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Original article

# A study of $\omega_1 - (\omega + n)$-projective QTAG-modules

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ABSTRACT

In this paper, we define a new class of $\omega_1 - (\omega + n)$-projective module and study their basic properties. We proved that a module $M$ is almost $\omega_1 - (\omega + n)$- projective if and only if there exists a countably generated nice submodule $K$ which satisfies the inequalities $H_{\omega+n}(M) \subseteq K \subseteq H_{\omega}(M)$ such that $M/K$ is almost $(\omega + n)$-projective.


## 1. Introduction

The concepts for groups such as projectivity, purity, height etc carried to modules preserving the sense. For getting those results of groups, which do not hold for modules; some constraints applied on the structure of module or underlying ring. After taking into effect these constraints many natural results of groups can be established for the QTAG-module structure, which are not true in general. The results of this paper are motivated from the results in (Danchev, 2014).

The study on the structure of QTAG-modules was started by Singh (1976). After that many researchers such as Khan, Mehdi, Abbasi etc. generalized different concepts of groups to QTAG-modules (Khan, 1979; Mehdi et al., 2006) etc. They introduced various notions and structures for QTAG-modules motivated from group structures and obtained some exciting results. Yet many concepts remain to generalize for modules. In this sequence of generalizations, we focus here on almost $\omega_1 - p^{(\omega+n)}$-projective abelian p-groups and generalize them. For them literature on this one can go through (Danchev, 2008; Danchev and keef, 2009; keef, 2010)

Some of the fundamental definitions used in this manuscript have already appeared in one of the co-authors' previous works; these are offered as quotations and are duly cited here.

"A module $M$ over an associative ring $R$ with unity is a *QTAG*-module if every finitely generated submodule of any homomorphic image of $M$ is a direct sum of uniserial modules (Singh, 1987). All the rings $R$ considered here are associative with unity and modules $M$ are unital *QTAG*-modules. An element $x \in M$ is uniform, if $xR$ is a non-zero uniform (hence uniserial) module and for any $R$-module $M$ with a unique composition series, $d(M)$ denotes its composition length. For a uniform element $x \in M, e(x) = d(xR)$ and $H_M(x) = \sup\left\{d\left(\frac{yR}{xR}\right) | y \in M, x \in yR \text{and} y \text{uniform}\right\}$ are the exponent and height of $x$ in $M$, respectively. $H_k(M)$ denotes the submodule of $M$ generated by the elements of height at least $k$ and $H^k(M)$ is the submodule of $M$ generated by the elements of exponents at most $k$. $M$ is $h$-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ (Khan, 1979) and it is $h$-reduced if it does not contain any $h$-divisible submodule. In

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other words it is free from the elements of infinite height. A *QTAG*-module $M$ is said to be separable, if $M^1 = 0$. Let $M$ be a module, then the sum of all simple submodules of $M$ is called the socle of $M$ and is denoted by $\mathrm{Soc}(M)$. If $M, M\prime$ are QTAG-modules then a homomorphism $f : M \to M\prime$ is an isometry if it is 1–1, onto and $H_{M\prime}(f(x)) = H_M(x)$, for all $x \in M$. A submodule $N$ of a QTAG-module $M$ is a nice submodule if every nonzero coset $a + N$ is proper with respect to $N$ *i.e.* for every nonzero $a + N$ there is an element $b \in N$ such that $H_M(a + b) = H_{M/N}(a + N)$."

"A family $\mathscr{N}$ of submodules of $M$ is called a nice system in $M$ if

(i) $0 \in \mathscr{N}$;
(ii) If $\{N_i\}_{i \in I}$ is any subset of $\mathscr{N}$, then $\Sigma_I N_i \in \mathscr{N}$;
(iii) Given any $N \in \mathscr{N}$ and any countable subset $X$ of $M$, there exists $K \in \mathscr{N}$ containing $N \cup X$, such that $K/N$ is countably generated (Mehdi et al., 2006).

Every submodule in a nice system is nice submodule. A $h$-reduced *QTAG*-module $M$ is called totally projective if it has a nice system and direct sums and direct summands of totally projective modules are also totally projective. A submodule $N$ of $M$ is $h$-pure in $M$ if $N \cap H_k(M) = H_k(N)$, for every integer $k \geqslant 0$ and a submodule $N$ of $M$ is said to be *isotype* in $M$, if it is $\sigma$- pure for every ordinal $\sigma$ (Singh, 1976). A QTAG- module $M$ is $(\omega + n)$- projective, if there exists a submodule $N \subset H^n(M)$ such that $M/N$ is a direct sum of uniserial modules or equivalently, if and only if there is a direct sum of uniserial module $K$ with a submodule $L \subseteq H^n(K)$ such that $M \cong K/L$. $M$ is $\omega$-projective if and only if it is a direct sum of uniserial modules. Also two $(\omega + n)$-projective QTAG-modules $M_1, M_2$ are isometric if and only if there is a height preserving isomorphism between $H^n(M_1)$ and $H^n(M_2)$ (Mehdi et al., 2006). A module $M$ is almost simply presented if it is the direct sum of $h$ divisible module and an almost totally projective module. For any QTAG-module $M, g(M)$ denotes the smallest cardinal number $\lambda$ such that $M$ admits a generating set $X$ of uniform elements of cardinality $\lambda$ *i.e.*, $|X| = \lambda$. A homomorphism $f : M \to N$ is said to be $\omega_1$-bijective if $g(\ker f), g(N/f(M)) < \omega_1$."

## 2. Main Results

We will first establish some fundamental findings that will serve as the foundation for our key findings. We begin by stating the following Proposition which is the generalization of result by P.D Hill and W.Ullery (Hill and Ullery, 1996).

**Proposition 1.** Suppose $K$ is an isotype submodule of a QTAG-module $M$. Then $K$ is almost totally projective provided that $K$ is separable in $M$.

**Proof.** Let $K$ be an almost totally projective QTAG-module and suppose to the contrary that it is not separable in $M$. Then, there exists $m \in M$ such that, for each countably generated submodules $T$ of $K$, we can find an element $k^* \in K$ such that $H(m + k^*) > H(m + t)|$ for every $t \in T$. Therefore we can find an ascending chain

$$0 = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_n \subseteq \ldots$$

of countably generated submodules $K_n$ of $K$ such that $K_n \in \mathscr{A}$ for each $n$ and such that the following condition holds.

($\star$) For every finite $n$, there exists $k_{n+1} \in K_{n+1}$ such that $H(m + k_{n+1}) > H(m + k)$ for all $k \in K_n$.

Now set $K_\omega = \bigcup_{n < \omega_0} K_n$ and observe that $K_\omega$ is a countably generated submodule of $K$ belonging to $\mathscr{A}$. Since $K_\omega$ is countably generated, there exists $k^* \in K$ such that $H(m + t) < H(m + k^*)$ for every $t \in K_\omega$. Since $K_\omega$ is nice in $K$, there exists $k' \in K_\omega$ such that $H_K(k^* - t) \leqslant H_K(k^* - k')$ for all $t \in K_\omega$. Moreover, since $k' \in K_n$ for some $n$, there exists by condition ($\star$) an element $k'' \in K_{n+1}$ such that $H(m + k'') < H(m + k^*)$. So, we now have $H_K(k^* - k\prime\prime) \leqslant H_K(k^* - k')$. Since $K$ is isotype in $M$, we have $H(m + k\prime) < H((m + k^*) - (m + k\prime\prime)) = H(k^* - k'') = H_K(k^* - k'')$ $\leqslant H_K(k^* - k\prime) = H(k^* - k\prime) = H(m + k\prime)$ which leads to a contradiction and proving our desired result.

The next two propositions as analogous of the corresponding well-known results for totally projective modules.

**Proposition 2.** If $H_\sigma(M)$ and $M/H_\sigma(M)$ are almost totally projective, for any ordinal $\sigma$ then $M$ is also almost totally projective.

**Proof.** We know that a submodule $K$ is a nice submodule of $M$ if and only if $H_\sigma(K)$ is a nice submodule of $H_\sigma(M)$ and $K + H_\sigma(M)/H_\sigma(M)$ is a nice submodule of $M/H_\sigma(M)$. Hence, the properties satisfying the three conditions for a family of nice submodules to be almost totally projective, for $H_\sigma(M)$ and $M/H_\sigma(M)$ lead to satisfying the same conditions for the module $M$.

**Proposition 3.** The arbitrary direct sums of almost totally projective QTAG-modules are almost totally projective.

**Proof.** Suppose $M = A_{i \in I} M_i$. As it is well known, if $K = A_{i \in I} K_i$ with $K_i \subseteq M_i$ for all $i \in I$, then $K$ is a nice submodule of $M$ if and only if $K_i$ is a nice submodule of $M_i$ for each $i \in I$. Hence, the three properties of almost totally projectives modules satisfying by $M_i$ ensures that $M$ will certainly satisfies the same and hence the result follows.

The concept of almost direct sum of uniserial QTAG-modules were defined in (Hasan, 2018) as follows:

**Definition 1.** "The separable *QTAG*-module $M$ is said to be almost direct sum of uniserial modules if it possesses a collection $\mathscr{N}$ consisting of nice submodules of $M$ which satisfies the following three conditions:

(i) $\{0\} \in \mathscr{N}$
(ii) $\mathscr{N}$ is closed with respect to ascending unions, i.e., if $N_i \in \mathscr{N}$ with $N_i \subseteq N_j$ whenever $i \leqslant j (i, j \in I)$ then $\cup_{i \in I} N_i \in \mathscr{N}$;
(iii) If $K$ is a countably generated submodule of $M$, then there is $L \in \mathscr{N}$ (that is, a nice submodule $L$ of $M$) such that $K \subseteq L$ and $L$ is countably generated.

When $M$ is $h$-reduced and satisfies clauses $(i), (ii)$ and $(iii)$, it is called almost totally projective, and when $M$ has length not exceeding $\omega_1$, it is called almost direct sum of countably generated modules.

The last concept can be generalized to the following one.

**Definition 2** Hasan, 2018. The *QTAG*-module $M$ is said to be almost $(\omega + n)$-projective if there exists $N \subseteq H^n(M)$ such that $M/N$ is almost direct sum of uniserial modules.

Clearly, $H_{\omega+n}(M) = \{0\}$.".

We will now illustrate how the aforementioned definition can be equivalently classified into the following:

**Proposition 4.** The module $M$ is almost $(\omega + n)$-projective if and only if $M \cong S/T$ for some almost direct sum of uniserial module $S$ and $T \subseteq H^n(S)$.

**Proof.**

"⇒. Let $N$ be a module with $H_n(N) = M$ and let $S = N/L$ where $L \subseteq H^n(M)$ such that $M/L$ is almost direct sum of uniserial modules. Consequently, $H_n(S) = H_n(N/L) = M/L$ is almost direct sum of uniserial module and hence, by Proposition 2, $S$ is almost direct sum of uniserial module too. Supposing $T = H^n(N)/L$, we deduce that $S/T \cong N/H^n(N) \cong H_n(N) = M$, as required.⇐. Assume that $M \cong S/T$, which we without loss of generality interpret as an equality, whence we get $L = H^n(S)/T \subseteq H^n(M)$. Thus, again in view of Proposition 2, $M/L \cong S/H^n(S) \cong H_n(S)$ is almost direct sum of uniserial module, as desired.

The notion of $\omega_1 - (\omega + n)$-projective modules were defined in Sikander (2019) and showed that a module $M$ is $\omega_1 - (\omega + n)$-projective if and only if there exists a countably generated nice submodule $L$ such that $M/L$ is $(\omega + n)$ -projective. So, formulating the last in terms of almost $(\omega + n)$-projective modules, we obtain a common strengthening of Definition 2 as follows:

**Definition 3.** A QTAG-module $M$ is called almost $\omega_1 - (\omega + n)$-projective if there is a countably generated nice submodule $N$ such that $M/N$ is almost $(\omega + n)$-projective.

Apparently, $H_{\omega+n}(M)$ is countably generated.

Therefore, the main goal of this article is to study some of the distinctive qualities of the mentioned new class of modules in Definition 3 by using the information from above. Although at first glance there is an absolute analogue with (Sikander, 2019), which is not true and the fundamental reason why this is wrong is that the almost direct sum of uniserial modules lacks the crucial direct decomposition characteristic of the direct sum of uniserial modules.

**Proposition 5.** A submodule of an almost $(\omega + n)$-projective module is also almost $(\omega + n)$-projective.

**Proof.** . Let $L \subseteq H^n(M)$ such that $M/L$ is almost $\Sigma$-uniserial i.e, $M/L$ is almost direct sum of uniserial modules and suppose that $T \subseteq M$. Then $(T + L)/L \subseteq M/L$ is again almost $\Sigma$-uniserial, and $T/(T \cap L) \cong (T + L)/L$ with $T \cap L \subseteq H^n(T)$, as desired.

**Proposition 6.** The submodule of an almost $\omega_1 - (\omega + n)$-projective module is almost $\omega_1 - (\omega + n)$-projective as well.

**Proof.** Assume that $P \subseteq M$ where $M$ is almost $\omega_1 - (\omega + n)$-projective. Thus $M/N$ is almost $(\omega + n)$-projective for some countably generated submodule $N$. Moreover, $(N + P)/P \subseteq M/P$ is almost $(\omega + n)$-projective too by Proposition 5, and $P/(P \cap N) \cong (N + P)/P$. Since $N \cap P \subseteq N$ is countably generated, we are done.

**Remark 1.** Let $M$ be a separable module with a countably generated nice submodule $N$. Then $M$ is almost direct sum of uniserial modules if and only if $M/N$ is almost direct sum of uniserial modules.

**Lemma 1.** Let $N$ be a countably generated submodule of a QTAG module $M$ with $M/N$ as almost direct sum of uniserial submodules. Then $M$ is the sum of a countably generated submodule and an almost direct sum of uniserial modules.

**Proof.** Since $H_\omega(M) \subseteq N$ is countably generated, we may isomorphically embed it in an essential submodule of $M/L$ where $L$ is a high submodule of $M$ and thus $M/L$ will also be countably generated. In fact, $H_\omega(M) \cong (H_\omega(M) \oplus L)/L \subseteq M/L$ where it is easily checked that $(H_\omega(M) \oplus L)/L$is essential in $M/L$ because $L$ is maximal with respect to intersecting $H_\omega(M)$ trivially. This provide evidence to support our claim. Furthermore, one can write that $M = L + S$ for some countably generated submodule $S$. We know that by Begam, 2014, If $M/N$ is almost direct sum of uniserial submodules for some QTAG module $M$ and its countably generated submodule $N$, then $M$ is almost simply presented and a high submodule of an almost simply presented module is almost direct sum of uniserial module, we obtain that $M$ must be almost direct sum of uniserial module and so the required decomposition.

An immediate consequence of the above lemma is as follows:

**Corollary 1.** Suppose that $T$ is an almost direct sum of uniserial module and $S$ is its countably generated submodule. Then $T/S$ is the sum of a countably generated module and an almost direct sum of uniserial module.

**Proof.** Using Remark 1, one can infer, for any countably generated nice submodule $C$ of $T$ as $S \subseteq C$ that $T/C$ remain almost direct sum of uniserial module. But $T/C \cong (T/S)/(C/S)$, where $C/S$ is countably generated. Now use of Lemma 1 ensures the desired decomposition of $T/S$.

**Proposition 7.** A separable QTAG module $M$ is almost $(\omega + n)$-projective if and only if $M/N$ is almost $(\omega + n)$-projective, where $N$ be a countably generated nice submodule of the sepearable module $M$.

**Proof.** "⇒ Suppose that $M/L$ is almost direct sum of uniserial module, for some $L \subseteq H^n(M)$. Observe that the two isomorphisms hold:

$$(M/N)/(L + N)/N \cong M/(L + N) \cong (M/L)/(L + N)/L$$

Since $(L + N)/L \cong N/(L \cap N)$ is countably generated, it follows from Begam, 2014 that $T/D$ is almost simply presented where we put $M/N = T$ and $(L + N)/N = D$. Therefore,

$$\begin{aligned} T/[\cap_{n<\omega}(H_n(T) + D)] &\cong (T/D)/[\cap_{n<\omega}(H_n(T) + D)]/D \\ &= (T/D)/H_\omega(T/D) \end{aligned}$$

is almost direct sum of uniserial modules with $H_t(\cap_{n<\omega}(H_n(T) + D)) = H_\omega(T) = H_\omega(M/N) = (H_\omega(M) + N)/N = \{0\}$. Thus by Definition 2 the quotient module $T = M/N$ is almost $H_{\omega+n}$-projective, as claimed.

⇐. For the reverse implication suppose that $M/N$ be almost $H_{\omega+n}$-projective. Therefore, there is a quotient $S/N$ with $S \subseteq M$ and $H_n(S) \subseteq N$ such that $(M/N)/(S/N) \cong M/S$ is almost direct sum of uniserial modules. Hence $S$ is the direct sum of a countably generated and an $n$-bounded module, say $S = A \oplus Q$ where $A$ is countably generated and $H_n(Q) = \{0\}$. We further infer that

$$M/S = M/(A \oplus Q) \cong (M/Q)/(A \oplus Q)/Q$$

where $(A \oplus Q)/Q \cong A$ is countably generated and so $M/Q$ is almost simply presented. Therefore, $(M/Q)/H_\omega(M/Q) \cong M/[\cap_{n<\omega}(H_n(M) + Q)]$ is almost direct sum of uniserial modules with $H_t(\cap_{n<\omega}(H_n(M) + Q)) = H_\omega(M) = 0$. Finally, $M$ is almost $H_{\omega+n}$-projective, as expected.

**Corollary 2.** Suppose that $M$ is a separable almost $\omega_1 - (\omega + n)$-projective module. Then $M$ is almost $(\omega + n)$-projective.

**Proof.** Let $N$ be a countably generated submodule of the module $M$ such that $M/N$ is almost $(\omega+n)$-projective. Application of Proposition 7 gives that $M$ is almost $(\omega+n)$-projective, as asserted.

**Proposition 8.** (a) If $M$ is almost $(\omega+n)$-projective, then $M/H_{\gamma}(M)$ is almost $(\omega+n)$-projective for any ordinal $\gamma$.

(b) If $M$ is (nicely) almost $\omega_1-(\omega+n)$-projective, then $M/H_{\gamma}(M)$ is (nicely) almost $\omega_1-(\omega+n)$-projective for all ordinals $\gamma$.

**Proof.** The above statements are trivial for the ordinals $\gamma<\omega$. So, we will discuss the case $\gamma \geqslant \omega$.

(a) Suppose $M/L$ is almost direct sum of uniserial modules for some $L\subseteq H^{n}(M)$. So, $H_{\omega}(M)\subseteq L$ and hence $H_{\gamma}(M)\subseteq L$ for each ordinal $\gamma \geqslant \omega$. Furthermore, $M/L\cong (M/H_{\gamma}(M))/(L/H_{\gamma}(M))$ is almost direct sum of uniserial modules with $L/H_{\gamma}(M)\subseteq H^{n}((M/H_{\gamma}(M)))$, as desired.

(b) First, we will discuss the case for nice submodules. So if $M/N$ is almost $(\omega+n)$-projective for some countably generated nice submodule $N$, we deduce with the help of part (a), that

$$(M/N)/H_{\gamma}(M/N)=(M/N)/\big(H_{\gamma}(M)+N\big)/N\cong$$
$$M/\big(H_{\gamma}(M)+N\big)\cong\big(M/H_{\gamma}(M)\big)/\big(H_{\gamma}(M)+N\big)/p^{\lambda}G$$

is almost $(\omega+n)$-projective. Since $\big(H_{\gamma}(M)+N\big)/H_{\gamma}(M)\cong N/\big(H_{\gamma}(M)\cap N\big)$ is obviously countably generated and nice in $M/H_{\gamma}(M)$, Fuchs (1970 and 1973.), we are done.

Now we will discuss the part without *niceness*. To show it, suppose $M/N$ be almost $(\omega+n)$-projective for some countably generated submodule $N$. We claim that

$$(M/N)/\big(H_{\gamma}(M)+N\big)/N\cong M/\big(H_{\gamma}(M)+N\big)$$
$$\cong\big(M/H_{\gamma}(M)\big)/\big(H_{\gamma}(M)+N\big)/H_{\gamma}(M).$$

is almost $(\omega+n)$-projective. In fact, if $P$ is an almost $(\omega+n)$-projective module with $T\subseteq H_{\gamma}(P)$, then $P/T$ is also almost $(\omega+n)$-projective. To this goal, write $P/Q$ is almost direct sum of uniserial modules for some $Q\subseteq H^{n}(P)$. Thus $H_{\omega}P\subseteq Q$ whence $T\subseteq H_{\gamma}(P)\subseteq H_{\omega}(P)\subseteq Q$. This gives that $P/Q\cong(P/T)/(Q/T)$ is almost direct sum of uniserial modules for $Q/T\subseteq H^{n}(P/T)$ and means that $P/T$ is really as desired. We just apply this assertion to $P=M/N$ and $T=\big(H_{\gamma}(M)+N\big)/N\subseteq H_{\gamma}(M/N)=H_{\gamma}(P)$ and the claim is established. Furthermore, by what we have previously shown, $(M/H_{\gamma}(M))/(H_{\gamma}(M)+N)/H_{\gamma}(M)$ being almost $(\omega+n)$-projective with countably generated $(H_{\gamma}(M)+N)/H_{\gamma}(M)\cong N/\big(N\cap H_{\gamma}(M)\big)$ ensures that $M/H_{\gamma}(M)$ is almost $\omega_1-(\omega+n)$-projective, as formulated.

As an immediate consequence we have the following:

**Corollary 3.** If $M$ is almost $\omega_1-(\omega+n)$-projective, then $M/H_{\omega}(M)$ is almost $(\omega+n)$-projective.

For modules with countably generated first Ulm submodule, we have the following interesting result:

**Theorem 1.** Suppose $M$ is a module such that $H_{\omega}(M)$ is countably generated. Then $M$ is almost $\omega_1-(\omega+n)$-projective if and only if $M/H_{\omega}(M)$ is almost $(\omega+n)$- projective.

**Proof.** The direct part follows trivially by Definition 3 while the reverse implication part can be obtained using Corollary 3.

**Theorem 2.** The module $M$ is (nicely) almost $\omega_1-(\omega+n)$-projective if and only if $H_{\omega+n}(M)$ is countably generated and $M/H_{\omega+n}(M)$ is (nicely) almost $\omega_1-(\omega+n)$ projective.

**Proof.** The direct implications follows from Proposition 8 (b) by substituting $\gamma=\omega+n$. As for the other way round, suppose $(M/H_{\omega+n}(M))/(N/H_{\omega+n}(M))\cong M/N$ is almost $(\omega+n)$-projective for some countably generated (nice) quotient $N/H_{\omega+n}(M)$ such that $N\subseteq M$. But $N$ is countably generated (and nice) in $M$, so that $M$ is (nicely) almost $\omega_1-(\omega+n)$-projective, as claimed.

One can state the following by exploiting the above idea:

**Corollary 4.** Suppose that $H_{\alpha}(M)$ is countably generated for some ordinal $\alpha$. Then $M$ is almost $\omega_1-(\omega+n)$-projective if and only if $M/H_{\alpha}(M)$ is almost $\omega_1-(\omega+n)$-projective.

**Proof.** The direct implication can be obtained using Proposition 8 while the reverse implication follows on the same line of the proof of Theorem 2.

**Proposition 9.** The direct sums of almost $(\omega+n)$-projective modules are almost $(\omega+n)$-projective modules.

**Proof.** Suppose $M=\underset{i\in I}{A}M_i$ where all components $M_i$ are almost $(\omega+n)$- projective. So, $M_i/N_i$'s are almost direct sum of uniserial modules for some $N_i\subseteq H^{n}(M_i)$. Furthermore, putting $N=\underset{i\in I}{A}N_i$, we infer that $N\subseteq H^{n}(M)$ and that $M/N\cong\underset{i\in I}{A}M_i/N_i$ are almost direct sums of uniserial modules owing to Mehdi et al. (2006), as expected.

The following improves (Proposition 2.14, Sikander, 2019) to the new framework.

**Proposition 10.** Suppose $M=\underset{t\in T}{A}M_t$ is a module for some index set $T$. Then $M$ is almost $\omega_1-(\omega+n)$-projective if and only if $M_t$ is almost $\omega_1-(\omega+n)$-projective for each index $t\in T$, and there exists a countable subset $B\subseteq T$ such that $M_t$'s are almost $(\omega+n)$-projective for all $t\in T\setminus B$.

**Proof.** For the direct implication, suppose that $X$ be a countably generated submodule of $M$ such that $M/X$ is almost $(\omega+n)$-projective. From Proposition 6 it follows that all $M_t$'s are almost $\omega_1-(\omega+n)$-projective. Clearly, $X\subseteq\oplus_{t\in B}M_t$ for some $B\subseteq T$ with $|B|\leqslant\aleph_0$. Therefore,

$M/X\cong\left[\left(\underset{t\in B}{A}M_t\right)/X\right]\oplus\left[\underset{t\in T\setminus B}{A}M_t\right]$, so that $\underset{t\in T\setminus B}{A}M_t$, and hence $M_t$'s are almost $(\omega+n)$-projective for every $t\in T\setminus B$ in conjunction with Proposition 5.

For the reverse implication suppose all factors $M_t/X_t$ be almost $(\omega+n)$-projective for some countably generated submodules $X_t\subseteq M_t$. Set $X=\oplus_{t\in B}X_t$, whence $X$ is countably generated. However,

$$M/X\cong[\oplus_{t\in B}(M_t/X_t)]\oplus\left[\oplus_{t\in T\setminus B}M_t\right],$$

and so by Proposition 9 we conclude that $M/X$ is almost $(\omega+n)$-projective, as required.

We have the following consequence:

**Corollary 5.** The countable direct sum of almost $\omega_1-(\omega+n)$-projective module is an almost $\omega_1-(\omega+n)$- projective module.

Now we will discuss some equivalencies that give comprehensive characterizations of almost $\omega_1 - (\omega + n)$-projectivity.

**Theorem 3.** The following conditions are equivalent:

1. $M$ is almost $\omega_1 - (\omega + n)$-projective;
2. $M/N$ is the sum of a countably generated module and an almost direct sum of uniserial module where $H_n(N) = \{0\}$;
3. $M \cong \mathfrak{B}/S$ where $\mathfrak{B}$ is the sum of a countably generated module and an almost direct sum of uniserial module and $H_n(S) = \{0\}$;
4. $M/T$ is almost direct sum of uniserial module, where $H_n(T)$ is countably generated ($T$ is the direct sum of a countably generated module and an $n$-bounded module);
5. $M \cong W/H$, where $W$ is almost direct sum of uniserial module and $H_n(B)$ is countably generated ($B$ is the direct sum of a countably generated module and a $n$-bounded module);
6. $M \cong X/Y$, where $X$ is almost $(\omega + n)$-projective and $Y$ is countably generated;
7. $M/U$ is countably generated, where $U$ is almost $(\omega + n)$-projective.

**Proof.** We start with $(1) \Rightarrow (7)$: Suppose $M/Q$ is almost $(\omega + n)$-projective for some countably generated submodule $Q \subseteq M$. Let $U \subseteq M$ be maximal with respect to $U \cap Q = \{0\}$.

Clearly $U \cong (U \oplus Q)/Q \subseteq M/Q$, so that Proposition 5 applies to get that $U$is almost $(\omega + n)$-projective too.

On the other way round, $Q \cong (Q \oplus U)/U$ where the latter is an essential submodule of $M/U$, and thus $M/U$ will be countably generated. In fact, for any $A \subseteq M$ with $A \neq U$ we obtain by the modular law from Fuchs (1970 and 1973.) that $[(Q \oplus U)/U] \cap (A/U) = ((Q \oplus U) \cap A)/U = (U + Q \cap A)/U \neq \{0\}$ because $Q \cap A \nsubseteq U$ since $Q \cap A \neq \{0\}$. Thus (7) follows.

(1) $\Longleftrightarrow$ (4). Suppose that $M/T$ is almost direct sum of uniserial modules, where $T = Q \oplus N$ for some countably generated submodule $Q$ and $n$-bounded submodule $N$. But $M/T = M/(Q \oplus N) \cong M/Q/(Q \oplus N)/Q$, and $(Q \oplus N)/Q \cong N$ is $n$-bounded. Therefore, $M/Q$ is almost $(\omega + n)$-projective and (1) holds. Conversely suppose that $M/Q$ be almost $(\omega + n)$-projective for some countably generated submodule $Q$. Hence there is an $n$-bounded submodule $T/Q$ with $T \subseteq M$ such that $(M/Q)/(T/Q) \cong M/T$ is almost direct sum of uniserial modules. Since $H_n(T) \subseteq Q$ is countably generated, we are done.

(2) $\Longleftrightarrow$ (4). First, we note the following helpful fact: Setting $X = B + W$, where $B$ is countably generated and $W$ is almost direct sum of uniserial modules, there exists a countably generated module $C$ such that $X/C$ is almost direct sum of uniserial modules. Indeed, $B \cap W$ being a countably generated submodule of $W$ forces that $B \cap W \subseteq F$ where $F$ is a countably generated nice submodule of $W$ whence by Remark 1 we infer that $W/F$ is almost direct sum of uniserial module. It therefore follows that $X/F = [(B + F)/F] \oplus [W/F]$. That is why, $(X/F)/(B + F)/F \cong X/(B + F) \cong W/F$ is almost direct sum of uniserial modules. Denoting $C = B + F$, we are done. Furthermore, applying the last observation to $M/N$ we obtain that $(M/N)/(T/N) \cong M/T$ is almost direct sum of uniserial modules, where $T/N$ is countably generated. Hence $H_n(T)$ is countably generated, as stated. For the reverse part suppose that $M/T$ is almost direct sum of uniserial modules with $T = \mathfrak{B} \oplus N$ where $\mathfrak{B}$ is countably generated and $N$ is bounded by $n$. However, $M/T \cong (M/N)/(T/N)$ is almost direct sum of uniserial modules with countably generated $T/N \cong \mathfrak{B}$, so that Lemma 1 is applicable for $M/N$ to finish the equivalence.

(6) $\Longleftrightarrow$ (5). First, assume that $M \cong X/Y$ for some almost $(\omega + n)$- projective module $X$ and its countably generated submodule $Y$. Using Proposition 4, one may write that $X = W/N$ where $W$ is almost direct sum of uniserial modules with $H_n(N) = \{0\}$, and $Y = D/N$ is countably generated with $D \subseteq W$. Furthermore, $M \cong W/D$ and since $D = N + C$ for some countably generated submodule $C$, one may derive that $H_n(D) = H_n(C)$ is countably generated, as required.

For the reverse implication, let us assume that $M \cong W/D$ where $W$ is almost direct sum of uniserial modules and $H_n(D)$ is countably generated. Since $D$ is the direct sum of a countably generated submodule $B$ and an $H_n$-bounded module $V$, say $D = B \oplus V$, one may deduce that $M \cong W/(B \oplus V) \cong (W/V)/(B \oplus V)/V$. However, using Proposition 4, $W/V$ is almost $(\omega + n)$-projective, whereas $(B \oplus V)/V \cong B$ is countably generated. This ensures that (6) holds, thus completing the verification of the desired equivalence.

$(7) \Rightarrow (6)$. Suppose that $M/U$ is countably generated for some almost $(\omega + n)$- projective submodule $U$. Let $A$ be a countably generated submodule which is the direct sum of uniserial modules and $\beta : A \to M$ be a homomorphism such that $M = U + \beta(A)$. If we set $F = U \oplus A$, then $F$ is almost $(\omega + n)$-projective appealing to Proposition 9. If now we let $g : U \to M$ be the identity map, then we have a surjective homomorphism $\alpha : F \to M$. If $B$ is its kernel, then obviously $B \cap U = \{0\}$; in fact, $x \in B \cap U$ forces that $\alpha(x) = x = 0$. Hence $M \cong F/B$ and $B$ is isomorphic to a submodule of $A$. Thus $B$ is countably generated, and we are done.

$(5) \Rightarrow (3)$. Let $M \cong W/D$ where $W$ is almost direct sum of uniserial modules and $D = G \oplus N$ where $G$ is countably generated and $N$ is bounded by $n$. Since $M \cong (W/G)/(D/G)$ and $D/G \cong N$ is $n$-bounded, we just take into account Corollary 1 to conclude that $W/G$ is the sum of a countably generated submodule and an almost direct sum of uniserial modules, as desired.

$(3) \Rightarrow (2)$. Suppose $M = \mathfrak{B}/S$ where $\mathfrak{B} = B + W$ with countably generated submodule $B$ and almost direct sum of uniserial module $W$. Since $S \subseteq H^n(\mathfrak{B})$, set $N = H^n(\mathfrak{B})/S \subseteq M$. Thus $M/N \cong \mathfrak{B}/H^n(\mathfrak{B}) \cong H_n(\mathfrak{B}) = H_n(B) + H_n(W)S$ remains again the sum of a countably generated submodule and an almost direct sum of uniserial modules.

$(3) \Rightarrow (7)$. Let us express $M = \mathfrak{B}/S$, where $\mathfrak{B} = B + W$ and $B$ is countably generated whereas $W$ is almost direct sum of uniserial modules. But $\mathfrak{B}/S = (C + S)/S + (W + S)/S$. Observing that $(C + S)/S \cong C/(C \cap S)$ is countably generated and $(W + S)/S \cong S/(S \cap V)$ is almost $(\omega + n)$-projective in conjunction with Proposition 4, we routinely see that $M/U$ is countably generated for $U = (W + S)/S$, as needed.

(2) $\Longleftrightarrow$ (7). Suppose first that $M/U$ is countably generated for some almost $(\omega + n)$-projective module $U$. So, we write $M = U + C$ for some countably generated submodule $C \subseteq M$. Consequently, $U/N$ is almost direct sum of uniserial module for some $N \subseteq H^n(U)$ and thus $M/N = [U/N] + [(C + N)/N]$ where $(C + N)/N \cong C/(C \cap N)$ is countably generated. Thus (2) is satisfied. Conversely, let $M/N = (X/N) + (Y/N)$ where the first summand is countably generated while the second is almost direct sum of uniserial modules. So, $M = X + U$ and $X = N + G$ where $G$ is countably generated, which gives $M = U + G$. But $U$ is almost direct sum of uniserial modules and $M/U = (G + U)/U \cong G/(G \cap U)$ is countably generated. Hence (7) is fulfilled.

**Theorem 4.** Let $N$ be a submodule of $M$ such that $M/N$ is countaly generated. Then $M$ is almost $\omega_1 - (\omega + n)$-projective if and only if $N$ is almost $\omega_1 - (\omega + n)$ projective.

**Proof.** The direct part follows by using Proposition 6. For the reverse implication, according to point (7) of Theorem 3, we write that $N/T$ is countably generated for some almost $(\omega + n)$-projective module $T$. Therefore, $M/N \cong (M/T)/(N/T)$ being countably generated implies that $M/T$ is countably generated and again point (7) of Theorem 3 yields that $M$ is almost $\omega_1-(\omega+n)$-projective, as claimed.

**Theorem 5.** Let $L$ be a countably generated submodule of $M$. Then $M$ is almost $\omega_1-(\omega+n)$-projective if and only if $M/L$ is almost $\omega_1-(\omega+n)$ projective.

**Proof.** $\Leftarrow$: Suppose $M/L$ is almost $\omega_1-(\omega+n)$-projective for the countably generated submodule $L$. Therefore there is a countably generated submodule $P/L$ of $M/L$ with $P\subseteq M$ such that $(M/L)/(P/L)\cong M/P$ is almost $(\omega+n)$-projective. Since $P$ remains countably generated, $M$ must be almost $\omega_1-(\omega+n)$-projective, as claimed.

$\Rightarrow$: Suppose $M/Y$ be almost $(\omega+n)$-projective for some countably generated submodule $Y$. By part (6) of Theorem 3, we have that $(M/Y)/(L+Y)/Y\cong M/(L+Y)\cong(M/L)/(L+Y)/L$ is almost $(\omega+n)$-projective because $(L+Y)/Y\cong L/(L\cap Y)$ is countably generated. Since $(L+Y)/L\cong Y/(Y\cap L)$ remains countably generated, an appeal to the above part proved assures that $M/L$ is almost $(\omega+n)$-projective, as asserted.

One of the main result of the article is the following:

**Theorem 6.** The class of almost $\omega_1-(\omega+n)$-projective modules is closed under $\omega_1$-bijections, and is the minimal class containing almost $(\omega+n)$-projectives with that property.

**Proof.** The first part follows by a combination of Theorems 4 and 5 along with (Lemma 2.9, Sikander, 2019). The remaining part can be proved easily by using Theorem 3 and (Proposition 2.10, Sikander, 2019)

We now conclude this article by establishing the characterization of a module to be amost $\omega_1-(\omega+n)$-projective as follows:

**Theorem 7.** The module $M$ is almost $\omega_1-(\omega+n)$-projective if and only if there exists a countably generated nice submodule $K$ which satisfies the inequalities $H_{\omega+n}(M)\subseteq K\subseteq H_\omega(M)$ such that $M/K$ is almost $(\omega+n)$-projective.

**Proof.** In view of part (3) of Theorem 3, we write $M=L/P$ where $L$ is the sum of a countably generated module $B$ and an almost direct sum of uniserial modules $Q$, say $L=B+Q$, and $P\subseteq H^n(L)$. Setting $K=(H_\omega(L)+P)/P$, we observe that

$$M/K\cong L/(H_\omega(L)+P)\cong(L/H_\omega(L))/(H_\omega(L)+P)/H_\omega(L)$$

We now prove two things about $L$, that are, $L/H_\omega(L)$ is almost direct sum of uniserial modules, and $H_\omega(L)$ is countably generated whence so is $H_\omega(L)/(H_\omega(L)\cap P)\cong K$. In fact, since $B\cap Q\subseteq Q$ is countably generated, there is a countably generated nice submodule $\mathfrak{B}$ of $Q$ such that $\mathfrak{B}\supseteq B\cap Q$. That is why, $L/\mathfrak{B}=[(B+\mathfrak{B})/\mathfrak{B}]\oplus[\mathfrak{B}/R]$. However, $(B+\mathfrak{B})/\mathfrak{B}$ remains countably generated, whereas $Q/\mathfrak{B}$ remains separable. Consequently, $H_\omega(L/\mathfrak{B})=H_\omega$ is countably generated and hence the same holds for its submodule $(H_\omega(L)+\mathfrak{B})/\mathfrak{B}\cong H_\omega(L)/(\mathfrak{B}\cap H_\omega(L))$. But $(\mathfrak{B}\cap H_\omega(L))$ is countably generated, so that $H_\omega(L)$ is countably generated, indeed.

For the other claim by using Remark 1, we have that $(L/\mathfrak{B})/(B+\mathfrak{B})/\mathfrak{B}\cong L/(B+\mathfrak{B})\cong Q/\mathfrak{B}$ is almost direct sum of uniserial modules. Since $B+\mathfrak{B}$ remains countably generated Then $L/H_\omega(L)$ is really almost direct sum of uniserial modules Begam, 2014. Furthermore, since $(H_\omega(L)+P)/H_\omega(L)\cong P/(P\cap H_\omega(L))$ is bounded by $n$, it now follows from Proposition 4 that $M/K$ must be almost $(\omega+n)$- projective, as desired.

Finally, it is routinely seen that $K\subseteq H_\omega(M)$ and that $K\supseteq H_(\omega+n)(M)$ because $(H_{\omega+n}(M)+K)/K\subseteq H_{\omega+n}(M/K)=\{0\}$. It is next easily checked that such a submodule $K$ satisfying the above two inequalities should be nice in $M$, as desired.

## Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.


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## References


Begam, F., 2014. Transitive QTAG-modules and other structures, Ph.D. Thesis, A.M. U., Aligarh.

Danchev, P.V., 2008. On extensions of primary almost totally projective groups. Math. Bohemica 133 (2), 149–155.

Danchev, P.V., 2014. ON ALMOST $\omega_1-p^{(\omega+n)}$-PROJECTIVE ABELIAN p-GROUPS. Korean J. Math. 22 (3), 501–516. https://doi.org/10.11568/kjm.2014.22.3.501.

Danchev, P.V., Keef, P.W., 2009. Generalized Wallace theorems. Math. Scand. 104 (1), 33–50.

Fuchs, L., 1970 and 1973.. Infinite Abelian Groups, volumes I and II. Acad Press, New York and London.

Hasan, A., 2018. On Almost n-Layered QTAG-Modules. Iranian J. Math. Sci. Inform. 13 (2), 163–171. https://doi.org/10.7508/ijmsi.2018.13.013.

Hill, P.D., Ullery, W., 1996. Almost totally Projective groups. Czechoslovak Math. J. 46 (2), 249–258.

Keef, P.W., 2010. On $\omega_1-p^{\omega+n}$-projective primary abelian groups. J. Algebra Numb. Th. Acad. 1 (1), 41–75.

Khan, M.Z., 1979. h-divisible and basic submodules. Tamkang J. Math. 10 (2), 197–203.

Mehdi, A., Abbasi, M.Y., Mehdi, F., 2006. On ($\omega$+n)-projective modules. Ganita Sandesh 20 (1), 27–32.

Sikander, F., 2019. On projective QTAG-modules. Tbilisi Math. J. 12 (1), 55–68.

Singh, S., 1976. Some decomposition theorems in abelian groups and their generalizations, Ring Theory, Proc. of Ohio Univ. Conf. Marcel Dekker N.Y. 25, 183–189.

Singh, S., 1987. Abelian groups like modules. Act. Math. Hung. 50 (2), 85–95.