we show that there exists $\hat{\theta}:=\|q\|L_1\left(\Lambda+\frac{\Lambda_1}{\Gamma(2-\delta)}\right)<1$ such that

$H_d(\Omega_F(x),\Omega_F(\bar{x}))\leqslant\hat{\theta}\|x-\bar{x}\|_X$ foreach $x,\bar{x}\in C^3([0,1],\mathbb{R})$.

Let $x,\bar{x}\in C^3([0,1],\mathbb{R})$ and $h_1\in\Omega_F(x)$. Then there exists $v_1(t)\in F(t,x(t),{}^cD^{\delta}_{0+}x(t),I^{\gamma}x(t))$ such that, for each $t\in[0,1]$,

$$h_1(t)=\frac{1}{\Delta_1}\left[-\sum_{i=1}^{m}a_i\left\{\chi(\zeta_i)\int_0^1 e^{-k(1-s)}I^{q-1}v_1(s)ds+\int_0^{\zeta_i}e^{-k(\zeta_i-s)}I^{q-1}v_1(s)ds\right\}\right.$$
$$\left.+\lambda\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\chi(s)\int_0^1 e^{-k(1-u)}I^{q-1}v_1(u)du+\int_0^s e^{-k(s-u)}I^{q-1}v_1(u)du\right)ds\right]$$
$$+\chi(t)\int_0^1 e^{-k(1-s)}I^{q-1}v_1(s)ds+\int_0^t e^{-k(t-s)}I^{q-1}v_1(s)ds.$$

By (A_2), we have

$H_d(F(t,x,y,z),F(t,\bar{x},\bar{y},\bar{z}))\leqslant p(t)[|x-\bar{x}|+|y-\bar{y}|+|z-\bar{z}|].$

So, there exists $w\in F(t,\bar{x},\bar{y},\bar{z})$ such that

$|v_1(t)-w|\leqslant p(t)[|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|+|z(t)-\bar{z}(t)|],\ t\in[0,1].$

Define $U:[0,1]\to\mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leqslant q(t)[|x(t) - \bar{x}(t)| + |y(t) - \bar{y}(t)| + |z(t) - \bar{z}(t)|]\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{x}, \bar{y}, \bar{z})$ is measurable (Proposition III.4 Castaing and Valadier, 1977), there exists a function $v_2(t)$ which is a measurable selection for $U(t) \cap F(t, \bar{x}, \bar{y}, \bar{z})$. So $v_2(t) \in F(t, \bar{x}, \bar{y}, \bar{z})$ and for each $t \in [0,1]$, we have $|v_1(t) - v_2(t)| \leqslant q(t)[|x(t) - \bar{x}(t)| + |y(t) - \bar{y}(t)| + |z(t) - \bar{z}(t)|]$. For each $t \in [0,1]$, let us define

$$\begin{aligned} h_2(t) = & \frac{1}{\Delta_1}\left[-\sum_{i=1}^m a_i \left\{\chi(\zeta_i)\int_0^1 e^{-k(1-s)} I^{q-1} v_2(s)ds + \int_0^{\zeta_i} e^{-k(\zeta_i - s)} I^{q-1} v_2(s)ds\right\}\right. \\ & \left. + \lambda \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\chi(s)\int_0^1 e^{-k(1-u)} I^{q-1} v_2(u)du + \int_0^s e^{-k(s-u)} I^{q-1} v_2(u)du\right)ds\right] \\ & + \chi(t)\int_0^1 e^{-k(1-s)} I^{q-1} v_2(s)ds + \int_0^t e^{-k(t-s)} I^{q-1} v_2(s)ds. \end{aligned}$$

Thus

$$\begin{aligned} |h_1(t) - h_2(t)| = & \frac{1}{|\Delta_1|}\left[\sum_{i=1}^m |a_i|\left\{|\chi(\zeta_i)|\int_0^1 e^{-k(1-s)} I^{q-1}|v_1 - v_2|(s)ds\right.\right. \\ & \left. + \int_0^{\zeta_i} e^{-k(\zeta_i - s)} I^{q-1}|v_1 - v_2|(s)ds\right\} \\ & + \lambda \int_0^\eta \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)}\left(\chi(s)\int_0^1 e^{-k(1-u)} I^{q-1}|v_1 - v_2|(u)du\right. \\ & \left.\left. + \int_0^s e^{-k(s-u)} I^{q-1}|v_1 - v_2|(u)du\right)ds\right] \\ & + |\chi(t)|\int_0^1 e^{-k(1-s)} I^{q-1}|v_1 - v_2|(s)ds \\ & + \int_0^t e^{-k(t-s)} I^{q-1}|v_1 - v_2|(s)ds \\ \leqslant & \|p\|\left(\frac{1}{|\Delta_1|}\left\{\sum_{i=1}^m \frac{|a_i|}{k\Gamma(q)}\left(|\chi(\zeta_i)|(1 - e^{-k}) + \zeta_i^{q-1}(1 - e^{-k\zeta_i})\right)\right.\right. \\ & \left. + \frac{|\lambda|\eta^\beta}{k\Gamma(\beta+1)\Gamma(q)}\left((1 - e^{-k})|\chi(\eta)| + \eta^{q-1}(1 - e^{-k\eta})\right)\right\} \\ & \left. + \frac{2(1 - e^{-k})}{k\Gamma(q)}\right)\|x - \bar{x}\|_X, \end{aligned}$$

which yields $\|h_1 - h_2\| \leqslant \|p\|\Lambda L_1\|x - \bar{x}\|_X$.Further, in view of the estimate $|h_1'(t) - h_2'(t)| \leqslant \|p\||\Lambda_1 L_1\|x - \bar{x}\|_X$,we have

$$\begin{aligned} |{}^cD_{0+}^\delta h_1(t) - {}^cD_{0+}^\delta h_2(t)| & \leqslant \int_0^t \frac{(t-s)^{-\delta}}{\Gamma(1-\delta)}|h_1'(t) - h_2'(t)|ds \\ & \leqslant \frac{1}{\Gamma(2-\delta)}\|p\|\Lambda_1 L_1\|x - \bar{x}\|_X. \end{aligned}$$

In consequence, we get

$$\|h_1 - h_2\| \leqslant \|p\|L_1\left(\Lambda + \frac{\Lambda_1}{\Gamma(2-\delta)}\right)\|x - \bar{x}\|_X.$$

Analogously, interchanging the roles of x and $\bar{x}$, we can obtain

$$H_d(\Omega_F(x), \Omega_F(\bar{x})) \leqslant \|p\|L_1\left(\Lambda + \frac{\Lambda_1}{\Gamma(2-\delta)}\right)\|x - \bar{x}\|_X.$$

By the condition (4.3), it follows that Ω_F is a contraction and hence it has a fixed point x by Lemma 4.5, which is a solution of the problem (1.2) and (1.3). This completes the proof. □

Remark 4.14. In case the multivalued map F is not necessarily convex valued, we consider the following problem

$$\begin{cases} ({}^cD_{0+}^\alpha + k\,{}^cD_{0+}^{\alpha-1})x(t) = f(x(t)), \quad t \in [0,1],\ 2 < \alpha \leqslant 3, \\ x(0) = x(1), x'(0) = 0, \sum_{i=1}^m a_i x(\zeta_i) = \lambda \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}x(s)ds, 0 < \eta < \zeta_1 < \cdots < \zeta_m < 1. \end{cases} \tag{4.4}$$

and note that a solution $x \in C^3([0,1], \mathbb{R})$ of the problem (4.4) is a solution to the problem (1.2) and (1.3). In relation to the problem (4.4), we have the operator $\overline{\Omega}_F x(t)$ defined by

$$\begin{aligned} \overline{\Omega}_F x(t) = & \frac{1}{\Delta_1}\left[-\sum_{i=1}^m a_i\left\{\chi(\zeta_i)\int_0^1 e^{-k(1-s)} I^{q-1}(fx)(s)ds + \int_0^{\zeta_i} e^{-k(\zeta_i-s)} I^{q-1}(fx)(s)ds\right\}\right. \\ & \left. + \lambda\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\chi(s)\int_0^1 e^{-k(1-u)} I^{q-1}(fx)(u)du + \int_0^s e^{-k(s-u)} I^{q-1}(fx)(u)du\right)ds\right] \\ & + \chi(t)\int_0^1 e^{-k(1-s)} I^{q-1}(fx)(s)ds + \int_0^t e^{-k(t-s)} I^{q-1}(fx)(s)ds. \end{aligned}$$

In order to establish the existence of solutions for the given problem, we need the following assumption in addition to (C_2) and (C_3):

(C_4) $F : [0,1] \times \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that
- (a) $(t, x, y, z) \mapsto F(t, x, y, z)$ is $\mathcal{L} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}$ measurable,
- (b) $(x, y, z) \mapsto F(t, x, y, z)$ is lower semicontinuous for each $t \in [0,1]$.

The proof of the concerned existence result follows the method of proof for Theorem 4.12 and relies on the nonlinear alternative of Leray-Schauder type comined with the selection theorem of Bressan and Colombo (1988) for lower semi-continuous maps with decomposable values.

Remark 4.15. For $\beta = 1$, the results of this paper correspond to semi-periodic nonlocal classical integro-multipoint boundary conditions of the form: $x(0) = x(1), x'(0) = 0, \sum_{i=1}^m a_i x(\zeta_i) = \lambda \int_0^\eta x(s)ds$.

5. Examples

(a) Consider the following nonlocal multi-point boundary value problem of Caputo type sequential fractional integro-differential equations

$$\begin{cases} ({}^cD^{5/2} + \frac{1}{4}{}^cD_{0+}^{3/2})x(t) = f(t, x(t), {}^cD_{0+}^{4/5}x(t), I^{2/5}x(t)), \ 0 < t < 1, \\ x(0) = x(1), x'(0) = 0, \frac{3}{4}x(\frac{1}{2}) + \frac{5}{4}x(\frac{2}{3}) + x(\frac{3}{4}) = \int_0^{\frac{1}{4}} \frac{(\frac{1}{4}-s)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}x(s)ds. \end{cases} \tag{5.1}$$

Here $q = 5/2, k = 1/4, \delta = 4/5, \gamma = 2/5, a_1 = 3/4, a_2 = 5/4, a_3 = 1, a_4 = 3, \zeta_i = i/(i+1), i = 1, \ldots 3, \lambda = 1/2, \eta = 1/4, \beta = 3/2$. With the given values, it is found that $\Delta_1 \approx 2.9970615, \Lambda \approx 1.877424, \Lambda_1 \approx 2.196638, L_1 \approx 2.1270605$. Now we illustrate the obtained results by choosing different values of $f(t, x(t), {}^cD_{0+}^{4/5}x(t), I^{1/2}x(t))$.

(i) Let us consider

$$\begin{aligned} f(t, x(t), {}^cD_{0+}^{4/5}x(t), I^{1/2}x(t)) = & \frac{1}{\sqrt{t+121}}\left(\frac{|x(t)|}{1+|x(t)|} + \tan^{-1}\left({}^cD_{0+}^{4/5}x(t)\right)\right) \\ & + \frac{1}{11}I^{1/2}x(t) + \cos(\pi t/2). \end{aligned}$$

Obviously $L = 1/11$ as $|f(t, x(t), {}^cD_{0+}^{4/5}x(t), I^{1/2}x(t)) - f(t, y(t), {}^cD_{0+}^{4/5}y(t), I^{1/2}y(t))| \leqslant \frac{1}{11}\left(\|x - y\| + \|{}^cD_{0+}^{4/5}x - {}^cD_{0+}^{4/5}y\| + \|I^{1/2}x - I^{1/2}y\|\right)$. Further, $LL_1\left(\Lambda + \frac{\Lambda_1}{\Gamma(2-\delta)}\right) \approx 0.825654 < 1$.Thus all the conditions of Theorem 3.5 are satisfied. Therefore, by the conclusion of Theorem 3.5, we conclude that there exists a unique solution for the problem (5.1) on $[0,1]$.

(ii) To show the applicability of Theorem 3.7, we take the nonlinear function f of the form:

$$f(t,x(t),{}^cD_{0+}^{4/5}x(t),I^{1/2}x(t)) = \frac{3}{t+20}\left(\sin(x(t)) + \frac{|{}^cD_{0+}^{4/5}x(t)|}{1+|{}^cD^{4/5}x(t)|}\right) + \frac{3}{20}I^{1/2}x(t) + \frac{1}{10}, t \in [0,1].$$

Clearly $L = 3/20$ and $LL_1\left[\Lambda - \frac{(1-e^{-k})}{k\Gamma(q)} + \frac{(1-e^{-k})^2}{|1-e^{-k}-k|\Gamma(2-\delta)\Gamma(q)}\right] \approx 0.830740 < 1$. As all the conditions of Theorem 3.7 hold true, the conclusion of Theorem 3.7 applies. Hence the problem (5.1) with the given value of f has at least one solution on $[0,1]$.

(b) Let us consider the following inclusions problem:

$$\begin{cases} ({}^cD_{0+}^{5/2} + \frac{1}{4}{}^cD_{0+}^{3/2})x(t) \in F(t,x(t),{}^cD_{0+}^{4/5}x(t),I_{0+}^{2/5}x(t)), \quad 0 < t < 1, \\ x(0) = x(1), x'(0) = 0, \frac{3}{4}x\left(\frac{1}{2}\right) + \frac{5}{4}x\left(\frac{2}{3}\right) + x\left(\frac{3}{4}\right) = \int_0^{\frac{1}{4}} \frac{(\frac{1}{4}-s)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}x(s)ds. \end{cases} \tag{5.2}$$

(i) In order to demonstrate the application of Theorem 4.12, we consider

$$F(t,x(t),{}^cD_{0+}^{4/5}x(t),I^{1/2}x(t)) = \left[\frac{1}{2\sqrt{400+t^2}}\left(\sin x(t) + \frac{(\Gamma(3/2)+1)}{\Gamma(3/2)}{}^cD_{0+}^{4/5}x(t) + I^{1/2}x(t) + 1\right), \frac{1}{\sqrt{2500+t^2}}\left(x(t) + \sin({}^cD_{0+}^{4/5}x(t)) + \Gamma(3/2)I^{1/2}x(t) + \frac{1}{5}\right)\right]. \tag{5.3}$$

Obviously $\|\phi\| = \frac{(1+\Gamma(3/2))}{40\Gamma(3/2)}, \Omega(\|x\|_X) = 1 + \|x\|_X$ and Condition (H_3) is satisfied with $M > M_1 \approx 0.935206$. Thus, all the conditions of Theorem 4.12 are satisfied and consequently, there exists at least one solution for the problem (5.2) with F given by (5.3) on $[0,1]$.

(ii) For the illustration of Theorem 4.13, let us choose

$$F(t,x(t)) = \left[0, \frac{1}{12+t^2}\left(\frac{|x|}{8(4+|x|)} + \tan^{-1}({}^cD_{0+}^{4/5}x(t)) + \frac{\sqrt{\pi}}{4}I^{1/2}x(t)\right) + \frac{1}{15+t}\right]. \tag{5.4}$$

Clearly

$$H_d(F(t,x),F(t,\bar{x})) \leqslant \frac{1}{(12+t^2)}\|x-\bar{x}\|_X.$$

Letting $p(t) = 1/(12+t^2)$, it is easy to check that $d(0,F(t,0)) \leqslant p(t)$ holds for almost all $t \in [0,1]$ and that $\|p\|L_1\left(\Lambda + \frac{\Lambda_1}{\Gamma(2-\delta)}\right) \leqslant 0.756850 < 1$. As the hypotheses of Theorem 4.13 are satisfied, we conclude that the problem (5.2) with F given by (5.4) has at least one solution on $[0,1]$.

6. Conclusions

We have developed the existence theory for single-valued and multivalued problems of Caputo type sequential fractional differential equations and inclusions involving Riemann-Liouville integral equipped with semi-periodic and nonlocal multipoint Riemann-Liouville type integral boundary conditions. The nonlinearities in the given problems implicitly depend on the unknown function together with its fractional derivative of order $\delta \in (0,1)$ and its Riemann-Liouville integral of order $\gamma \in (0,1)$. We apply standard fixed theorems for single-valued and multivalued maps to establish the desired results. Our results are not only new in the given configuration but also yield some new special cases for specific choices of the parameters involved in the problem. For instance, the results associated with semi-periodic and nonlocal multipoint classical integral boundary conditions follow by taking $\beta = 1$ in the results of this paper. Letting $a_i = 0, i = 1,2,\ldots,m$, our results correspond to the three-point boundary conditions: $x(0) = x(1), x'(0) = 0, I^{\beta}x(\eta) = 0$. We can get the results for semi-periodic nonlocal multipoint boundary conditions of the form: $x(0) = x(1), x'(0) = 0, \sum_{i=1}^{m} a_i x(\zeta_i) = 0$ if we take $\lambda = 0$ in the obtained results.

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