Journal of King Saud University – Science 31 (2019) 519–524

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Circular strongly partially-balanced repeated measurement designs in periods of two different sizes using method of cyclic shifts (Rule II)

Rida Jabeen, Rashid Ahmed *, Muhammad Sajjad, H.M. Kashif Rashid, Abid Khan

Department of Statistics, The Islamia University of Bahawalpur, Pakistan

ARTICLE INFO

Article history:
Received 30 July 2017
Accepted 3 February 2019
Available online 7 February 2019

Mathematics Subject Classification (2010):
05B05
62K10
62K05

Keywords:
Repeated measurement designs
Balanced repeated measurement designs
Strongly balanced repeated measurement designs
Circular strongly partially-balanced repeated measurement designs

ABSTRACT

Strongly balanced repeated measurement designs may be used in medicine, pharmacology, animal sciences and psychology with unequal period sizes. To avoid a large number of subjects, strongly partially-balanced repeated measurement designs are preferred. In this article, some infinite series are developed to generate the minimal strongly partially-balanced repeated measurement designs in periods of two different sizes using method of cyclic shifts (Rule II).

1. Introduction

A repeated measurements design (RMD) is balanced with respect to the first-order residual effects if each treatment is immediately preceded same number of times, say λ' times by each other treatment (excluding itself). Williams (1949, 1950) first initiated RMDs. Magda (1980) introduced the idea of a circular balanced RMDs. Cheng and Wu (1980) constructed balanced and strongly balanced RMD. RMD is strongly balanced with respect to the first-order residual effects if each treatment is immediately preceded λ' times by each other treatment (including itself). Afsarinejed (1994) constructed balanced and strongly balanced minimal RMDs with unequal period sizes. RMDs in unequal period sizes are very useful if there is a restriction on the total number of treatments, some experimental units can receive on the total length of time while some experimental units can remain in the trial. Using method of cyclic shifts, Iqbal and Jones (1994) constructed (i) efficient RMDs with equal and unequal period sizes (ii) Strongly balanced RMDs for two unequal period sizes. Iqbal and Tahir (2009) constructed CSBRMD (circular strongly balanced RMDs) for some classes. Iqbal et al. (2010) constructed some first- and second-order CBRMD and CSBRMDs. Rasheed et al. (2018) developed some infinite series to obtain the minimal CSBRMDs in periods of three different sizes. The situations where minimal CSBRMDs cannot be constructed, minimal CSPBRMDs are preferred. RMD is strongly partially balanced if each treatment is not immediately preceded same number of times by each other treatment (including itself). If λ'_i takes only two values as $\lambda'_1 = \lambda'_2 + 1$ then it is very close to the balanced. Strongly balanced and strongly partially-balanced RMDs are useful for the estimation of direct and residual effects independently. Minimal CSPBRMDs are preferred for the cases where minimal SBRMDs cannot be constructed. Jabeen et al. (2019) constructed minimal CSPBRMDs in equal period sizes almost for every cases. Using method of cyclic shifts (Rule I), Nazeer et al. (2018) constructed these designs in periods of two different sizes only for a few cases of v. For the remaining cases of v, minimal CSPBRMDs in periods of two

* Corresponding author.
E-mail address: rashid.ahmed@iub.edu.pk (R. Ahmed).

Peer review under responsibility of King Saud University.

ELSEVIER Production and hosting by Elsevier

https://doi.org/10.1016/j.jksus.2019.02.002
1018-3647/

different sizes could be constructed through method of cyclic shifts (Rule II) which should be constructed. Therefore, in this article, these designs are constructed for the remaining cases through method of cyclic shifts (Rule II).

The rest of the paper is organized as follows: In Section 2, method of cyclic shifts (Rule II) is explained to generate the CSPBRMDs in periods of two different sizes. In Section 3, efficiency of proposed designs is discussed. Using method of cyclic shifts (Rule II), some infinite series are developed in Section 4 to obtain minimal CSPBRMDs. These series are very useful for researchers and experimenters. They can get the required designs just by putting the values needed for the series. Contribution of this research is discussed in Section 5.

2. Method of cyclic shifts

In this article, Method of Cyclic Shifts introduced by Iqbal (1991) is used as a methodology for construction of the proposed designs. This method is preferred because it provides an easy construction of several types of cyclic designs such as (i) balanced incomplete block designs, (ii) polygonal designs, (iii) neighbor balanced designs, (iv) strongly balanced neighbor designs, (v) balanced RMDs, (vi) strongly balanced RMDs, (vii) weakly balanced RMDs, and (viii) strongly partially-balanced RMDs. All these designs can be constructed through this method in linear and circular periods/blocks of equal and unequal sizes. Furthermore, this method has edge over the existing methods because without studying the complete design, one can check the standard property of treatment balance and other balance properties such as for neighbor effects and residual effects, etc. Method of cyclic shifts (Rule II) is explained here briefly only for the construction of CSPBRMDs. For detail, see Iqbal & Tahir (2009) and Iqbal et al. (2010).

Rule II: Let $S_1 = [q_{11}, q_{12}, \ldots, q_{1(p_1-1)}]$ and $S_2 = [q_{21}, q_{22}, \ldots, q_{2(p_2-2)}]$ t be sets of shifts, where $0 \le q_{ij} \le v-2$. If each element 0, 1, 2, …, $v-2$ appears an equal number of times, say λ' in a new set of shifts S*, where $S^* = [q_{11}, q_{12}, \ldots, q_{1(p_1-1)}, q_{21}, q_{22}, \ldots, q_{2(p_2-2)}, v-1\text{-}(q_{11}+q_{12}+\ldots+q_{1(p_1-1)}) \bmod v-1]$ then it will be CSBRMD in periods of sizes p_1 & p_2, otherwise CSPBRMD.

Example 2.1:. *CSPBRMDs is constructed for* $v = 10$, $p_1 = 6$ *and* $p_2 = 4$ *through the following two sets of shifts.*

$S_1 = [1,3,2,8,7]$, $S_2 = [4,0]$t

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}
0_3	1_4	2_5	3_6	4_7	5_8	6_0	7_1	8_2	0_9	1_9	2_9	3_9	4_9	5_9	6_9	7_9	8_9
1_0	2_1	3_2	4_3	5_4	6_5	7_6	8_7	0_8	4_0	5_1	6_2	7_3	8_4	0_5	1_6	2_7	3_8
4_1	5_2	6_3	7_4	8_5	0_6	1_7	2_8	3_0	4_4	5_5	6_6	7_7	8_8	0_0	1_1	2_2	3_3
6_4	7_5	8_6	0_7	1_8	2_0	3_1	4_2	5_3	9_4	9_5	9_6	9_7	9_8	9_0	9_1	9_2	9_3
5_6	6_7	7_8	8_0	0_1	1_2	2_3	3_4	4_5									
3_5	4_6	5_7	6_8	7_0	8_1	0_2	1_3	2_4									

3. Statistical model and efficiency of the design

In order to consider the efficiency of the constructed designs, the model for circular RMDs proposed by Davis and Hall (1969) is used.

$$\boldsymbol{Y} = \mu\boldsymbol{E} + \boldsymbol{D}\delta + \boldsymbol{R}\rho + \boldsymbol{U}\nu + \boldsymbol{P}\pi + \varepsilon \tag{1}$$

Here **Y** is the $np \times 1$ column vector of the np observations, μ is the overall mean, $\boldsymbol{\delta}$ is the vector of direct effects of order $v \times 1$, $\boldsymbol{\rho}$ is residual effect vector of order $v \times 1$, $\boldsymbol{\nu}$ is the unit vector of order $n \times 1$, $\boldsymbol{\pi}$ is the vector of period effects having order $p \times 1$ and $\boldsymbol{\varepsilon}$ is random error vector of order $np \times 1$ with mean zero and constant variance σ^2. **E** is the matrix of 1's with order $p \times q$. **D, R, U, P** are design matrices of observations versus direct effects, residual effects, unit effects and period effects of treatments with order $np \times v$, $np \times v$, $np \times bv$ and $np \times p$ respectively.

Using the identities, $\mathbf{D'D} = \mathbf{R'D} = bp\mathbf{I}_v$, $\mathbf{D'R} = \mathbf{L}$, $\mathbf{D'U} = \mathbf{N}$, $\mathbf{D'P} = b\mathbf{E}_{v,p}$, $\mathbf{R'U} = \mathbf{N}$, $\mathbf{U'U} = p\mathbf{I}_n$, $\mathbf{U'U} = \mathbf{E}_{k,q}$, $\mathbf{P'P} = n\mathbf{I}_n$. The reduced normal equations for $\hat{\delta}$ and $\hat{\rho}$ will be:

$$\boldsymbol{C}\begin{bmatrix}\hat{\delta}\\ \hat{\rho}\end{bmatrix} = \begin{bmatrix}\theta & \pi\\ \pi' & \Theta\end{bmatrix}\begin{bmatrix}\hat{\delta}\\ \hat{\rho}\end{bmatrix} = \begin{bmatrix}\boldsymbol{T}\\ \boldsymbol{S}\end{bmatrix}$$

$$\begin{bmatrix} bp\boldsymbol{I}_v - p^{-1}\boldsymbol{NN}' & \boldsymbol{L}' - p^{-1}\boldsymbol{NN}' \\ \boldsymbol{L}' - p^{-1}\boldsymbol{NN}' & bp\boldsymbol{I}_v - p^{-1}\boldsymbol{NN}' \end{bmatrix}\begin{bmatrix}\hat{\delta}\\ \hat{\rho}\end{bmatrix} = \begin{bmatrix}\boldsymbol{D}'\boldsymbol{Y} - p^{-1}\boldsymbol{NU}'\boldsymbol{Y} \\ \boldsymbol{R}'\boldsymbol{Y} - p^{-1}\boldsymbol{NU}'\boldsymbol{Y}\end{bmatrix}$$

$\boldsymbol{\theta} = bp\boldsymbol{I}_v - p^{-1}\boldsymbol{NN}'$, $\quad \boldsymbol{\Theta} = bp\boldsymbol{I}_v - p^{-1}\boldsymbol{NN}'$, $\quad \boldsymbol{\pi} = \boldsymbol{L}' - p^{-1}\boldsymbol{NN}'$, $\boldsymbol{T} = \boldsymbol{D}'\boldsymbol{Y} - p^{-1}\boldsymbol{NU}'\boldsymbol{Y}$, $\boldsymbol{S} = \boldsymbol{R}'\boldsymbol{Y} - p^{-1}\boldsymbol{NU}'\boldsymbol{Y}$,

For the period of two different sizes information matrix can be presented as:

$$\boldsymbol{C}^* = \begin{bmatrix} bp\boldsymbol{I}_v - p_1^{-1}\boldsymbol{N}_1\boldsymbol{N}'_1 - p_2^{-1}\boldsymbol{N}_2\boldsymbol{N}'_2 & \boldsymbol{L}' - p_1^{-1}\boldsymbol{N}_1\boldsymbol{N}'_1 - p_2^{-1}\boldsymbol{N}_2\boldsymbol{N}'_2 \\ \boldsymbol{L}' - p_1^{-1}\boldsymbol{N}_1\boldsymbol{N}'_1 - p_2^{-1}\boldsymbol{N}_2\boldsymbol{N}'_2 & bp\boldsymbol{I}_v - p_1^{-1}\boldsymbol{N}_1\boldsymbol{N}'_1 - p_2^{-1}\boldsymbol{N}_2\boldsymbol{N}'_2 \end{bmatrix}$$

The information matrix for direct and residual effects denoted by θ and $\boldsymbol{\Theta}$ respectively can be specified by their initial rows:

$\theta = [\theta_0, \theta_1, \ldots, \theta_{t-1}]$ and $\Theta = [\Theta_0, \Theta_1, \ldots, \Theta_{t-1}]$

According to the duality presented in the model (1), both direct and residual effects share the same information matrix. The non-zero Eigen values of information matrix C^* are called the canonical efficiency factors, see James and Wilkinson (1971) and Pearce et al. (1974). The canonical efficiency factor is calculated by working out harmonic mean of non-zero Eigen values of their respective information matrix relative to that of an orthogonal with the same number of treatments having same number of replications. It is further assume that σ^2 is the same for the proposed design and the orthogonal design to which it is compared. The high value of E_r shows that design is suitable for the estimation of residual effects. Our proposed designs have high value of E_r, therefore, these designs are suitable for the estimation of residual effects while using periods of two different sizes.

4. Infinite Series to generate CSPBRMDs in periods of two different sizes

In this section, some infinite series are developed by method of cyclic shifts (Rule II) to generate minimal CSPBRMDs in periods of two different sizes. In the all following series, S* contains all values from 0, 1, 2, …, $v-1$ either 0 or 1 time, therefore, all series provide minimal CSPBRMDs. Here ordered pairs {(0, $v/2$), (1, $(v+2)/2$), …, ($(v-4)/2$, $v-2$), ($(v-2)/2$, 0), ($v/2$, 1), …, ($v-2$, $(v-2)/2$), ($v-1$, $v-1$)} do not appear together while all other appear once in Series 3.1–3.9 while ordered pairs {(0, $(v+1)/2$), (1, $(v+3)/2$), …, ($(v-3)/2$, $v-2$), ($(v-1)/2$, 0), ($(v+1)/2$, 1), …, ($v-2$, $(v-3)/2$), ($v-1$, $v-1$)} do not appear together while all other appear once in Series 3.10 to 3.20. In these series sum of any two, three, …, $(p-2)$ consecutive elements should not be 0 (mod v). If so, reorder the elements.

Series 4.1: CSPBRMDs can be constructed for $v = 2mi + 4$, $m > 2$, i integer, $p_1 = 2m$ and $p_2 = 4$ through the following $(i+1)$ sets of shifts.

$S_{j+1} = [mj+1, mj+2, \ldots, mj+m, v-2\text{-}mj, v-3\text{-}mj, \ldots, v-m\text{-}mj]$; $j = 0, 1, \ldots, i-1$.
$S_{i+1} = [0, (v-2)/2]$t

Example 4.1:.

v	m	i	p_1	p_2	Sets of Shifts	Er
10	3	1	6	4	[1,2,3,8,7] + [0,4]t	0.83
16	3	2	6	4	[1,2,3,14,13] + [4,5,6,11,10] + [0,7]t	0.83
12	4	1	8	4	[1,2,3,4,10,9,8] + [0,5]t	0.86
20	4	2	8	4	[1,2,3,4,18,17,16] + [5,6,7,8,14,13,12] + [0,9]t	0.86

***Series* 4.2:** *CSPBRMDs can be constructed for* $v = 2mi + 6$, $m > 3$, *i integer,* $p_1 = 2\,m$ *and* $p_2 = 6$ *through the following* $(i + 1)$ *sets of shifts.*

$S_{j+1} = [mj + 1,\ mj + 2,\ \ldots,\ mj + m,\ v - 2\text{-}mj,\ v - 3\text{-}mj,\ \ldots,\ v - m\text{-}mj]$; $j = 0, 1, \ldots, i - 1$.
$S_{i+1} = [0, (v - 4)/2, (v - 2)/2, (v + 2)/2]\mathrm{t}$

Example 4.2:.

v	m	i	p_1	p_2	Sets of Shifts	Er
14	4	1	8	6	[1,2,3,4,12,11,10] + [0,5,6,8]t	0.89
22	4	2	8	6	[1,2,3,4,20,19,18] + [5,6,7,8,16,15,14] + [0,9,10,12]t	0.88
16	5	1	10	6	[1,2,3,4,5,14,13,12,11] + [0,6,7,9]t	0.91
26	5	2	10	6	[1,2,3,4,5,24,23,22,21] + [6,7,8,9,10,19,18,17,16] + [0,11,12,14]t	0.90

***Series* 4.3:** *CSPBRMDs can be constructed for* $v = 2mi + 8$, $m > 4$, *i integer,* $p_1 = 2\,m$ *and* $p_2 = 8$ *through the following* $(i + 1)$ *sets of shifts.*

$S_{j+1} = [mj + 1,\ mj + 2,\ \ldots,\ mj + m,\ v - 2\text{-}mj,\ v - 3\text{-}mj,\ \ldots,\ v - m\text{-}mj]$; $j = 0, 1, \ldots, i - 1$.
$S_{i+1} = [0, (v - 6)/2, (v - 4)/2, (v - 2)/2, (v + 2)/2, (v + 4)/2]\mathrm{t}$

Example 4.3:.

v	m	i	p_1	p_2	Sets of Shifts	Er
18	5	1	10	8	[1,2,3,4,5,16,15,14,13] + [0,6,7,8,10,11]t	0.92
28	5	2	10	8	[1,2,3,4,5,26,25,24,23] + [6,7,8,9,10,21,20,19,18] + [0,11,12,13,15,16]t	0.91
20	6	1	12	8	[1,2,3,4,5,6,18,17,16,15,14] + [0,7,8,9,11,12]t	0.93
32	6	2	12	8	[1,2,3,4,5,6,30,29,28,27,26] + [7,8,9,10,11,12,24,23,22,21,20] + [0,13,14,15,17,18]t	0.93

***Series* 4.4:** *CSPBRMDs can be constructed for* $v = 2mi + 10$, $m > 5$, *i integer,* $p_1 = 2\,m$ *and* $p_2 = 10$ *through the following* $(i + 1)$ *sets of shifts.*

$S_{j+1} = [mj + 1,\ mj + 2,\ \ldots,\ mj + m,\ v - 2\text{-}mj,\ v - 3\text{-}mj,\ \ldots,\ v - m\text{-}mj]$; $j = 0, 1, \ldots, i - 1$.
$S_{i+1} = [0, (v - 6)/2, (v - 4)/2, (v - 2)/2, (v + 2)/2, (v + 4)/2]\mathrm{t}$
$S_{i+1} = [0, (v - 8)/2, (v - 6)/2, (v - 4)/2, (v - 2)/2, (v + 2)/2, (v + 4)/2, (v + 6)/2]\mathrm{t}$

Example 4.4:.

v	m	i	p_1	p_2	Sets of Shifts	Er
22	6	1	12	10	[1,2,3,4,5,6,20,19,18,17,16] + [0,7,8,9,10,12,13,14]t	0.94
34	6	2	12	10	[1,2,3,4,5,6,32,31,30,29,28] + [7,8,9,10,11,12,26,25,24,23,22] + [0,13,14,15,16,18,19,20]t	0.93
24	7	1	14	10	[1,2,3,4,5,6,7,22,21,20,19,18,17] + [0,8,9,10,11,13,14,15]t	0.95
38	7	2	14	10	[1,2,3,4,5,6,7,36,35,34,33,32,31] + [8,9,10,11,12,13,14,29,28,27, 26,25,24] + [0,15,16,17,18,20,21,22]t	0.94

***Series* 4.5:** *CSPBRMDs can be constructed for* $v = 2mi + 12$, $m > 6$, *i integer,* $p_1 = 2\,m$ *and* $p_2 = 12$ *through the following* $(i + 1)$ *sets of shifts.*

$S_{j+1} = [mj + 1,\ mj + 2,\ \ldots,\ mj + m,\ v - 2\text{-}mj,\ v - 3\text{-}mj,\ \ldots,\ v - m\text{-}mj]$; $j = 0, 1, \ldots, i - 1$.
$S_{i+1} = [0, (v - 10)/2, (v - 8)/2, (v - 6)/2, (v - 4)/2, (v - 2)/2, (v + 2)/2, (v + 4)/2, (v + 6)/2, (v + 8)/2]\mathrm{t}$

Example 4.5:.

v	m	i	p_1	p_2	Sets of Shifts	Er
26	7	1	14	12	[1,2,3,4,5,6,7,24,23,22,21,20,19] + [0,8,9,10,11,12,14,15,16,17]t	0.81
40	7	2	14	12	[1,2,3,4,5,6,7,38,37,36,35,34,33] + [8,9,10,11,12,13,14,31,30,29,28,27,26] + [0,15,16,17,18,19,21,22,23,24]t	0.94
28	8	1	16	12	[1,2,3,4,5,6,7,8,26,25,24,23,22,21,20] + [0,9,10,11,12,13,15,16,17,18]t	0.96
44	8	2	16	12	[1,2,3,4,5,6,7,8,42,41,40,39,38,37,36] + [9,10,11,12,13,14,15,16,34,33,32,31,30, 29,28] + [0,17,18,19,20,21,23,24,25,26]t	0.95

***Series* 4.6:** *CSPBRMDs can be constructed for* $v = 2\,m + 4$, $m > 1$ *integer,* $p_1 = 2\,m + 1$ *and* $p_2 = 3$ *through the following* $(i + 1)$ *sets of shifts.*

$S_1 = [1, 2, \ldots, m, v - 2, v - 3, \ldots, v - 1\text{-}m]$;
$S_2 = [(v - 2)/2]\mathrm{t}$

Example 4.6:.

v	m	p_1	p_2	Sets of Shifts	Er
8	2	5	3	[1,2,6,5] + [3]t	0.80
10	3	7	3	[1,2,3,8,7,6] + [4]t	0.85
12	4	9	3	[1,2,3,4,10,9,8,7] + [5]t	0.89

***Series* 4.7:** *CSPBRMDs can be constructed for* $v = 2\,m + 6$, $m > 2$ *integer,* $p_1 = 2\,m + 4$ *and* $p_2 = 5$ *through the following* $(i + 1)$ *sets of shifts.*

$S_1 = [1, 2, \ldots, m, v - 2, v - 3, \ldots, v - 1\text{-}m]$;
$S_2 = [(v - 4)/2, (v - 2)/2, (v + 2)/2]\mathrm{t}$

Example 4.7:.

v	m	p_1	p_2	Sets of Shifts	Er
12	3	7	5	[1,2,3,10,9,8] + [4,5,7]t	0.88
14	4	9	5	[1,2,3,4,12,11,10,9] + [5,6,8]t	0.90
16	5	11	5	[1,2,3,4,5,14,13,12,11,10] + [6,7,9]t	0.89

***Series* 4.8:** *CSPBRMDs can be constructed for* $v = 2m + 8$, $m > 3$ *integer,* $p_1 = 2m + 1$ *and* $p_2 = 7$ *through the following* $(i + 1)$ *sets of shifts.*

$S_1 = [1, 2, \ldots, m, v-2, v-3, \ldots, v-1\text{-}m]$;
$S_2 = [(v-6)/2, (v-4)/2, (v-2)/2, (v+2)/2, (v+4)/2]\mathrm{t}$

Example 4.8:.

v	m	p_1	p_2	Sets of Shifts	Er
16	4	9	7	[1,2,3,4,14,13,12,11] + [5,6,7,9,10]t	0.92
18	5	11	7	[1,2,3,4,5,16,15,14,13,12] + [6,7,8,10,11]t	0.93
20	6	13	7	[1,2,3,4,5,6,18,17,16,15,14,13] + [7,8,9,11,12]t	0.94

***Series* 4.9:** *CSPBRMDs can be constructed for* $v = 2m + 10$, $m > 4$ *integer,* $p_1 = 2m + 1$ *and* $p_2 = 9$ *through the following* $(i + 1)$ *sets of shifts.*

$S_1 = [1, 2, \ldots, m, v-2, v-3, \ldots, v-1\text{-}m]$;
$S_2 = [(v-8)/2, (v-6)/2, (v-4)/2, (v-2)/2, (v+2)/2, (v+4)/2, (v+6)/2]\mathrm{t}$

Example 4.9:.

v	m	p_1	p_2	Sets of Shifts	Er
20	5	11	9	[1,2,3,4,5,18,17,16,15,14] + [6,7,8,9,11,12,13]t	0.94
22	6	13	9	[1,2,3,4,5,6,20,19,18,17,16,15] + [7,8,9,10,12,13,14]t	0.93
24	7	15	9	[1,2,3,4,5,6,7,22,21,20,19,18,17,16] + [8,9,10,11,13,14,15]t	0.95

***Series* 4.10:** *CSPBRMDs can be constructed for* $v = 2mi + 5$, $m > 2$, i *integer,* $p_1 = 2m$ *and* $p_2 = 5$ *through the following* $(i + 1)$ *sets of shifts.*

$S_{j+1} = [mj + 1, mj + 2, \ldots, mj + m, v-2\text{-}mj, v-3\text{-}mj, \ldots, v-m\text{-}mj]$; $j = 0, 1, \ldots, i-1$.
$S_{i+1} = [0, (v-3)/2, (v-1)/2]\mathrm{t}$

Example 4.10:.

v	m	i	p_1	p_2	Sets of Shifts	Er
11	3	1	6	5	[1,2,3,9,8] + [0,4,5]t	0.85
17	3	2	6	5	[1,2,3,15,14] + [4,5,6,12,11] + [0,7,8]t	0.85
13	4	1	8	5	[1,2,3,4,11,10,9] + [0,5,6]t	0.88
21	4	2	8	5	[1,2,3,4,19,18,17] + [5,6,7,8,15,14,13] + [0,9,10]t	0.88

***Series* 4.11:** *CSPBRMDs can be constructed for* $v = 2mi + 7$, $m > 3$, i *integer,* $p_1 = 2m$ *and* $p_2 = 7$ *through the following* $(i + 1)$ *sets of shifts.*

$S_{j+1} = [mj + 1, mj + 2, \ldots, mj + m, v-2\text{-}mj, v-3\text{-}mj, \ldots, v-m\text{-}mj]$; $j = 0, 1, \ldots, i-1$.
$S_{i+1} = [0, (v-5)/2, (v-3)/2, (v-1)/2, (v+3)/2]\mathrm{t}$

Example 4.11:.

v	m	i	p_1	p_2	Sets of Shifts	Er
15	4	1	8	7	[1,2,3,4,13,12,11] + [0,5,6,7,9]t	0.91
23	4	2	8	7	[1,2,3,4,21,20,19] + [5,6,7,8,17,16,15] + [0,9,10,11,13]t	0.89
17	5	1	10	7	[1,2,3,4,5,15,14,13,12] + [0,6,7,8,10]t	0.92
27	5	2	10	7	[1,2,3,4,5,25,24,23,22] + [6,7,8,9,10,20,19,18,17] + [0,11,12,13,15]t	0.91

***Series* 4.12:** *CSPBRMDs can be constructed for* $v = 2mi + 9$, $m > 4$, i *integer,* $p_1 = 2m$ *and* $p_2 = 9$ *through the following* $(i + 1)$ *sets of shifts.*

$S_{j+1} = [mj + 1, mj + 2, \ldots, mj + m, v-2\text{-}mj, v-3\text{-}mj, \ldots, v-m\text{-}mj]$; $j = 0, 1, \ldots, i-1$.
$S_{i+1} = [0, (v-7)/2, (v-5)/2, (v-3)/2, (v-1)/2, (v+3)/2, (v+5)/2]\mathrm{t}$

Example 4.12:.

v	m	i	p_1	p_2	*Sets of Shifts*	*Er*
19	5	1	10	9	[1,2,3,4,5,17,16,15,14] + [0,6,7,8,9,11,12]t	0.93
29	5	2	10	9	[1,2,3,4,5,27,26,25,24] + [6,7,8,9,10,22,21,20,19] + [0,11,12,13,14,16,17]t	0.92
21	6	1	12	9	[1,2,3,4,5,6,19,18,17,16,15] + [0,7,8,9,10,12,13]t	0.94
33	6	2	12	9	[1,2,3,4,5,6,31,30,29,28,27] + [7,8,9,10,11,12,25,24,23,22,21] + [0,13,14,15,16,18,19]t	0.93

***Series* 4.13:** *CSPBRMDs can be constructed for* $v = 2mi + 11$, $m > 5$, i *integer,* $p_1 = 2m$ *and* $p_2 = 11$ *through the following* $(i + 1)$ *sets of shifts.*

$S_{j+1} = [mj + 1, mj + 2, \ldots, mj + m, v-2\text{-}mj, v-3\text{-}mj, \ldots, v-m\text{-}mj]$; $j = 0, 1, \ldots, i-1$.
$S_{i+1} = [0, (v-9)/2, (v-7)/2, (v-5)/2, (v-3)/2, (v-1)/2, (v+3)/2, (v+5)/2, (v+7)/2]\mathrm{t}$

Example 4.13:.

v	m	i	p_1	p_2	*Sets of Shifts*	*Er*
23	6	1	12	11	[1,2,3,4,5,6,16,17,18,19,20] + [0,7,8,9,10,11,13,14,15]t	0.94
35	6	2	12	11	[1,2,3,4,5,6,33,32,31,30,29] + [7,8,9,10,11,12,27,26,25,24,23] + [0,13,14,15,16,17,19,20,21]t	0.93
25	7	1	14	11	[1,2,3,4,5,6,7,23,22,21,20,19,18] + [0,8,9,10,11,12,14,15,16]t	0.95
39	7	2	14	11	[1,2,3,4,5,6,7,37,36,35,34,33,32] + [8,9,10,11,12,13,14,30,29,28,27,26,25] + [0,15,16,17,18,19,21,22,23]t	0.94

***Series* 4.14:** *CSPBRMDs can be constructed for* $v = 2mi + 13$, $m > 6$, *i integer*, $p_1 = 2m$ *and* $p_2 = 13$ *through the following* $(i + 1)$ *sets of shifts.*

$S_{j+1} = [mj + 1, mj + 2, \ldots, mj + m, v - 2\text{-}mj, v - 3\text{-}mj, \ldots, v - m\text{-}mj]$; $j = 0, 1, \ldots, i - 1$.
$S_{i+1} = [0, (v - 11)/2, (v - 9)/2, \ldots, (v - 1)/2, (v + 3)/2, (v + 5)/2, (v + 7)/2, (v + 9)/2]$t

Example 4.14:.

v	m	i	p_1	p_2	Sets of Shifts	Er
27	7	1	14	13	[1,2,3,4,5,6,7,25,24,23,22,21,20] + [0,8,9,10,11,12,13,15,16,17,18]t	0.82
41	7	2	14	13	[1,2,3,4,5,6,7,39,38,37,36,35,34] + [8,9,10,11,12,13,14,32,31,30,29,28,27] + [0,15,16,17,18,19,20,22,23,24,25]t	0.85
29	8	1	16	13	[1,2,3,4,5,6,7,8,27,26,25,24,23,22,21] + [0,9,10,11,12,13,14,16,17,18,19]t	0.83
45	8	2	16	13	[1,2,3,4,5,6,7,8,43,42,41,40,39,38,37] + [9,10,11,12,13,14,15,16,35,34,33,32,31,30, 29] + [0,17,18,19,20,21,22,24,25,26,27]t	0.87

***Series* 4.15:** *CSPBRMDs can be constructed for* $v = 4i + 2m + 1$, $m > 1$, *i integer*, $p_1 = 2m + 1$ *and* $p_2 = 4$ *through the following* $(i + 1)$ *sets of shifts.*

$S_1 = [1, 2, \ldots, m, v - 2, v - 3, \ldots, v - 1\text{-}m]$;
$S_{j+2} = [2j + m + 1, 2j + m + 2, v - 2\text{-}m\text{-}2j]$; $j = 0, 1, \ldots, i - 2$.
$S_{i+1} = [(v - 3)/2, (v - 1)/2]$t

Example 4.15:.

v	m	i	p_1	p_2	Sets of Shifts	Er
9	2	1	5	4	[1,2,7,6] + [3,4]t	0.82
13	2	2	5	4	[1,2,11,10] + [3,4,9] + [5,6]t	0.80
11	3	1	7	4	[1,2,3,9,8,7] + [4,5]t	0.86
15	3	2	7	4	[1,2,3,13,12,11] + [4,5,10] + [6,7]t	0.86

***Series* 4.16:** *CSPBRMDs can be constructed for* $v = 6i + 2m + 1$, $m > 2$, *i integer*, $p_1 = 2m + 1$ *and* $p_2 = 6$ *through the following* $(i + 1)$ *sets of shifts.*

$S_1 = [1, 2, \ldots, m, v - 2, v - 3, \ldots, v - 1\text{-}m]$;
$S_{j+2} = [3j + m + 1, 3j + m + 2, 3j + m + 3, v - 2\text{-}m\text{-}3j, v - 3\text{-}m\text{-}3j]$; $j = 0, 1, \ldots, i - 2$.
$S_{i+1} = [(v - 5)/2, (v - 3)/2, (v - 1)/2, (v + 3)/2]$t

Example 4.16:.

v	m	i	p_1	p_2	Sets of Shifts	Er
13	3	1	7	6	[1,2,3,11,10,9] + [4,5,6,8]t	0.89
19	3	2	7	6	[1,2,3,17,16,15] + [4,5,6,14,13] + [7,8,9,11]t	0.88
15	4	1	9	6	[1,2,3,4,13,12,11,10] + [5,6,7,9]t	0.91
21	4	2	9	6	[1,2,3,4,19,18,17,16] + [5,6,7,15,14] + [8,9,10,12]t	0.90

***Series* 4.17:** *CSPBRMDs can be constructed for* $v = 8i + 2m + 1$, $m > 3$, *i integer*, $p_1 = 2m + 1$ *and* $p_2 = 8$ *through the following* $(i + 1)$ *sets of shifts.*

$S_1 = [1, 2, \ldots, m, v - 2, v - 3, \ldots, v - 1\text{-}m]$;
$S_{j+2} = [4j + m + 1, 4j + m + 2, 4j + m + 3, 4j + m + 4, v - 2\text{-}m\text{-}4j, v - 3\,m\text{-}4j, v - 4\text{-}m\text{-}4j]$; $j = 0,1, \ldots, i - 2$.
$S_{i+1} = [(v - 7)/2, (v - 5)/2, (v - 3)/2, (v - 1)/2, (v + 5)/2, (v + 3)/2]$t

Example 4.17:.

v	m	i	p_1	p_2	Sets of Shifts	Er
17	4	1	9	8	[1,2,3,4,15,14,13,12] + [5,6,7,8,11,10]t	0.92
25	4	2	9	8	[1,2,3,4,23,22,21,20] + [5,6,7,8,19,18,17] + [9,10,11,12,15,14]t	0.91
19	5	1	11	8	[1,2,3,4,5,17,16,15,14,13] + [6,7,8,9,12,11]t	0.93
27	5	2	11	8	[1,2,3,4,5,25,24,23,22,21] + [6,7,8,9,20,19,18] + [10,11,12,13,16,15]t	0.92

***Series* 4.18:** *CSPBRMDs can be constructed for* $v = 10i + 2m + 1$, $m > 4$, *i integer*, $p_1 = 2m + 1$ *and* $p_2 = 10$ *through the following* $(i + 1)$ *sets of shifts.*

$S_1 = [1, 2, \ldots, m, v - 2, v - 3, \ldots, v - 1\text{-}m]$;
$S_{j+2} = [5j + m + 1, 5j + m + 2, \ldots, 5j + m + 5, v - 2\text{-}m\text{-}5j, v - 3\text{-}m\text{-}5j, \ldots, v - 5\text{-}m\text{-}5j]$; $j = 0, 1, \ldots, i - 2$.
$S_{i+1} = [(v - 9)/2, (v - 7)/2, (v - 5)/2, (v - 3)/2, (v - 1)/2, (v + 7)/2, (v + 3)/2, (v + 5)/2]$t

Example 4.18:.

v	m	i	p_1	p_2	Sets of Shifts	Er
21	5	1	11	10	[1,2,3,4,5,19,18,17,16,15] + [6,7,8,9,10,14,13,12]t	0.84
31	5	2	11	10	[1,2,3,4,5,29,28,27,26,25] + [6,7,8,9,10,24,23,22,21] + [11,12,13,14,15,17,18,19]t	0.81
23	6	1	13	10	[1,2,3,4,5,6,21,20,19,18,17,16] + [7,8,9,10,11,13,14,15]t	0.86
33	6	2	13	10	[1,2,3,4,5,6,31,30,29,28,27,26] + [7,8,9,10,11,25,24,23,22] + [12,13,14,15,16,18,19,20]	0.82

***Series* 4.19:** *CSPBRMDs can be constructed for* $v = 12i + 2m + 1$, $m > 5$, *i integer*, $p_1 = 2m + 1$ *and* $p_2 = 12$ *through the following* $(i + 1)$ *sets of shifts.*

$S_1 = [1, 2, \ldots, m, v - 2, v - 3, \ldots, v - 1\text{-}m]$;
$S_{j+2} = [6j + m + 1, 6j + m + 2, \ldots, 6j + m + 6, v - 2\text{-}m\text{-}6j, v - 3\text{-}m\text{-}6j, \ldots, v - 6\text{-}m\text{-}6j]$; $j = 0, 1, \ldots, i - 2$.
$S_{i+1} = [(v - 11)/2, (v - 9)/2, \ldots, (v - 1)/2, (v + 9)/2, (v + 7)/2, (v + 5)/2, (v + 3)/2]$t

Example 4.19:.

v	m	i	p_1	p_2	Sets of Shifts	Er
25	6	1	13	12	[1,2,3,4,5,6,23,22,21,20,19,18] + [7,8,9,10,11,12,17,16,15,14]*t*	0.80
37	6	2	13	12	[1,2,3,4,5,6,35,34,33,32,31,30] + [7,8,9,10,11,12,29,28,27,26,25] + [13,14,15,16,17,18,23,22,21,20]*t*	0.84
27	7	1	15	12	[1,2,3,4,5,6,7,25,24,23,22,21,20,19] + [8,9,10,11,12,13,18,17,16,15]*t*	0.88
39	7	2	15	12	[1,2,3,4,5,6,7,37,36,35,34,33,32,31] + [8,9,10,11,12,13,30,29,28,27,26] + [14,15,16,17,18,19,24,23,22,21]*t*	0.90

***Series* 4.20:** *CSPBRMDs can be constructed for* $v = 14i + 2m + 1$, $m > 6$, *i integer,* $p_1 = 2m + 1$ *and* $p_2 = 14$ *through the following* $(i + 1)$ *sets of shifts.*

$S_1 = [1, 2, \ldots, m, v-2, v-3, \ldots, v-1\text{-}m]$;
$S_{j+2} = [7j + m + 1, 7j + m + 2, \ldots, 7j + m + 7, v-2\text{-}m\text{-}7j, v-3\text{-}m\text{-}7j, \ldots, v-7\text{-}m\text{-}7j]$; $j = 0, 1, \ldots, i-2$.
$S_{i+1} = [(v-13)/2, (v-11)/2, \ldots, (v+11)/2, (v+9)/2, (v+7)/2, (v+5)/2, (v+3)/2]t$

Example 4.20:.

v	m	i	p_1	p_2	Sets of Shifts	Er
29	7	1	15	14	[1,2,3,4,5,6,7,27,26,25,24,23,22,21] + [8,9,10,11,12,13,14,20,19,18,17,16]*t*	0.96
43	7	2	15	14	[1,2,3,4,5,6,7,41,40,39,38,37,36,35] + [8,9,10,11,12,13,14,34,33,32,31,30,29] + [15,16,17,18,19,20,21,27,26,25,24,23]*t*	0.90
31	8	1	17	14	[1,2,3,4,5,6,7,8,29,28,27,26,25,24,23,22] + [9,10,11,12,13,14,15,21,20,19,18,17]*t*	0.96
45	8	2	17	14	[1,2,3,4,5,6,7,8,43,42,41,40,39,38,37,36] + [9,10,11,12,13,14,15,35,34,33,32,31,30] + [16,17,18,19,20,21,22,28,27,26,25,24]*t*	0.95

5. Contribution of this research

Strongly balanced and strongly partially-balanced RMDs are useful for the estimation of direct and residual effects independently. The situations where minimal CSBRMDs cannot be constructed, minimal CSPBRMDs are preferred. In this article, some series are developed which are new one and provide the minimal CSPBRMDs in periods of two different sizes with high efficiency to balance out the residual effects. These designs can be used in the experiments to investigate (i) whether the perceived velocity of a moving point on a computer screen is affected by relative cues such as the presence of either vertical or horizontal lines and the amount of spacing between them, (ii) chronic conditions in clinical trials, (iii) the effect of chlorhexidine gluconate in dental plaque regrowth, (iv) different methods of preoxygenation, and (iv) cellulose membranes. These designs are also useful in asthma trial to compare doses of Budesonide and Fluticasone. Other areas where proposed designs may be applied are agricultural animal feeding trials, bioassay, bioequivalence studies, biomedical or physiological measurements, consumer trials, questionnaires, taste testing experiments, etc.

Acknowledgements

Authors are thankful to the editor and reviewers for their valuable suggestions. Authors Rashid Ahmed and H. M. Kashif Rasheed are highly grateful also to the Higher Education Commission Pakistan Research and Development Division National Research Program for Universities for providing grant for this research. Grant No: 6294/ Punjab/NRPU/R&D/HEC/2016.

References

Afsarinejed, K., 1994. Repeated measurements designs with unequal periods sizes. J. Ital. Stat. Soc. 2, 161–168.
Cheng, C.S., Wu, C.F., 1980. Balanced repeated measurements designs. Ann. Stat. 8, 1272–1283.
Davis, A.W., Hall, W.B., 1969. Cyclic change over designs. Biometrika 56, 283–293.
Iqbal, I., 1991. Construction of experimental designs using cyclic shifts (Ph.D thesis). University of Kent at Canterbury, UK.
Iqbal, I., Jones, B., 1994. Efficient repeated measurements designs with equal and unequal period sizes. J. Stat. Plann. Inference 42, 79–88.
Iqbal, I., Tahir, M.H., 2009. Circular strongly balanced repeated measurements designs. Commun. Stat.-Theory Methods 38, 3686–3696.
Iqbal, I., Tahir, M.H., Ghazali, S.S.A., 2010. Circular first-and second-order balanced repeated measurements designs. Commun. Stat.-Theory Methods 39, 228–240.
Jabeen, R., Rasheed, H.M.K., Ahmed, R., Shehzad, F., 2019. Construction of circular strongly partially-balanced repeated measurements designs. J. King Saud Univ.-Sci. 31, 345–351.
James, A.T., Wilkinson, G.N., 1971. Factorization of the residual operator and canonical decomposition of non-orthogonal factors in the analysis of variance. Biometrika 58, 258–294.
Pearce, S.C., Calinski, T., de C. Marshall, T.F., 1974. The basic contrasts of an experimental design with special reference to the analysis of data. Biometrika 61, 449–460.
Magda, C.G., 1980. Circular balanced repeated measurements designs. Commun. Stat.-Theory Methods 9, 1901–1918.
Nazeer, Y., Ahmed, R., Jabeen, R., Daniyal, M., Tahir, M.H., 2018. Circular strongly partially-balanced repeated measurements designs in periods of two different sizes. J. Prob. Stat. Sci. (in press)
Rasheed, U., Rasheed, H.M.K., Rasheed, M., Ahmed, R., 2018. Minimal circular strongly balanced repeated measurements designs in periods of three different sizes. Communications in Statistics-Theory and Methods 47, 4088–4094.
Williams, E.R., 1949. Experimental designs balanced for the estimation of residual effects of treatments. Austr. J. Sci. Res. A2, 149–168.
Williams, E.R., 1950. Experimental designs balanced for pairs of residual effects. Austr. J. Sci. Res. A3, 351–363.