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ORIGINAL ARTICLE

Numerical computation of BCOPs[1] in two variables for solving the vibration problem of a CF-elliptical plate

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Abstract Boundary characteristic orthogonal polynomials in *xy*-coordinates have been built up over an elliptical domain occupied by a thin elastic plate. Half of the plate boundary is taken clamped while the other half is kept free. Coefficients of these polynomials have been computed once and for all so that an orthogonal polynomial sequence is generated from a set of linearly independent functions satisfying the essential boundary conditions of the problem. Use of this sequence in Rayleigh–Ritz method for solving the free vibration problem of the plate makes it faster in convergence and leads to a simplified system whose solution is comparatively easier. Three-dimensional solution surfaces and the associated contour lines have been plotted in some selected cases. Comparison have been made with known results whenever available.

1. Introduction

Use of orthogonal polynomials in the Rayleigh–Ritz method for solving most of the important differential equations has attracted the researcher's interest since 19th century. Many studies on the vibration of non-rectangular plates assuming various deflection shape functions in the Rayleigh–Ritz method have been reported by Leissa (1969). Following the publications of Szego's well known treatise Szego (1967) and Singh and Chakraverty (1991, 1992, 1993, 1994a,b) there has been tremendous growth of literature covering various aspects of the subject but, unfortunately, for plates of uniform boundary conditions. Sato (1973) presented experimental as well as theoretical results for elliptic plates but again with uniform free edge. An interesting contribution in this regard has been done by Bhat et al. (1998) and Chakraverty et al. (1999). They presented a recurrence scheme that makes the generation of two-dimensional boundary characteristic orthogonal polynomials for a variety of geometries straight forward and quite efficient. They also provide a survey of the application of BCOPs method in vibration problems. Some important books on orthogonal polynomials and its applications are Beckmann (1973), Chihara (1978), and Gautschi et al. (1999).

There is no analytical solution to the vibration problems of plates with non-uniform boundary conditions even for plates of simple geometrical shapes like rectangles (Wei et al., 2001).

[1] Boundary characteristic orthogonal polynomials.

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Notation

BCOPs	boundary characteristic orthogonal polynomials	ν	Poisson ratio
CC	for a plate with uniform fully clamped boundary	λ	non-dimensional frequency parameter
CF	for a plate with half of the boundary, $y \leqslant 0$, clamped and the rest free	∇^2	Laplacian operator
FF	for a plate with uniform completely free boundary	N	the approximation order
$a,\ b$	semi major and minor axes of the elliptical domain	c_j	the unknown coefficients used in the solution expansion
r	aspect ratio b/a	$\phi_i(X,Y)$	orthogonal functions over R
$x,\ y$	cartesian coordinates	$\hat{\phi}_i(X,Y)$	orthonormal functions over R
$X,\ Y$	non-dimensional coordinates $X = x/a,\ \ Y = y/a$	β_{ij}	coefficients of the orthogonal polynomials $\phi_i(X,Y)$
R'	domain occupied by the plate in xy-coordinate	$f,\ g$	functions of x and y
R	domain occupied by the plate in XY-coordinates	$\langle f,g\rangle$	inner product of f and g
$W(X,Y)$	displacement	$\Vert f\Vert$	norm of f
ρ	density of the material of the plate	$[a_{ij}],\ [b_{ij}]$	$N \times N$ matrices
E	Young's modulus		
ω	angular natural frequency		

Very little is available in literature on elliptical plates with non-uniform boundary conditions and, whenever available, it is mostly on circular plates. That is why this kind of problems has become a challenging problem for scientists and engineers. Some available references are Eastep and Hemmig (1982), Hemmig (1975), Leissa et al. (1979), Laura and Ficcadenti (1981) and Narita and Leissa (1981). Hassan (2007) has generated BCOPs to compute natural frequencies of an elliptical plate with half of the boundary simply supported and the rest free and gave numerical and graphical results for this case. In Hassan (2004) he solved the vibration problem under consideration by using traditional basis functions that satisfy the essential boundary conditions of the CF-elliptical plate in the Rayleigh–Ritz method. Explicit numerical and graphical results have been given and reported for the first time. Other publications dealing with plates with mixed boundary conditions have been recently appeared by Boborykin (2006), Czernous (2006), and Zovatto and Nicolini (2006). They investigated the bending problem of a rectangular plate with mixed boundary conditions. No numerical results are available for vibrations of elliptical plates with mixed boundary conditions.

The aim of the present work is to generate a sequence of boundary characteristic orthogonal polynomials over an elliptical domain occupied by a thin elastic plate with half of the boundary, $y \leqslant 0$, clamped and the rest free. These polynomials should satisfy at least the essential boundary conditions of the problem. The coefficients of these polynomials will be generated and tabulated in advance, once and for all, with the desired precision. Use of these polynomials in Rayleigh–Ritz method helped in presenting explicit numerical results for the problem under consideration. This method reduces ill-conditioning of the resulting system whose solution has become comparatively easier and faster in convergence. Three-dimensional solution surfaces, mode shapes, and the associated contour lines of the problem have been plotted in some selected cases. Comparison of results have been made with known results in literature whenever available.

2. Generation of boundary characteristic orthogonal polynomials

As has been done by Bhat (1985) for one-dimensional orthogonal polynomials and by Liew et al. (1990) for rectangular plates one will follow the same procedures to generate a set of orthogonal polynomials in two variables over an elliptical domain R' occupied by a thin elastic plate in the xy-plane with half of the boundary, $y \leqslant 0$, clamped and the rest free. For this one can start with the set of linearly independent functions

$$\{F_i(x,y) = u f_i(x,y)\}_{i=1}^{N}, \tag{1}$$

with (x,y) is a point in $R' = \left\{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1\right\}$ and $a,\ b$ as the semi major and semi minor axes of the elliptical domain. The functions u and f are chosen to be of the form

$$u = \begin{cases} (y^2 - r^2 z^2)^2 & \text{for CC-elliptical plate,} \\ (y + rz)^2 z & \text{for CF-elliptical plate,} \\ 1 & \text{for FF-elliptical plate,} \end{cases} \tag{2}$$

with $z = \sqrt{1 - x^2},\ \ r = \frac{b}{a}$, and

$$\{f_i, i = 1,2,\ldots\} = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, \ldots\}, \tag{3}$$

so that the essential boundary conditions of the problem are satisfied. To obtain an orthogonal set we define the inner product of two functions f and g by

$$\langle f,g\rangle = \int\!\!\int_{R'} f(x,y) g(x,y)\, dx\, dy \tag{4}$$

and the norm of a function f is then defined by

$$\Vert f\Vert^2 = \langle f,f\rangle = \int\!\!\int_{R'} f^2(x,y)\, dx\, dy \tag{5}$$

The orthogonal functions $\phi_i(x,y)$ are generated by using Gram–Schmidt process the algorithm for which may be summarized as follows:

$$\left.\begin{aligned} \phi_1 &= F_1, \\ \phi_i &= F_i - \sum_{j=1}^{i-1} \alpha_{ij}\phi_j, \\ \text{where} \quad & \\ \alpha_{ij} &= \langle F_i, \phi_j\rangle / \langle \phi_j, \phi_j\rangle, \quad j = 1,2,\ldots,i-1 \end{aligned}\right\}, \quad i = 2,3,\ldots,N. \tag{6}$$

The functions ϕ_i can be normalized by using the equation

$$\hat{\phi}_i = \phi_i / \Vert\phi_i\Vert = \phi_i / \langle \phi_i, \phi_i\rangle^{\frac{1}{2}}. \tag{7}$$

Computations of α_{ij} are greatly simplified if u and f_i are chosen as simple polynomials in x and y such that the essential boundary conditions of the problem are satisfied. The functions ϕ_i and $\hat{\phi}_i$ can be expressed in terms of $f_1, f_2, \ldots$ if desired. Thus coefficients β_{ij} and $\hat{\beta}_{ij}$ can be found such that:

$$\phi_i = u\sum_{j=1}^{i}\beta_{ij}f_j, \quad \hat{\phi}_i = u\sum_{j=1}^{i}\hat{\beta}_{ij}f_j, \quad i = 1, 2, \ldots, N. \tag{8}$$

3. Rayleigh–Ritz procedures

For a plate undergoing simple harmonic motion equating the maximum strain energy V_{max} and the maximum kinetic energy T_{max} of the deformed plate the Rayleigh quotient (see Siddiqi, 2004) is

$$\omega^2 = \frac{D\int\int_{R'}\left[\left(\nabla^2 W\right)^2 + 2(1-v)\{W_{xy}^2 - W_{xx}W_{yy}\}\right]dx\,dy}{\rho h\int\int_{R'} W^2 dx\,dy}, \tag{9}$$

where $W(x, y)$ is the deflection of the plate. Subscripts denote differentiation with respect to subscripted variables. $D = Eh^3/[12(1-v^2)]$ is the flexural rigidity, E is Young's modulus, ρ is mass density, v is Poisson ratio, h is the thickness of the plate which has been taken to be unity all over the plate in this work, and ω is the radian natural frequency of vibration. Introducing the non-dimensional variables $X = x/a$ and $Y = y/a$ the new domain R is then defined by

$$\{(X, Y) : X^2 + Y^2/r^2 \leqslant 1\}, \quad r = \frac{b}{a}. \tag{10}$$

Assuming the plate deflection to be in the form

$$W(X, Y) = \sum_{j=1}^{N} c_j\phi_j(X, Y), \tag{11}$$

and applying the stationary conditions of ω^2 with respect to the coefficients $c_1, c_2, \ldots, c_N$ in the form

$$\frac{\partial\omega^2}{\partial c_j} = 0, \quad j = 1, 2, \ldots, N, \tag{12}$$

results in the eigenvalue problem

$$\sum_{j=1}^{N}\left(a_{ij} - \lambda^2 b_{ij}\right)c_j = 0, \quad i = 1, 2, \ldots, N, \tag{13}$$

where

$$a_{ij} = \int\int_R \left[(\phi_i)_{XX}(\phi_j)_{XX} + (\phi_i)_{YY}(\phi_j)_{YY} + v\left((\phi_i)_{XX}(\phi_j)_{YY} + (\phi_i)_{YY}(\phi_j)_{XX}\right) + 2(1-v)(\phi_i)_{XY}(\phi_j)_{XY}\right] dXdY, \tag{14}$$

$$b_{ij} = \int\int_R \phi_i\phi_j dXdY, \tag{15}$$

$$\lambda^2 = a^4\omega^2\rho h/D. \tag{16}$$

Solving the resulting eigenvalue problem (13) for λ and c_j one gets the frequencies and mode shapes.

4. Numerical results and discussion

The function u in (2) has been so chosen that the essential boundary conditions of the elliptical plate are satisfied. Consequently the essential boundary conditions of the problem are thus satisfied by the functions $F_i(X, Y)$ also. Finally BCOPs can be expressed in terms of f_i by computing β_{ij} in (8). All the computations have been worked out by using Mathematica 5.2 on a PC. This greatly simplifies and reduces the huge effort spent in preparing lengthy computations and cumbersome programs in any programming high level language. Also one can examine directly and easily the validity of the chosen function u whether it is suitable to our case or not without repeating the hall calculations from the very beginning in case one face any integration problems. Moreover, it enables one to use variation functions other than polynomial variations without fear of the integrals involved (for further work). The coefficients $\hat{\beta}_{ij}$ of the orthonormal polynomials have been computed and reported in Tables 1–4 which correspond to the aspect ratios $r = 0.5, 1.0, 1.5$ and 2.0, respectively, for CC-elliptical plate. Tables 5–8 are for CF-elliptical plate and Tables 9–12 are for FF-elliptical plate. The case $r = 1.0$ corresponds to a circular plate. It is to be noted that the program can generate results

Table 1 Coefficients $\hat{\beta}_{ij}$ of first 10-polynomials $\hat{\phi}_i$ for CC-elliptical plate with $r = 0.5$, and $v = 0.3$.

j	$i \to 1$	2	3	4	5	j
1	28.546	0.0	0.0	−22.7718	0.0	1
		98.8862	0.0	0.0	0.0	2
10	5511.19		197.772	0.0	0.0	3
9	0.0	4390.37		273.262	0.0	4
8	317.954	0.0	2172.04		739.997	5
7	0.0	99.7811	0.0	681.635		
6	0.0	0.0	0.0	0.0	1097.59	6
5	0.0	0.0	0.0	0.0	0.0	5
4	0.0	0.0	0.0	0.0	24.9453	4
3	−317.954	0.0	−155.145	0.0	0.0	3
2	0.0	−99.7811	0.0	−146.065	0.0	2
1	0.0	0.0	0.0	0.0	−24.9453	1
j	$i \to 10$	9	8	7	6	j

Table 2 Coefficients $\hat{\beta}_{ij}$ of first 10-polynomials $\hat{\phi}_i$ for CC-elliptical plate with $r = 1$, and $v = 0.3$.

j	$i \to 1$	2	3	4	5	j
1	1.26157	0.0	0.0	−1.00638	0.0	1
		4.37019	0.0	0.0	0.0	2
10	30.4453		4.37019	0.0	0.0	3
9	0.0	48.5072		12.0766	0.0	4
8	7.02585	0.0	47.9957		16.3518	5
7	0.0	4.40974	0.0	30.1243		
6	0.0	0.0	0.0	0.0	12.1268	6
5	0.0	0.0	0.0	0.0	0.0	5
4	0.0	0.0	0.0	0.0	1.10244	4
3	−7.02585	0.0	−3.42826	0.0	0.0	3
2	0.0	−4.40974	0.0	−6.4552	0.0	2
1	0.0	0.0	0.0	0.0	−1.10244	1
j	$i \to 10$	9	8	7	6	j

Table 3 Coefficients $\hat{\beta}_{ij}$ of first 10-polynomials $\hat{\phi}_i$ for CC-elliptical plate with $r = 1.5$, and $v = 0.3$.

j	$i \to 1$	2	3	4	5	j
1	0.20347	0.0	0.0	−0.162313	0.0	1
		0.704839	0.0	0.0	0.0	2
10	1.45491		0.469893	0.0	0.0	3
9	0.0	3.47707		1.94775	0.0	4
8	0.755434	0.0	5.1606		1.75818	5
7	0.0	0.711218	0.0	4.85854		
6	0.0	0.0	0.0	0.0	0.869266	6
5	0.0	0.0	0.0	0.0	0.0	5
4	0.0	0.0	0.0	0.0	0.177804	4
3	−0.755434	0.0	−0.368614	0.0	0.0	3
2	0.0	−0.711218	0.0	−1.04112	0.0	2
1	0.0	0.0	0.0	0.0	−0.177804	1
j	$i \to 10$	9	8	7	6	j

Table 4 Coefficients $\hat{\beta}_{ij}$ of first 10-polynomials $\hat{\phi}_i$ for CC-elliptical plate with $r = 2$, and $v = 0.3$.

j	$i \to 1$	2	3	4	5	j
1	0.0557539	0.0	0.0	−0.0444762	0.0	1
		0.193137	0.0	0.0	0.0	2
10	0.168188		0.0965686	0.0	0.0	3
9	0.0	0.535934		0.533715	0.0	4
8	0.155251	0.0	1.06056		0.361326	5
7	0.0	0.194885	0.0	1.33132		
6	0.0	0.0	0.0	0.0	0.133983	6
5	0.0	0.0	0.0	0.0	0.0	5
4	0.0	0.0	0.0	0.0	0.0487212	4
3	−0.155251	0.0	−0.0757546	0.0	0.0	3
2	0.0	−0.194885	0.0	−0.285282	0.0	2
1	0.0	0.0	0.0	0.0	−0.0487212	1
j	$i \to 10$	9	8	7	6	j

Table 5 Coefficients $\hat{\beta}_{ij}$ of first 10-polynomials $\hat{\phi}_i$ for CF-elliptical plate with $r = 0.5$, and $v = 0.3$.

j	$i \to 1$	2	3	4	5	j
1	2.41258	0.0	−5.60759	−2.98204	0.0	1
		7.62924	0.0	0.0	−17.2778	2
10	485.054		17.7818	3.11455	0.0	3
9	0.0	360.597		19.9985	0.0	4
8	42.9796	0.0	164.751		60267.3	5
7	0.0	14.5061	0.0	48.3590		
6	−280.018	0.0	16.1507	0.0	98.4336	6
5	0.0	−158.944	0.0	16.6373	0.0	5
4	−9.38614	0.0	−43.1758	0.0	3.90257	4
3	19.0568	0.0	−21.5259	0.0	−47.0246	3
2	0.0	6.52485	0.0	−16.8595	0.0	2
1	4.35020	0.0	4.47922	0.0	2.83810	1
j	$i \to 10$	9	8	7	6	j

Table 6 Coefficients $\hat{\beta}_{ij}$ of first 10-polynomials $\hat{\phi}_i$ for CF-elliptical plate with $r = 1.0$, and $v = 0.3$.

j	$i \to 1$	2	3	4	5	j
1	0.426487	0.0	−0.991291	−0.527155	0.0	1
		1.34867	0.0	0.0	−3.05432	2
10	10.7183		1.57171	0.275290	0.0	3
9	0.0	15.9363		3.53527	0.0	4
8	3.79889	0.0	14.5621		5.32693	5
7	0.0	2.56435	0.0	8.54875		
6	−12.3752	0.0	0.713766	0.0	4.35019	6
5	0.0	−14.0488	0.0	1.47054	0.0	5
4	−1.65925	0.0	−7.63247	0.0	0.689883	4
3	1.68440	0.0	1.90264	0.0	−4.15643	3
2	0.0	1.15344	0.0	−2.98036	0.0	2
1	0.769013	0.0	0.791822	0.0	0.501709	1
j	$i \to 10$	9	8	7	6	j

Table 7 Coefficients $\hat{\beta}_{ij}$ of first 10-polynomials $\hat{\phi}_i$ for CF-elliptical plate with $r = 1.5$, and $v = 0.3$.

j	$i \to 1$	2	3	4	5	j
1	0.154767	0.0	−0.359727	−0.191298	0.0	1
		0.489416	0.0	0.0	−1.10837	2
10	1.15245		0.380235	0.0665994	0.0	3
9	0.0	2.57025		1.28290	0.0	4
8	0.919047	0.0	3.52293		1.28872	5
7	0.0	0.930569	0.0	3.10223		
6	−1.99591	0.0	0.115119	0.0	0.701613	6
5	0.0	−3.39875	0.0	0.355761	0.0	5
4	−0.602121	0.0	−2.76973	0.0	0.250350	4
3	0.407498	0.0	−0.460296	0.0	−1.00554	3
2	0.0	0.418569	0.0	−1.08153	0.0	2
1	0.279065	0.0	0.287342	0.0	0.182064	1
j	$i \to 10$	9	8	7	6	j

Table 8 Coefficients $\hat{\beta}_{ij}$ of first 10-polynomials $\hat{\phi}_i$ for CF-elliptical plate with $r = 2.0$, and $v = 0.3$.

j	$i \to 1$	2	3	4	5	j
1	0.0753930	0.0	−0.175237	−0.0931887	0.0	1
		0.238414	0.0	0.0	−0.539932	2
10	0.236843		0.138920	0.0243324	0.0	3
9	0.0	0.704290		0.624953	0.0	4
8	0.335778	0.0	1.28712		0.470838	5
7	0.0	0.453317	0.0	1.51122		
6	−0.546910	0.0	0.0315443	0.0	0.192253	6
5	0.0	−01.24175	0.0	0.129979	0.0	5
4	−0.293317	0.0	−1.34924	0.0	0.121955	4
3	0.148881	0.0	−0.168171	0.0	−0.367380	3
2	0.0	0.203902	0.0	−0.526858	0.0	2
1	0.135944	0.0	0.139976	0.0	0.0886905	1
j	$i \to 10$	9	8	7	6	j

Table 9 Coefficients $\hat{\beta}_{ij}$ of first 10-polynomials $\hat{\phi}_i$ for FF-elliptical plate with $r = 0.5$, and $v = 0.3$.

j	$i \to 1$	2	3	4	5	j
1	0.797885	0.0	0.0	−0.797885	0.0	1
		1.59577	0.0	0.0	0.0	2
10	57.092		3.19154	0.0	0.0	3
9	0.0	38.2985		3.19154	0.0	4
8	8.5638	0.0	17.1276		7.81764	5
7	0.0	3.19154	0.0	6.38308		
6	0.0	0.0	0.0	0.0	13.5406	6
5	0.0	0.0	0.0	0.0	0.0	5
4	0.0	0.0	0.0	0.0	1.12838	4
3	−8.5638	0.0	−2.8546	0.0	0.0	3
2	0.0	−3.19154	0.0	−3.19154	0.0	2
1	0.0	0.0	0.0	0.0	−1.12838	1
j	$i \to 10$	9	8	7	6	j

Table 10 Coefficients $\hat{\beta}_{ij}$ of first 10-polynomials $\hat{\phi}_i$ for FF-plate with $r = 1$, and $v = 0.3$.

j	$i \to 1$	2	3	4	5	j
1	0.56419	0.0	0.0	−0.56419	0.0	1
		1.12838	0.0	0.0	0.0	2
10	5.04627		1.12838	0.0	0.0	3
9	0.0	6.77028		2.25676	0.0	4
8	3.02776	0.0	6.05552		2.76395	5
7	0.0	2.25676	0.0	4.51352		
6	0.0	0.0	0.0	0.0	2.39365	6
5	0.0	0.0	0.0	0.0	0.0	5
4	0.0	0.0	0.0	0.0	0.797885	4
3	−3.02776	0.0	−1.00925	0.0	0.0	3
2	0.0	−2.25676	0.0	−2.25676	0.0	2
1	0.0	0.0	0.0	0.0	−0.797885	1
j	$i \to 10$	9	8	7	6	j

Table 11 Coefficients $\hat{\beta}_{ij}$ of first 10-polynomials $\hat{\phi}_i$ for FF-elliptical plate with $r = 1.5$, and $v = 0.3$.

j	$i \to 1$	2	3	4	5	j
1	0.460659	0.0	0.0	−0.460659	0.0	1
		0.921318	0.0	0.0	0.0	2
10	1.22082		0.614212	0.0	0.0	3
9	0.0	2.45685		1.84264	0.0	4
8	1.6481	0.0	3.29621		1.50451	5
7	0.0	1.84264	0.0	3.68527		
6	0.0	0.0	0.0	0.0	0.868627	6
5	0.0	0.0	0.0	0.0	0.0	5
4	0.0	0.0	0.0	0.0	0.65147	4
3	−1.6481	0.0	−0.549368	0.0	0.0	3
2	0.0	−1.84264	0.0	−1.84264	0.0	2
1	0.0	0.0	0.0	0.0	−0.65147	1
j	$i \to 10$	9	8	7	6	j

Table 12 Coefficients $\hat{\beta}_{ij}$ of first 10-polynomials $\hat{\phi}_i$ for FF-elliptical plate with $r = 2$, and $v = 0.3$.

j	$i \to 1$	2	3	4	5	j
1	0.398942	0.0	0.0	−0.398942	0.0	1
		0.797885	0.0	0.0	0.0	2
10	0.446031		0.398942	0.0	0.0	3
9	0.0	1.19683		1.59577	0.0	4
8	1.07047	0.0	2.14095		0.977205	5
7	0.0	1.59577	0.0	3.19154		
6	0.0	0.0	0.0	0.0	0.423142	6
5	0.0	0.0	0.0	0.0	0.0	5
4	0.0	0.0	0.0	0.0	0.56419	4
3	−1.07047	0.0	−0.356825	0.0	0.0	3
2	0.0	−1.59577	0.0	−1.59577	0.0	2
1	0.0	0.0	0.0	0.0	−0.56419	1
j	$i \to 10$	9	8	7	6	j

for any value of the aspect ratio $r > 0$. The approximation order N has been increased from 1 to 28 for CC and FF-cases but from 1 to 10 only in the CF-case. It is a gigantic task to go through approximations beyond this because of the singularities arising in some integrals due to discontinuities at $X = \pm 1$. If it is necessary the recurrence scheme mentioned in Bhat et al. (1998) is recommended, for further work, which makes the generation of orthogonal polynomials easier and straight forward. For need of space only 10 polynomials have been reported in all cases. The tabulated coefficients have been computed once for all and the reader can use these directly without repeating the calculations again and again. As a check on accuracy of the results it has been verified that

$$\langle \hat{\phi}_i, \hat{\phi}_j \rangle = \begin{cases} 0 & \text{for} \quad i \neq j, \\ 1 & \text{for} \quad i = j. \end{cases} \tag{17}$$

Use of these BCOPs as basis functions in Rayleigh–Ritz method greatly simplifies the resulting eigenvalue problem since the matrix in (15) becomes a unit matrix. Following these procedures the first four frequencies of the plate vibration have been computed and reported in Table 13 for the specified values of r. All the computations have been worked out for $v = 0.3$. Results corresponding to some other values have been computed and reported for comparisons. Note that the values of λ_1 given in FF-case is actually the value of λ_2. For this case the first frequency corresponds to the rigid body motion of translation and rotation of corresponding frequency 0.0 (Narita and Leissa, 1981). In Table 13 the abbreviation BC denotes the type of boundary conditions, * used for $v = 0.25$, and ** for $v = 0.33$. Comparison of these results with others and with those computed by using the traditional basis functions (Hassan, 2004) have been made and found to be better and on the lower side for the same approximation order. The present method has faster rate of convergence as compared to the traditional polynomials because the resulting eigenvalue problem is no longer the generalized one. The trends of convergence of the fundamental frequency parameter as computed by using BCOPs for $r = 0.5$ and 1.0 are investigated and reported in Table 14 for the three cases of boundary conditions. It is clear from

Table 13 First four frequency parameters of an elliptical plate ($\nu = 0.3$), * for $\nu = 0.25$ and ** for $\nu = 0.33$.

BC	Ref.	r	N	λ_1	λ_2	λ_3	λ_4
CC	Present	0.5	28	27.3776	39.5000	56.3275	69.8841
	Hassan (2004)	0.5	36	27.3774	39.4974	55.9757	69.8580
	Singh and Chakraverty (1994a)	0.5	36	27.377	39.497	55.985	69.858
	Present	1.0	28	10.2158	21.2605	34.8777	39.7733
	Hassan (2004)	1.0	36	10.2158	21.2604	34.8770	39.7711
	Narita and Leissa (1981)	1.0	36	10.2144	21.2613	34.8808	39.7656
	Exact	1.0		10.2158	21.2604	34.8770	39.7711
	Present	1.5	28	7.6131	12.6542	18.4388	19.7298
	Present	2.0	28	6.8444	9.8748	13.9962	17.4656
CF	Present	0.5	10	6.0249	14.1261	27.2132	27.8854
	Hassan (2004)	0.5	10	6.0832			
	Hassan (2004)	0.5	78	5.9937	13.7321	25.6766	27.6245
	Present	1.0	10	3.1552	9.7090	10.4854	19.8298
	Hassan (2004)	1.0	10	3.2002			
	Hassan (2004)	1.0	78	2.8781	8.9854	9.4516	18.4377
	Present	1.5	10	2.5092	6.6619	9.4444	11.5434
	Present	2.0	10	2.2332	5.4693	8.8450	8.9237
FF	Present	0.5	28	6.67058	10.5478	17.2116	22.3526
	Singh and Chakraverty (1994a)	0.5	20	6.6706	10.548	16.923	22.019
	Hassan (2004)	0.5	36	6.6706	10.548	16.923	22.019
	Present	1.0	28	5.3583	9.0034	12.4645	21.0331
	Hassan (2004)	1.0	36	5.3583	9.0031	12.439	—
	Exact	1.0		5.3583	9.0031	12.439	20.475
	Sato (1973)	1.0	20	5.3583	9.0031	12.439	20.475
*	Present	1.0	28	5.51119	8.89018	12.8811	21.158
*	Sato (1973)	1.0	20	5.5112	8.8899	12.744	20.409
**	Present	1.0	28	5.26205	9.06923	12.2625	21.0775
**	Sato (1973)	1.0	20	5.262	9.0689	12.244	20.513
**	Narita and Leissa (1981)	1.0	36	5.2624	9.0721	12.243	20.512
**	Exact	1.0		5.262	9.0689	12.244	20.513
	Present	1.5	28	2.87855	3.54941	7.24836	7.33783
	Present	2.0	28	1.66765	2.63694	4.3029	5.58816

Table 14 Convergence of the fundamental frequency parameter of an elliptical plate ($\nu = 0.3$).

CC			CF			FF		
N	$r = 0.5$	$r = 1.0$	N	$r = 0.5$	$r = 1$	N	$r = 0.5$	$r = 1.0$
6	27.3954	10.217	3	6.0755	3.3499	10	7.39485	5.79655
10	27.3954	10.217	4	6.0753	3.3278	11	6.70475	5.54263
11	27.3792	10.2166	5	6.0753	3.3278	13	6.70268	5.54200
12	27.3792	10.2166	6	6.0607	3.3161	15	6.70264	5.38067
13	27.3782	10.2163	7	6.0607	3.3161	22	6.67388	5.37167
14	27.3782	10.2163	8	6.0260	3.1553	24	6.67208	5.36885
15	27.3776	10.2158	9	6.0260	3.1553	27	6.67067	5.35834
16	27.3776	10.2158	10	6.0249	3.1552	28	6.67058	5.35834

the table that the present results converge to at least three significant figures in all the cases at a relatively low approximation order. Thus in view of the present results one indicates that there is a significant improvement in the rate of convergence if an orthonormal basis is used instead of the traditional one.

5. Mode shapes

Fig. 1a–f depict the first six mode shapes and the associated contour lines for a CF-elliptical plate with $r = 0.5$ and $\nu = 0.33$. Figures corresponding to $\nu = 0.3$ are roughly the same. These have been plotted by using tools of Computer Graphics under Turbo C++. The author has prepared his own software for that purpose. Other more figures corresponding to different aspect ratios are available in Hassan (2004).

6. Conclusion

The author has presented a set of orthonormal bases functions that can help in solving numerically the vibration problem of an elliptical plate clamped on lower half of the boundary

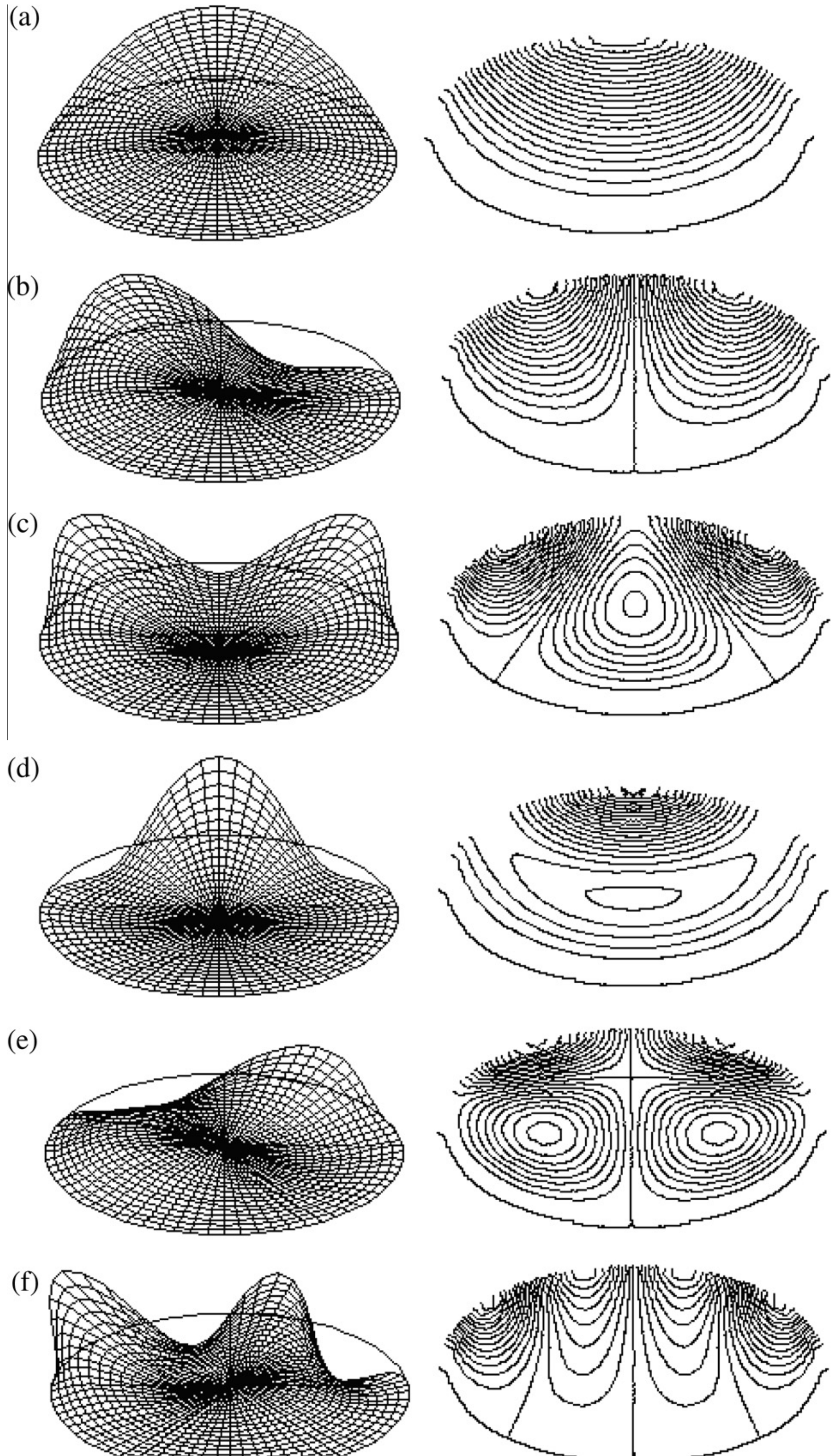

Figure 1 First six mode shapes and the associated contour lines for CF-elliptical plate with $r = 0.5$ and $v = 0.33$.

and free on the upper half. Interested readers can use these directly without repeating the calculations again and again for similar problems. Those polynomials are not only simplifying the calculations but also minimizes the effects of ill-conditioning which frequently occurs with such problems since the resulting eigenvalue problem is no longer the generalized one now.

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