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The complex pulsating $(a_1, a_2, \ldots, a_m, c)$-Fibonacci sequence

Prathomjit Khachorncharoenkul [a], Kiattiyot Phibul [a], Kittipong Laipaporn [a,*]

[a] School of Science, Walailak University, Nakhon Si Thammarat 80160, Thailand

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ABSTRACT

Several researchers have looked at pulsating Fibonacci sequences in the last ten years, which are generalizations of the Fibonacci sequence. They verify the closed form of these sequences via mathematical induction. This approach is beautiful, but it can only be utilized when patterns of the closed forms are predicted. In this paper, we introduce the complex pulsating $(a_1, a_2, \ldots, a_m, c)$-Fibonacci sequence and apply matrix theory, particularly eigenvalues, eigenvectors, and block matrices, as well as basic properties of the floor function to bridge the gap and obtain the closed form of the complex pulsating. Moreover, the golden ratios of this sequence are provided.

1. Introduction and Literature Review

The Fibonacci sequence $\{F_n\}$ is defined as $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$, and Binet's formulas for Fibonacci numbers are given by $F_n = \frac{\Phi^n - \phi^n}{\Phi - \phi} = \frac{\Phi^n - \phi^n}{\sqrt{5}}$, where $\Phi = \frac{1+\sqrt{5}}{2} \approx 1.6180339887\ldots$, namely, the golden ratio, and $\phi = 1 - \Phi$. This sequence has been extended in various versions and has also been studied by many mathematicians. For example, Miles (1960) defined the k-generalized Fibonacci numbers $f_{j,k}$ by

$$f_{j,k} = 0, \quad 0 \leqslant j \leqslant k-2, \; f_{k-1,k} = 1, \; f_{jk} = \sum_{n=1}^{k} f_{j-n,k}, \quad j \geqslant k.$$

Horadam (1961) defined the generalized Fibonacci sequence by

$$H_n = H_{n-1} + H_{n-2}, \; n \geqslant 3, \quad \textit{with} \;\; H_1 = p, \; H_2 = p + q$$

where p and q are arbitrary integers. Kalman and Mena (2003) generalized the Fibonacci sequence by

$$F_n = aF_{n-1} + bF_{n-2}, \; n \geqslant 2, \quad \text{with} \;\; F_0 = 0, \; F_1 = 1.$$

Gupta et al. (2012) defined the generalized Fibonacci sequence by

$$F_k = pF_{k-1} + qF_{k-2}, \; k \geqslant 2, \quad \text{with} \;\; F_0 = a, \; F_1 = b$$

where p, q, a and b are positive integers. Wani et al. (2017) defined the generalized k-Fibonacci sequence $\{S_{k,n}\}$ by

$$S_n = kS_{k,n-1} + S_{k,n-2}, \; n \geqslant 2, \textit{with} \;\; S_{k,0} = q, \; S_{k,1} = qk$$

where q and k are positive integers. Javaheri and Krylov (2020) generalized the Fibonacci sequence by

$$F_{n+1} = PF_n - QF_{n-1}, \quad \text{with} \;\; F_0 = 0, \; F_1 = 1$$

where P and Q are nonzero integers. However, Atanassov et al. (1985, 2001) defined four 2-Fibonacci sequences as follows:

$$\alpha_0 = a, \beta_0 = b, \alpha_1 = c, \beta_1 = d,$$

* Corresponding author.
E-mail address: lkittipo@wu.ac.th (K. Laipaporn).
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$$\alpha_{n+2} = \beta_{n+1} + \beta_n,\quad \beta_{n+2} = \alpha_{n+1} + \alpha_n \tag{1.1}$$

$$\alpha_{n+2} = \alpha_{n+1} + \beta_n,\quad \beta_{n+2} = \beta_{n+1} + \alpha_n \tag{1.2}$$

$$\alpha_{n+2} = \beta_{n+1} + \alpha_n,\quad \beta_{n+2} = \alpha_{n+1} + \beta_n \tag{1.3}$$

$$\alpha_{n+2} = \alpha_{n+1} + \alpha_n,\quad \beta_{n+2} = \beta_{n+1} + \beta_n \tag{1.4}$$

for every nonnegative integer n. We observe that most of them are related to the form of single-line sequences except for the work of K. Atanassov is remarkable. So, it is the inspiration for us to study and ascertain the behavior of this particular sequence.

Atanassov (2013) extended his sequence (1.2) by establishing the $(\boldsymbol{a}, \boldsymbol{b})$**-pulsated Fibonacci sequence**, which is defined by

$$\begin{aligned} &\alpha_0 = a,\ \beta_0 = b,\\ &\alpha_{2k+1} = \beta_{2k+1} = \alpha_{2k} + \beta_{2k},\\ &\alpha_{2k+2} = \alpha_{2k+1} + \beta_{2k},\\ &\beta_{2k+2} = \beta_{2k+1} + \alpha_{2k} \end{aligned} \tag{1.5}$$

where $a, b \in \mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$. One year later, he also proposed a generalized version of (1.5), so called the $(\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_m})$**-pulsated Fibonacci sequence**, which was published in Atanassov (2014) and is defined as follows:

$$\begin{aligned} &\alpha_{1,0} = a_1,\ \alpha_{2,0} = a_2,\ \ldots,\ \alpha_{m,0} = a_m,\\ &\alpha_{1,2k+1} = \alpha_{2,2k+1} = \ldots = \alpha_{m,2k+1} = \sum_{i=1}^{m} \alpha_{i,2k},\\ &\alpha_{j,2k+2} = \alpha_{j,2k+1} + \alpha_{m-j+1,2k} \end{aligned} \tag{1.6}$$

for any nonnegative integers j, k, and m such that $a_1, a_2, \ldots, a_m \in \mathbb{R}$ and $1 \leqslant j \leqslant m$. Moreover, the latest version of pulsating Fibonacci sequence was reported by Halici and Karatas (2019). They defined a new sequence, **complex pulsating Fibonacci sequence**, as follows:

$$\begin{aligned} &P_0 = a + ci,\ Q_0 = b + ci,\\ &\mathrm{Re}(P_{n+1}) = \mathrm{Im}(P_n),\ \mathrm{Re}(Q_{n+1}) = \mathrm{Im}(Q_n),\\ &\mathrm{Im}(P_{2n+2}) = \mathrm{Im}(Q_{2n+2}) = \mathrm{Im}(P_{2n+1} + Q_{2n+1}),\\ &\mathrm{Im}(P_{2n+1}) = \mathrm{Im}(P_{2n}) + \mathrm{Re}(Q_{2n}),\\ &\mathrm{Im}(Q_{2n+1}) = \mathrm{Im}(Q_{2n}) + \mathrm{Re}(P_{2n}) \end{aligned} \tag{1.7}$$

where a, b and c are real numbers and $n \in \mathbb{N} \cup \{0\}$.

We note that it was sheer coincidence that the sequences in (1.1)-(1.7) were confirmed in their closed form by using mathematical induction.

In this paper, we use matrix theory, especially eigenvalues, eigenvectors and block matrices, and basic properties of the floor function to find the closed form of the Fibonacci sequence that merges (1.6) and (1.7), named the **complex pulsating** $(\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_m}, \boldsymbol{c})$**-Fibonacci sequence**. This sequence is defined as follows: Let $a_1, a_2, \ldots, a_m$ and c be real numbers. Then,

$$\begin{aligned} &P_{1,0} = a_1 + ci,\ P_{2,0} = a_2 + ci,\ \ldots,\ P_{m,0} = a_m + ci,\\ &\mathrm{Re}(P_{j,k+1}) = \mathrm{Im}(P_{j,k}),\\ &\mathrm{Im}(P_{1,2k+2}) = \mathrm{Im}(P_{2,2k+2}) = \cdots = \mathrm{Im}(P_{m,2k+2}) = \mathrm{Im}\left(\sum_{i=1}^{m} P_{i,2k+1}\right)\\ &\mathrm{Im}(P_{j,2k+1}) = \mathrm{Im}(P_{j,2k}) + \mathrm{Re}(P_{m-j+1,2k}), \end{aligned} \tag{1.8}$$

for all nonnegative integers j, k, and m such that $1 \leqslant j \leqslant m$. To make it easier to visualize the complex pulsating $(a_1, a_2, \ldots, a_m, c)$-Fibonacci sequence, we show in case m is equal to 2, that is equivalent to (1.7), as Fig. 1.

Outline of the paper. In Section 2, we give some results which affect the main obtained result. Section 3 is devoted to our main results. The complex pulsating $(a_1, a_2, \ldots, a_m, c)$-Fibonacci sequence is given in the closed form. Besides, the golden ratios of this sequence are investigated. Finally, in Section 4, we summarize and discuss our results.

2. Preliminaries

Throughout this paper, let I_m be an m-by-m identity matrix and J_m be an m-by-m matrix in which every entry is one. Let K_m be an m-by-m reversal matrix that is a permutation matrix in which $k_{i,m-i+1} = 1$ for $i = 1, 2, \ldots, m$ and all other entries are zero.

In this section, to simplify the process of finding the closed form in Theorem 3.2, we construct the following lemmas.

Lemma 2.1. *For any integer $m \geqslant 2$, the eigenvalues of matrix $U = J_m + K_m$ are $m+1$ of multiplicity 1, -1 of multiplicity $\lfloor \frac{m}{2} \rfloor$ and 1 of multiplicity $\lfloor \frac{m-1}{2} \rfloor$. Moreover, the eigenvectors of matrix U are*

1. $[1]_{m\times 1}$ *with eigenvalue* $\lambda = m+1$,
2. $[v^h]_{m\times 1}$ *with eigenvalue* $\lambda = -1$ *for* $h = 1, 2, \ldots, \lfloor \frac{m}{2} \rfloor$, *where*

$$v^h_{i1} = \begin{cases} 1 & ;\ i = h \\ -1 & ;\ i = m-h+1 \\ 0 & ;\ otherwise, \end{cases}$$

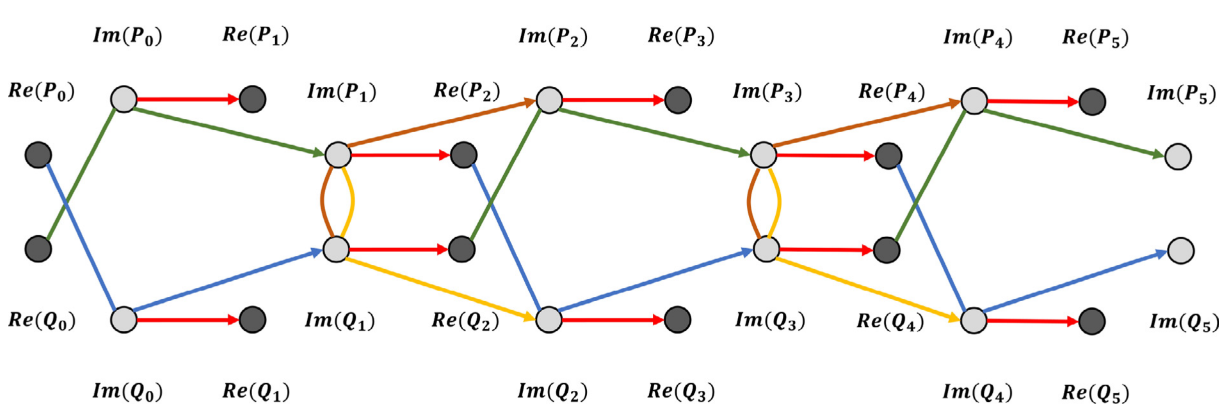

Fig. 1. The complex pulsating (a_1, a_2, c)-Fibonacci sequence.

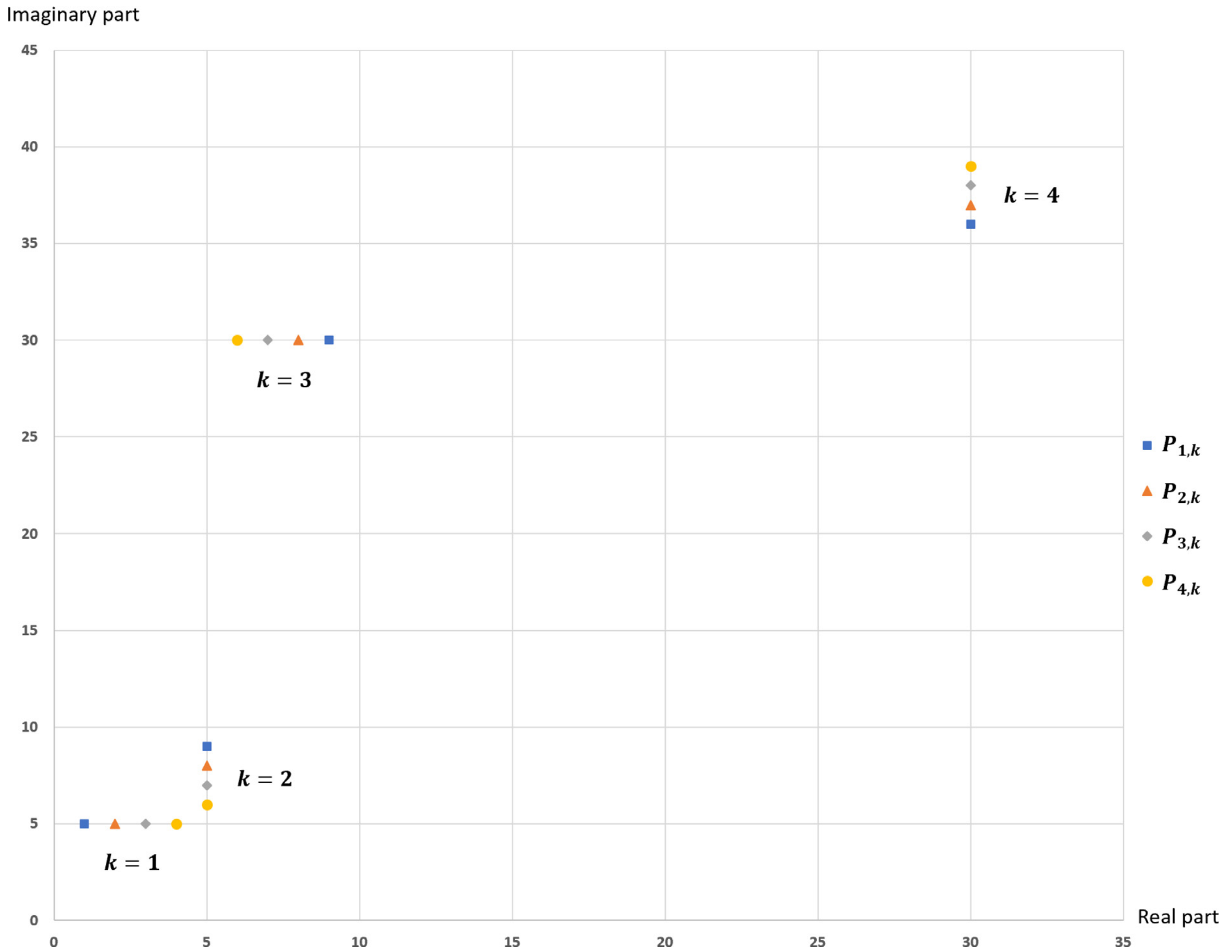

Fig. 2. The complex pulsating $(1, 2, 3, 4, 5)$-Fibonacci sequence.

Table 1
The complex pulsating $(a_1, a_2, \ldots, a_m, c)$-Fibonacci sequence in case $m = 4, a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4$ and $c = 5$.

k	$P_{1,k}$		$P_{2,k}$		$P_{3,k}$		$P_{4,k}$	
	$\mathrm{Re}(P_{1,k})$	$\mathrm{Im}(P_{1,k})$	$\mathrm{Re}(P_{2,k})$	$\mathrm{Im}(P_{2,k})$	$\mathrm{Re}(P_{3,k})$	$\mathrm{Im}(P_{3,k})$	$\mathrm{Re}(P_{4,k})$	$\mathrm{Im}(P_{4,k})$
1	1	5	2	5	3	5	4	5
2	5	9	5	8	5	7	5	6
3	9	30	8	30	7	30	6	30
4	30	36	30	37	30	38	30	39
5	36	150	37	150	38	150	39	150
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

3. $[w^h]_{m\times 1}$ *with eigenvalue* $\lambda = 1$ *for* $h = 1, 2, \ldots, \lfloor\frac{m-1}{2}\rfloor$, *where if m is even, then*

$$w_{i1}^h = \begin{cases} 1 & ; \; i = h \;\; and \;\; i = m - h + 1 \\ -1 & ; \; i = \frac{m}{2} \;\; and \;\; i = \frac{m}{2} + 1 \\ 0 & ; \; otherwise, \end{cases}$$

and if m is odd, then

$$w_{i1}^h = \begin{cases} 1 & ; \; i = h \;\; and \;\; i = m - h + 1 \\ -2 & ; \; i = \frac{m+1}{2} \\ 0 & ; \; otherwise. \end{cases}$$

Proof. Obviously, by the properties of the floor function, we have

$$1 + \left\lfloor\frac{m}{2}\right\rfloor + \left\lfloor\frac{m-1}{2}\right\rfloor = m.$$

It is easily observed that

$U[1]_{m\times 1} = [(m-1) + 2] = (m+1)[1]_{m\times 1}$;

thus, $m + 1$ is the eigenvalue of the matrix U with the corresponding eigenvector $[1]_{m\times 1}$.

Next, to show that $U[v^h] = -[v^h]$ for all $h \in \{1, 2, \ldots, \lfloor\frac{m}{2}\rfloor\}$, let $U = [u_{ij}]$ and z_i^h be the ith column of $U[v^h]$. Then,

$$\begin{aligned} z_i^h &= \sum_{j=1}^{m} u_{ij} v_{j1}^h = u_{ih} - u_{i,m-h+1} = \begin{cases} 2 - u_{m-h+1,m-h+1} \ ; \ i = m-h+1 \\ 1 - u_{i,m-h+1} \ ; \ i \neq m-h+1 \end{cases} \\ &= \begin{cases} 2-1 \ ; \ i = m-h+1 \\ 1-2 \ ; \ i = h \\ 1-1 \ ; \ \text{otherwise} \end{cases} = \begin{cases} -1 \ ; \ i = h \\ 1 \quad ; \ i = m-h+1 \\ 0 \quad ; \ \text{otherwise} \end{cases} = -v_{i1}^h. \end{aligned}$$

Finally, we show that $U[w^h] = [w^h]$ for all $h \in \{1, 2, \dots, \lfloor \frac{m-1}{2} \rfloor\}$. Let $U = [u_{ij}]$ and z_i^h be the ith column of $U[w^h]$. If m is even, then

$$\begin{aligned} z_i^h &= \sum_{j=1}^{m} u_{ij} w_{j1}^h = u_{ih} + u_{i,m-h+1} - u_{i,\frac{m}{2}} - u_{i,\frac{m}{2}+1} \\ &= \begin{cases} 2 + u_{m-h+1,m-h+1} - u_{m-h+1,\frac{m}{2}} - u_{m-h+1,\frac{m}{2}+1} \ ; i = m-h+1 \\ 1 + u_{i,m-h+1} - u_{i,\frac{m}{2}} - u_{i,\frac{m}{2}+1} \quad ; i \neq m-h+1 \end{cases} \\ &= \begin{cases} 2+1-1-1 \ ; i = m-h+1 \\ 1+2-1-1 \ ; i = h \\ 1+1-1-2 \ ; i = \frac{m}{2} \\ 1+1-2-1 \ ; i = \frac{m}{2}+1 \\ 1+1-1-1 \ ; \text{otherwise} \end{cases} = \begin{cases} 1 \ ; i = h \text{ or } i = m-h+1 \\ -1 \ ; i = \frac{m}{2} \text{ or } i = \frac{m}{2}+1 \\ 0 \ ; \text{otherwise} \end{cases} = w_{i1}^h. \end{aligned}$$

If m is odd, then

$$\begin{aligned} z_i^h &= \sum_{j=1}^{m} u_{ij} w_{j1}^h = u_{ih} + u_{i,m-h+1} - 2u_{i,\frac{m+1}{2}} \\ &= \begin{cases} 2 + u_{m-h+1,m-h+1} - 2u_{m-h+1,\frac{m+1}{2}} \ ; i = m-h+1 \\ 1 + u_{i,m-h+1} - 2u_{i,\frac{m+1}{2}} \quad ; i \neq m-h+1 \end{cases} \\ &= \begin{cases} 2+1-2(1) \ ; i = m-h+1 \\ 1+2-2(1) \ ; i = h \\ 1+1-2(2) \ ; i = \frac{m+1}{2} \\ 1+1-2(1) \ ; \text{otherwise} \end{cases} = \begin{cases} 1 \ ; i = h \text{ or } i = m-h+1 \\ -2 \ ; i = \frac{m+1}{2} \\ 0 \ ; \text{otherwise} \end{cases} = w_{i1}^h. \end{aligned}$$

Hence, we have the desired result.

Lemma 2.2. *Let U be the matrix which defined in Lemma 2.1 and $n \in \mathbb{N}$. Then,*

$$U^n = xJ_m + I_m$$

when n is even, and

$$U^n = xJ_m + K_m$$

when n is odd, where $x = \frac{(m+1)^n - 1}{m}$.

Proof. By Lemma 2.1, we know that $U^n = PD^nP^{-1}$, where P is an m-by-m matrix such that each column vector is an eigenvector of U associated with the eigenvalues $m+1, -1$ and 1 and $D = \text{diag}(m+1, \underbrace{-1, \dots, -1}_{\lfloor \frac{m}{2} \rfloor}, \underbrace{1, \dots, 1}_{\lfloor \frac{m-1}{2} \rfloor})$. To understand the following process more easily, we separate the proof into two cases.

Case 1 If m is odd, then it is easily found that $\lfloor \frac{m-1}{2} \rfloor = r = \lfloor \frac{m}{2} \rfloor$, where $m = 2r+1$. A useful expression for the correspondingly partitioned presentation of m-by-m matrices P, D^n and P^{-1} is

$$P = \begin{bmatrix} [1]_{r\times 1} & I_r & I_r \\ 1 & 0_{1\times r} & [-2]_{1\times r} \\ [1]_{r\times 1} & -K_r & K_r \end{bmatrix}, \quad D^n = \begin{bmatrix} (m+1)^n & 0_{1\times r} & 0_{1\times r} \\ 0_{r\times 1} & (-1)^n I_r & 0_r \\ 0_{r\times 1} & 0_r & I_r \end{bmatrix}$$

and

$$P^{-1} = \begin{bmatrix} [\frac{1}{m}]_{1\times r} & \frac{1}{m} & [\frac{1}{m}]_{1\times r} \\ \frac{1}{2} I_r & 0_{r\times 1} & -\frac{1}{2} K_r \\ \frac{1}{2} I_r - \frac{1}{m} J_r & [-\frac{1}{m}]_{r\times 1} & \frac{1}{2} K_r - \frac{1}{m} J_r \end{bmatrix}.$$

We know that $JK = J = KJ$ and $K^2 = I$. Let $x = \frac{(m+1)^n - 1}{m}$. Then, by carrying out a partitioned multiplication and then simplifying, we write U^n as follows:

$$U^n = \begin{bmatrix} U_1 & U_2 & U_3 \\ U_4 & U_5 & U_4 \\ U_3 & U_2 & U_1 \end{bmatrix}$$

where

$$\begin{aligned} U_1 &= [\tfrac{1}{m}(m+1)^n]_r + \tfrac{(-1)^n}{2} I_r + \tfrac{1}{2} I_r - \tfrac{1}{m} J_r = xJ_r + \left(\tfrac{(-1)^n + 1}{2}\right) I_r, \\ U_2 &= [\tfrac{1}{m}(m+1)^n]_{r\times 1} + [-\tfrac{1}{m}]_{r\times 1} = [x]_{r\times 1}, \\ U_3 &= [\tfrac{1}{m}(m+1)^n]_r + \tfrac{(-1)^{n+1}}{2} K_r + \tfrac{1}{2} K_r - \tfrac{1}{m} J_r = xJ_r + \left(\tfrac{(-1)^{n+1} + 1}{2}\right) K_r, \\ U_4 &= [\tfrac{1}{m}(m+1)^n]_{1\times r} + [-1]_{1\times r} + [\tfrac{2r}{m}]_{1\times r} = \left[\tfrac{(m+1)^n + 2r}{m} - 1\right]_{1\times r} = [x]_{1\times r}, \\ U_5 &= \tfrac{1}{m}(m+1)^n + \tfrac{2r}{m} = x + 1. \end{aligned}$$

Then, we can rewrite

$$U^n = xJ_m + \left(\frac{(-1)^n + 1}{2}\right) I_m + \left(\frac{(-1)^{n+1} + 1}{2}\right) K_m$$

to conclude that if n is even, then $U^n = xJ_m + I_m$, and if n is odd, then $U^n = xJ_m + K_m$.

Case 2 If m is even, then it is easy to see that $\lfloor \frac{m-1}{2} \rfloor = r$ and $\lfloor \frac{m}{2} \rfloor = r+1$, where $m = 2(r+1)$. By the same argument as in Case 1, we obtain the matrix U^n, where

$$P = \begin{bmatrix} [1]_{r\times 1} & I_r & 0_{r\times 1} & I_r \\ [1]_{2\times 1} & 0_{2\times r} & \begin{bmatrix} 1 \\ -1 \end{bmatrix} & [-1]_{2\times r} \\ [1]_{r\times 1} & -K_r & 0_{r\times 1} & K_r \end{bmatrix},$$

$$D^n = \begin{bmatrix} (m+1)^n & 0_{1\times r} & 0 & 0_{1\times r} \\ 0_{r\times 1} & (-1)^n I_r & 0_{r\times 1} & 0_r \\ 0 & 0_{1\times r} & (-1)^n & 0_{1\times r} \\ 0_{r\times 1} & 0_r & 0_{r\times 1} & I_r \end{bmatrix}$$

and

$$P^{-1} = \begin{bmatrix} [\frac{1}{m}]_{1\times r} & [\frac{1}{m}]_{1\times 2} & [\frac{1}{m}]_{1\times r} \\ \frac{1}{2} I_r & 0_{r\times 2} & -\frac{1}{2} K_r \\ 0_{1\times r} & [\frac{1}{2} \ \ -\frac{1}{2}] & 0_{1\times r} \\ \frac{1}{2} I_r - \frac{1}{m} J_r & [-\frac{1}{m}]_{r\times 2} & \frac{1}{2} K_r - \frac{1}{m} J_r \end{bmatrix}.$$

we can rewrite

$$U^n = xJ_m + \left(\frac{(-1)^n + 1}{2}\right) I_m + \left(\frac{(-1)^{n+1} + 1}{2}\right) K_m$$

and hence, if n is even, then $U^n = xJ_m + I_m$, and if n is odd, then $U^n = xJ_m + K_m$, where $x = \frac{(m+1)^n - 1}{m}$, as desired.

3. Main results

Two portions are divided in this section. The closed form of the complex pulsating $(a_1, a_2, \dots, a_m, c)$-Fibonacci sequence, which is the focus of the study, is demonstrated in the first section. The golden ratios of this sequence are discussed in the second half.

3.1. The closed form of the complex pulsating $(a_1, a_2, \ldots, a_m, c)$-Fibonacci sequence

In this section, matrix theory is an essential proof technique. We will reveal the reasoning behind this choice in Section 4. However, to visualize it more clearly, we would like to provide an example of the following sequence.

Example 3.1. In the circumstances $m = 4, a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4$, and $c = 5$, we shall demonstrate an example of the complex pulsating $(a_1, a_2, \ldots, a_m, c)$-Fibonacci sequence. Table 1 and Fig. 2 provide more information.

Theorem 3.2. The closed form of the complex pulsating $(a_1, a_2, \ldots, a_m, c)$-Fibonacci sequence is

$$\mathrm{Im}(P_{1,2k+2}) = \cdots = \mathrm{Im}(P_{m,2k+2}) = (m+1)^k \sum_{i=1}^{m} a_i + m(m+1)^k c,$$

$$\mathrm{Im}(P_{j,2k+1}) = \begin{cases} \frac{(m+1)^k - 1}{m} \sum_{i=1}^{m} a_i + a_j + (m+1)^k c \ ; k \text{ is odd} \\ \frac{(m+1)^k - 1}{m} \sum_{i=1}^{m} a_i + a_{m-j+1} + (m+1)^k c \ ; k \text{ is even} \end{cases}$$

for all $k \in \mathbb{N} \cup \{0\}$.

Proof. Define a linear transformation $T : \mathbb{R}^{4m} \to \mathbb{R}^{4m}$ by

$$\begin{aligned} &T(\delta_1, \ldots, \delta_m, \kappa_1, \ldots, \kappa_m, \zeta_1, \ldots, \zeta_m, \mu_1, \ldots, \mu_m) \\ &= (\kappa_1 + \delta_m, \kappa_2 + \delta_{m-1}, \ldots, \kappa_m + \delta_1, \underbrace{p, \ldots, p}_{m}, \underbrace{q, \ldots, q}_{m}, \\ &\mu_m + q, \mu_{m-1} + q, \ldots, \mu_1 + q), \end{aligned} \tag{3.1}$$

where $p = \sum_{k=1}^{m} \delta_k + \kappa_k$ and $q = \sum_{k=1}^{m} \mu_k$. Clearly, the matrix representation of T with respect to the standard basis is

$$Q = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \text{where } A = \begin{bmatrix} K_m & I_m \\ J_m & J_m \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & J_m \\ 0 & U \end{bmatrix}.$$

Note that U is the matrix in Lemma 2.1. Then, the matrix A can be rewritten in the following form:

$$A = \begin{bmatrix} I_m & 0_m \\ -K_m & I_m \end{bmatrix} \begin{bmatrix} 0_m & I_m \\ 0_m & U \end{bmatrix} \begin{bmatrix} I_m & 0_m \\ K_m & I_m \end{bmatrix}.$$

We are now ready to find the closed form. For each $k \in \mathbb{N} \cup \{0\}$, we let $X_k = \begin{bmatrix} X_k' \\ X_k'' \end{bmatrix}$, where $X_k' = [\mathrm{Re}(P_{1,k}) \ \cdots \ \mathrm{Re}(P_{m,k})]^t$ and $X_k'' = [\mathrm{Im}(P_{1,k}) \ \cdots \ \mathrm{Im}(P_{m,k})]^t$. Moreover, we observe that $Q\begin{bmatrix} X_{2k} \\ X_{2k+1} \end{bmatrix} = \begin{bmatrix} X_{2k+2} \\ X_{2k+3} \end{bmatrix}$. By computing directly, we have

$$\begin{bmatrix} X_{2k} \\ X_{2k+1} \end{bmatrix} = Q^k \begin{bmatrix} X_0 \\ X_1 \end{bmatrix},$$

where $X_0 = [a_1 \ \cdots \ a_m \ c \ \cdots \ c]^t_{1\times 2m}$, $X_1 = [c \ \cdots \ c \ c + a_m \ c + a_{m-1} \cdots c + a_1]^t_{1\times 2m}$ and $k \in \mathbb{N}$. Furthermore, by block matrix multiplication, we obtain

$$Q^k = \begin{bmatrix} A^k & 0 \\ 0 & B^k \end{bmatrix},$$

where

$$A^k = \begin{bmatrix} I_m & 0_m \\ -K_m & I_m \end{bmatrix} \begin{bmatrix} 0_m & U^{k-1} \\ 0_m & U^k \end{bmatrix} \begin{bmatrix} I_m & 0_m \\ K_m & I_m \end{bmatrix} = \begin{bmatrix} U^{k-1} K_m & U^{k-1} \\ \left(-K_m U^{k-1} + U^k\right) K_m & -K_m U^{k-1} + U^k \end{bmatrix}$$

and

$$B^k = \begin{bmatrix} 0 & J_m U^{k-1} \\ 0 & U^k \end{bmatrix}.$$

As a result,

$$\begin{aligned} \begin{bmatrix} X_{2k} \\ X_{2k+1} \end{bmatrix} &= Q^k \begin{bmatrix} X_0 \\ X_1 \end{bmatrix} \\ &= \begin{bmatrix} U^{k-1} K_m & U^{k-1} & 0_m & 0_m \\ \left(-K_m U^{k-1} + U^k\right) K_m & -K_m U^{k-1} + U^k & 0_m & 0_m \\ 0_m & 0_m & 0_m & J_m U^{k-1} \\ 0_m & 0_m & 0_m & U^k \end{bmatrix} \begin{bmatrix} X_0' \\ X_0'' \\ X_1' \\ X_1'' \end{bmatrix} \\ &= \begin{bmatrix} U^{k-1} K_m X_0' + U^{k-1} X_0'' \\ \left(-K_m U^{k-1} + U^k\right) K_m X_0' + \left(-K_m U^{k-1} + U^k\right) X_0'' \\ J_m U^{k-1} X_1'' \\ U^k X_1'' \end{bmatrix}. \end{aligned}$$

We know that $J_m^2 = mJ_m$. Next, we let $x = \frac{(m+1)^k - 1}{m}$ and $x' = \frac{(m+1)^{k-1} - 1}{m}$, and consider the terms of matrix multiplication in two cases:

Case 1 If k is odd, we obtain that $U^k = xJ_m + K_m$ and $U^{k-1} = x'J_m + I_m$. Thus,

1)

$$\begin{aligned} &U^{k-1} K_m X_0' + U^{k-1} X_0'' = (x'J_m + K_m) X_0' + (x'J_m + I_m) X_0'' \\ &= \begin{bmatrix} x' \sum_{i=1}^{m} a_i + a_m \\ \vdots \\ x' \sum_{i=1}^{m} a_i + a_1 \end{bmatrix} + \begin{bmatrix} mx'c + c \\ \vdots \\ mx'c + c \end{bmatrix} \\ &= \begin{bmatrix} x' \sum_{i=1}^{m} a_i + a_m + (m+1)^{k-1} c \\ \vdots \\ x' \sum_{i=1}^{m} a_i + a_1 + (m+1)^{k-1} c \end{bmatrix}_{m\times 1}, \end{aligned}$$

2)

$$\begin{aligned} &\left(-K_m U^{k-1} + U^k\right) K_m X_0' + \left(-K_m U^{k-1} + U^k\right) X_0'' \\ &= (-K_m(x'J_m + I_m) K_m + (xJ_m + K_m) K_m) X_0' + (-K_m(x'J_m + I_m) \\ &\quad + (xJ_m + K_m)) X_0'' \\ &= (-x'J_m - I_m + xJ_m + I_m) X_0' + (-x'J_m - K_m + xJ_m + K_m) X_0'' \\ &= (-x' + x) J_m X_0' + (-x' + x) J_m X_0'' = (m+1)^{k-1} J_m (X_0' + X_0'') \\ &= \left[(m+1)^{k-1} \sum_{i=1}^{m} a_i + m(m+1)^{k-1} c \right]_{m\times 1}, \end{aligned}$$

3)

$$J_m U^{k-1} X_1' \prime = J_m (x' J_m + I_m) X_1'' = (mx' J_m + J_m) X_1'' = (mx'+1) J_m X_1'' = (m+1)^{k-1} J_m X_1'' = \left[(m+1)^{k-1} \sum_{i=1}^{m} a_i + m(m+1)^{k-1} c \right]_{m\times 1},$$

4)

$$U^k X_1'' = (x J_m + K_m) X_1'' = \begin{bmatrix} x\sum_{i=1}^{m} a_i + a_1 + (m+1)^k c \\ \vdots \\ x\sum_{i=1}^{m} a_i + a_m + (m+1)^k c \end{bmatrix}_{m\times 1}.$$

We conclude that $\begin{bmatrix} X_{2k} \\ X_{2k+1} \end{bmatrix} = Q^k \begin{bmatrix} X_0 \\ X_1 \end{bmatrix} = [p_j]_{4m\times 1}$, where p_j is in the pattern as follows:

$$p_j = \begin{cases} \frac{(m+1)^{k-1}-1}{m} \sum_{i=1}^{m} a_i + a_{m-j+1} + (m+1)^{k-1} c \ ; \ 1 \leqslant j \leqslant m \\ (m+1)^{k-1} \sum_{i=1}^{m} a_i + m(m+1)^{k-1} c \ ; \ m+1 \leqslant j \leqslant 3m \\ \frac{(m+1)^{k}-1}{m} \sum_{i=1}^{m} a_i + a_{j-3m} + (m+1)^{k} c \ ; \ 3m+1 \leqslant j \leqslant 4m. \end{cases}$$

Case 2 If k is even, we obtain that $U^k = xJ_m + I_m$ and $U^{k-1} = x'J_m + K_m$. In the same manner, we have both $\left(-K_m U^{k-1} + U^k\right) K_m X_0' + \left(-K_m U^{k-1} + U^k\right) X_0''$ and $J_m U^{k-1} X_1''$ are equal to $\left[(m+1)^{k-1} \sum_{i=1}^{m} a_i + m(m+1)^{k-1} c \right]_{m\times 1}$. However,

$$U^{k-1} K_m X_0' + U^{k-1} X_0'' = \begin{bmatrix} x' \sum_{i=1}^{m} a_i + a_1 + (m+1)^{k-1} c \\ \vdots \\ x' \sum_{i=1}^{m} a_i + a_m + (m+1)^{k-1} c \end{bmatrix}_{m\times 1}$$

and

$$U^k X_1'' = \begin{bmatrix} x\sum_{i=1}^{m} a_i + a_m + (m+1)^k c \\ \vdots \\ \sum_{i=1}^{m} a_i + a_1 + (m+1)^k c \end{bmatrix}_{m\times 1}.$$

As a result,

$$p_j = \begin{cases} \frac{(m+1)^{k-1}-1}{m} \sum_{i=1}^{m} a_i + a_j + (m+1)^{k-1} c \ ; \ 1 \leqslant j \leqslant m \\ (m+1)^{k-1} \sum_{i=1}^{m} a_i + m(m+1)^{k-1} c \ ; \ m+1 \leqslant j \leqslant 3m \\ \frac{(m+1)^{k}-1}{m} \sum_{i=1}^{m} a_i + a_{4m-j+1} + (m+1)^{k} c \ ; \ 3m+1 \leqslant j \leqslant 4m. \end{cases}$$

3.2. The golden ratio of the complex pulsating $(a_1, a_2, \ldots, a_m, c)$-Fibonacci sequence

In mathematics, the Fibonacci sequence and the golden ratio are closely related. This ratio is the limit of the ratios of successive terms of the Fibonacci sequence. In the previous section, we discover the closed form of the complex pulsating $(a_1, a_2, \ldots, a_m, c)$-Fibonacci sequence, which is a generalization of the Fibonacci sequence. As a consequence, it is no surprise that we will look at the golden ratio of this sequence in this section. The consequence of our closed form in Section 3.1 is the ratios

$$\lim_{k\to\infty} \frac{\mathrm{Re}(P_{j,2k+2}) + i\mathrm{Im}(P_{j,2k+2})}{\mathrm{Re}(P_{j,2k+1}) + i\mathrm{Im}(P_{j,2k+1})}$$

and

$$\lim_{k\to\infty} \frac{\mathrm{Re}(P_{j,2k+1}) + i\mathrm{Im}(P_{j,2k+1})}{\mathrm{Re}(P_{j,2k}) + i\mathrm{Im}(P_{j,2k})}$$

which seem to be the well-known golden ratio. For further results, look at the following proposition.

Proposition 3.3. For each $j = 1, 2, \ldots, m$, let $(P_{j,n})$ be a complex pulsating $(a_1, a_2, \ldots, a_m, c)$-Fibonacci sequence as the sequence (1.8). Then, the golden ratios of this sequence are

- $\lim_{k\to\infty} \frac{\mathrm{Re}(P_{j,2k+2}) + i\mathrm{Im}(P_{j,2k+2})}{\mathrm{Re}(P_{j,2k+1}) + i\mathrm{Im}(P_{j,2k+1})} = \frac{(m+1)(m^2+2m)}{m^2+(m+1)^2} + i\frac{(m+1)(m^2-m-1)}{m^2+(m+1)^2}$,
- $\lim_{k\to\infty} \frac{\mathrm{Re}(P_{j,2k+1}) + i\mathrm{Im}(P_{j,2k+1})}{\mathrm{Re}(P_{j,2k}) + i\mathrm{Im}(P_{j,2k})} = \frac{m^2+2m}{m^2+1} - i\frac{m^2-m-1}{m^2+1}$.

Proof. First, for each $j = 1, 2, \ldots, m$, we consider the ratio by using Theorem 3.2

$$\lim_{k\to\infty} \frac{\mathrm{Im}(P_{j,2k+2})}{\mathrm{Im}(P_{j,2k+1})} \leqslant \max\left\{ \lim_{k\to\infty} \frac{(m+1)^k \sum_{i=1}^{m} a_i + m(m+1)^k c}{\frac{(m+1)^k-1}{m}\sum_{i=1}^{m} a_i + a_j + (m+1)^k c}, \lim_{k\to\infty} \frac{(m+1)^k \sum_{i=1}^{m} a_i + m(m+1)^k c}{\frac{(m+1)^k-1}{m}\sum_{i=1}^{m} a_i + a_{m-j+1} + (m+1)^k c} \right\} = \frac{\sum_{i=1}^{m} a_i + mc}{\frac{1}{m}\sum_{i=1}^{m} a_i + c} = m,$$

and in the same manner, we have

$$\lim_{k\to\infty} \frac{\mathrm{Im}(P_{j,2k+2})}{\mathrm{Im}(P_{j,2k+1})} \geqslant \min\left\{ \lim_{k\to\infty} \frac{(m+1)^k \sum_{i=1}^{m} a_i + m(m+1)^k c}{\frac{(m+1)^k-1}{m}\sum_{i=1}^{m} a_i + a_j + (m+1)^k c}, \lim_{k\to\infty} \frac{(m+1)^k \sum_{i=1}^{m} a_i + m(m+1)^k c}{\frac{(m+1)^k-1}{m}\sum_{i=1}^{m} a_i + a_{m-j+1} + (m+1)^k c} \right\} = m.$$

As a result, $\lim_{k\to\infty} \frac{\mathrm{Im}(P_{j,2k+2})}{\mathrm{Im}(P_{j,2k+1})} = m$. In addition, by the same argument, we obtain

$$\lim_{k\to\infty} \frac{\mathrm{Im}(P_{j,2k})}{\mathrm{Im}(P_{j,2k+1})} = \frac{m}{m+1}$$

for all $j = 1, 2, \ldots, m$. So, we have

$$\lim_{k\to\infty} \frac{\mathrm{Re}(P_{j,2k+2}) + i\mathrm{Im}(P_{j,2k+2})}{\mathrm{Re}(P_{j,2k+1}) + i\mathrm{Im}(P_{j,2k+1})} = \lim_{k\to\infty} \frac{\mathrm{Im}(P_{j,2k+1}) + i\mathrm{Im}(P_{j,2k+2})}{\mathrm{Im}(P_{j,2k}) + i\mathrm{Im}(P_{j,2k+1})} = \lim_{k\to\infty} \frac{1 + i\left(\frac{\mathrm{Im}(P_{j,2k+2})}{\mathrm{Im}(P_{j,2k+1})}\right)}{\frac{\mathrm{Im}(P_{j,2k})}{\mathrm{Im}(P_{j,2k+1})} + i} = \frac{1+im}{\frac{m}{m+1}+i} = \frac{(m+1)(m^2+2m)}{m^2+(m+1)^2} + i\frac{(m+1)(m^2-m-1)}{m^2+(m+1)^2}.$$

And,

$$\lim_{k\to\infty}\frac{\mathrm{Re}\left(P_{j,2k+1}\right)+i\mathrm{Im}\left(P_{j,2k+1}\right)}{\mathrm{Re}\left(P_{j,2k}\right)+i\mathrm{Im}\left(P_{j,2k}\right)}=\lim_{k\to\infty}\frac{\mathrm{Im}\left(P_{j,2k}\right)+i\mathrm{Im}\left(P_{j,2k+1}\right)}{\mathrm{Im}\left(P_{j,2k-1}\right)+i\mathrm{Im}\left(P_{j,2k}\right)}$$

$$=\lim_{k\to\infty}\frac{1+i\left(\frac{\mathrm{Im}\left(P_{j,2k+1}\right)}{\mathrm{Im}\left(P_{j,2k}\right)}\right)}{\frac{\mathrm{Im}\left(P_{j,2k-1}\right)}{\mathrm{Im}\left(P_{j,2k}\right)}+i}$$

$$=\frac{1+i\left(\frac{m+1}{m}\right)}{\frac{1}{m}+i}=\frac{m^2+2m}{m^2+1}-i\frac{m^2-m-1}{m^2+1}.$$

4. Conclusion and Discussion

Actually, the heart of this paper is to find the appropriate matrix Q to use in the proof of Theorem 3.2 but the matrix Q can take many different forms. The variety of Q depends on a linear map T which obeys the rule in (1.8). Here is one of the examples of Q that we ever used to solve (1.8). Let $X_n=[\mathrm{Re}(P_{1,n})\ \mathrm{Im}(P_{1,n})\cdots \mathrm{Re}(P_{m,n})\mathrm{Im}(P_{m,n})]^t$ for $n\in\mathbb{N}\cup\{0\}$ and $T:\mathbb{R}^{4m}\to\mathbb{R}^{4m}$ be defined by $T(x_1,x_2,\ldots,x_{4m})=(x_2+x_{2m-1},y,x_4+x_{2m-3},y,\ldots,x_{2m}+x_1,y,y,x_{4m}+y,y,x_{4m-2}+y,\ldots,y,x_{2m+2}+y)$ where $y=\sum_{k=1}^{m}x_{2(m+k)}$. Then the matrix Q is in the form $\begin{bmatrix}A & C\\ 0 & B\end{bmatrix}$ where

$$A_{2m\times 2m}=\left[a_{ij}\right]=\begin{cases}1\ ;\ (j\text{ is odd and }i=2m-j),\\ \quad\text{or }(j\text{ is even and }i=j-1),\\ 0\ ;\ \text{otherwise},\end{cases}$$

$$B_{2m\times 2m}=\left[b_{ij}\right]=\begin{cases}0\ ;\ j\text{ is odd},\\ 2\ ;\ j\leqslant m,\ j\text{ is even and }i=2m+j-2,\\ 0\ ;\ \text{otherwise},\end{cases}$$

$$\text{and } C_{2m\times 2m}=\left[c_{ij}\right]=\begin{cases}0\ ;\ j\text{ is odd},\\ 2\ ;\ j\geqslant m+1,\ j\text{ is even and }i=2m-j+2,\\ 0\ ;\ \text{otherwise}.\end{cases}$$

In the same manner, we know that $\begin{bmatrix}X_{2n}\\ X_{2n+1}\end{bmatrix}=Q^n\begin{bmatrix}X_0\\ X_1\end{bmatrix}$, where $X_0=[a_1\quad c\quad a_2\quad c\quad\cdots\quad a_m\quad c]^t_{1\times 2m}$, $X_1=[c\quad c+a_m\quad c\quad c+a_{m-1}\quad\cdots\quad c\quad c+a_1]^t_{1\times 2m}$. The matrix Q^n looks uneasily to compute its closed form.So, here is the reason that we choose the map T in (3.1) because its matrix representation is in the form $Q=\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}$ which is easily compute Q^n. Even though, the matrix $A=\begin{bmatrix}K_m & I_m\\ J_m & J_m\end{bmatrix}$ looks so complicated but it can be decomposed as follows: $A=\begin{bmatrix}I_m & 0_m\\ -K_m & I_m\end{bmatrix}\begin{bmatrix}0_m & I_m\\ 0_m & U\end{bmatrix}\begin{bmatrix}I_m & 0_m\\ K_m & I_m\end{bmatrix}$. Moreover, the matrix $\begin{bmatrix}I_m & 0_m\\ -K_m & I_m\end{bmatrix}$ is the inverse of $\begin{bmatrix}I_m & 0_m\\ K_m & I_m\end{bmatrix}$, which makes it easier to see A^n. Then, the computation of Q^n is simplified to only find the matrix U^n. So, we have to collect all of the eigenvalues and eigenvectors in Lemma 2.1 and complete our work in Lemma2.2 for finding the closed form of U^n. Until now, we can see that the construction of Q relies on the rearrangement in the entries of the matrix X_n and the formula of a map T. Both of these may lead us to vary directions for getting Q.

By the way, we believe that our tactics to create the matrix Q in Section 3.1 are not the best. So, we wish to see a friendly matrix Q that can be computed favorably.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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