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Existence and stability of fractional integro differential equation with non-instantaneous integrable impulses and periodic boundary condition on time scales

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ABSTRACT

The present paper is devoted to the study of existence and stability of fractional integro differential equation with non-instantaneous impulses and periodic boundary condition on time scales. This paper consists of two segments: the first segment of the work is concerned with the theory of existence, uniqueness and the other segment is to Hyer's-Ulam type's stability analysis. The tools for study include the Banach fixed point theorem and nonlinear functional analysis. Finally, in support, an example is presented to validate the obtained results.

1. Introduction

In this paper, we consider the following fractional integro-differential equation with non-instantaneous impulses and periodic boundary condition on time scale:

$$\begin{aligned} &{}^{c}\Delta^{q}u(\theta)=\mathscr{M}(\theta,u(\theta),\mathscr{N}(u(\theta))),\quad \theta\in\cup_{k=0}^{p}(\eta_k,\theta_{k+1}]_{\mathbb{T}},\\ &u(\theta)=\frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta-z)^{q-1}g_k(z,u(\theta_k^-))\Delta z,\quad \theta\in(\theta_k,\eta_k]_{\mathbb{T}},\quad k=1,2,\ldots,p,\\ &u(0)=u(T),\end{aligned} \tag{1}$$

where $\mathbb{T}$ is a time scale with $\theta_k,\eta_k\in\mathbb{T}$ are right dense points with $0=\eta_0=\theta_0<\theta_1<\eta_1<\theta_2<\ldots\eta_p<\theta_{p+1}=T$, $u(\theta_k^+)=\lim_{h\to 0^+}u(\theta_k+h)$, $u(\theta_k^-)=\lim_{h\to 0^+}u(\theta_k-h)$ represent the right and left limits of $u(\theta)$ at $\theta=\theta_k$ in the sense of time scale. ${}^{c}\Delta^{q}$ is the Caputo delta fractional derivative with $q\in(0,1)$. $g_k(\theta,u(\theta_k^-))\in C(I,\mathbb{R})$ are the impulses in the intervals $(\theta_k,\eta_k]$, $k=1,2,\ldots,p$.$\mathscr{M}:I=[0,T]\times\mathbb{R}\to\mathbb{R}$ and $h:\mathscr{Q}\times\mathbb{R}\to\mathbb{R}$ are suitably defined functions satisfying certain conditions to be specified later, where $\mathscr{Q}=\{(\theta,s)\in I\times I:0\leqslant s\leqslant\theta\leqslant T\}$ and

$$\mathscr{N}(u(\theta))=\int_0^{\theta}h(\theta,s,u(s))\Delta s.$$

Hilger, 1988, introduced the calculus of time scales. The calculus of time scales encapsulates the continuous and discrete analysis, therefore the study of dynamical systems on time scales is a very potential area for researchers as well as engineers. For more details about the time scales and the dynamic equations on time scales one can go through the books (Bohner and Peterson, 2001, 2003) and papers (Agarwal and Bohner, 1999; Agarwal et al., 2002). In the past couple of years, few authors discussed the existence, uniqueness, and stability of fractional dynamical equations on time scales (Ahmadkhanlu and Jahanshahi, 2012; Bastos et al., 2011; Benkhettou et al., 2015, 2016; Shen, 2017).

A lot of certifiable issues are seen with sudden changes in their states, for example, cataclysmic events, stuns, and heartbeats. Such sudden changes are called impulses. Some of the times, these abrupt changes stay over a period of time and that impulses are known as non-instantaneous impulses. For the comprehensive studies of non-instantaneous impulsive systems, one can see (Abbas et al., 2017; Feĉkan and Wang, 2015; Gautam and Dabas, 2016; Hernández and O'Regan, 2013; Kumar et al., 2016; Pandey et al., 2014; Muslim et al., 2018) and the references therein. Further, the theory of fractional calculus is an extended version of the theory of integer order. Since fractional differential equations define the fundamental properties of the system more accurately, therefore fractional calculus plays a significant role in the qualitative theory

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of differential equations. In addition, the stability analysis is an important feature of the research area of fractional calculus. Moreover, an interesting type of stability was introduced by Hyers and Ulam which is known as Hyers-Ulam stability. The Hyers-Ulam stability for several dynamical equations of the integer as well as the fractional order has been reported in Agarwal et al. (2017) and Wang and Li (2016). However to the best of author's knowledge, there is no work related to existence and stability analysis of integro fractional differential equations with non-instantaneous impulses on time scales. Motivated by the above facts, in this paper we obtain existence and Ulam type stability results for the Eq. (1).

The paper is organized in the following manner, in Section 2, we give some preliminaries, fundamental definitions, useful lemmas and some important results. In Sections 3 and 4, the main results of the manuscript are discussed. In the last, an example is given to illustrate the implementation of the obtained results.

2. Preliminaries

Below, we briefly described basic notations, fundamental definitions and useful lemmas. Let $(X, \|.\|)$ be a Banach space. $C(I, \mathbb{R})$ be the set of all continuous functions and $PC(I, \mathbb{R})$ be the space of piecewise continuous functions. We define the space of piecewise continuous functions as $PC(I, \mathbb{R}) = \{u : I \to \mathbb{R} : u \in C((\theta_k, \theta_{k+1}], \mathbb{R}),$ $k = 0, 1, \dots, p$ and there exists $u(\theta_k^-)$ and $u(\theta_k^+)$, $k = 1, 2, \dots, p$, with $u(\theta_k^-) = u(\theta_k)\}$. Moreover, $PC(I, \mathbb{R})$ forms a Banach space endowed with the norm $\|u\|_0 = \sup_{\theta \in [a,b]} |u(\theta)|$. Further, we define $PC^1(I, \mathbb{R}) = \{u \in PC(I, \mathbb{R}) : u^\Delta \in PC(I, \mathbb{R})\}$. $PC^1(I, \mathbb{R})$ form a Banach space endowed with the norm $\|u\|_1 = \max\{\|u\|_0, \|u^\Delta\|_0\}$.

An arbitrary non-empty closed subset of the real numbers is called a Time scales. As usual, we denote a time scales by $\mathbb{T}$. The examples of time scales are $\mathbb{R}, \mathbb{N}, h\mathbb{Z}$, where $h > 0$. A time scale interval such that $[a, b] = \{\theta \in \mathbb{T} : a \leqslant \theta \leqslant b\}$, accordingly, we define $(a, b), [a, b), (a, b]$ and so on. Also, $\mathbb{T}^\kappa = \mathbb{T} \setminus \{\max \mathbb{T}\}$ if $\max \mathbb{T}$ exists, otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

The forward jump operator $\sigma : \mathbb{T}^\kappa \to \mathbb{T}$ is defined by $\sigma(\theta) := \inf\{s \in \mathbb{T} : s > \theta\}$ with the substitution $\inf\{\phi\} = \sup \mathbb{T}$ and the graininess function $\mu : \mathbb{T}^\kappa \to [0, \infty)$ by $\mu(\theta) := \sigma(\theta) - \theta, \forall \theta \in \mathbb{T}^\kappa$.

Definition 2.1. (Bohner and Peterson, 2001) Let $\phi : \mathbb{T} \to \mathbb{R}$ and $\theta \in \mathbb{T}^\kappa$. The delta derivative $\phi^\Delta(\theta)$ is the number (when it exists) such that given any $\epsilon > 0$, there is a neighborhood U of θ such that

$$|[\phi(\sigma(\theta)) - \phi(\tau)] - \phi^\Delta(\theta)[\sigma(\theta) - \tau]| \leqslant \epsilon|\sigma(\theta) - \tau|, \quad \forall\ \tau \in U.$$

Definition 2.2. (Bohner and Peterson, 2001) Function Φ is called the antiderivative of $\phi : \mathbb{T} \to \mathbb{R}$ provided $\phi^\Delta(\theta) = \phi(\theta)$ for each $\theta \in \mathbb{T}^\kappa$, then the delta integral is defined by

$$\int_{\theta_0}^{\theta} \phi(z)\Delta z = \Phi(\theta) - \Phi(\theta_0).$$

A function $\phi : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous on $\mathbb{T}$, if ϕ is continuous at points $\theta \in \mathbb{T}$ with $\sigma(\theta) = \theta$ and has finite left-sided limits at points $\theta \in \mathbb{T}$ with

$$\sup\{r \in \mathbb{T} : r < \theta\} = \theta$$

and the set of all rd-continuous functions $\phi : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.3. (Bohner and Peterson, 2001) A function $w : \mathbb{T} \to \mathbb{R}$ is said to be regressive if $1 + \mu(\theta)w(\theta) \neq 0,\ \forall\ \theta \in \mathbb{T}$ and the set of all regressive functions are denoted by $\mathscr{R}$. Also, w is said to be positive regressive function if $1 + \mu(\theta)w(\theta) > 0, \forall\ \theta \in \mathbb{T}$ and the set of such type of functions are denoted by $\mathscr{R}^+$.

Theorem 2.4. *(Bohner and Peterson, 2001) Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous, nondecreasing function, and let $\mathbb{T}$ be an arbitrary time scale with $\theta_1, \theta_2 \in \mathbb{T}$, such that $\theta_1 \leqslant \theta_2$ then,*

$$\int_{\theta_1}^{\theta_2} \phi(z)\Delta z \leqslant \int_{\theta_1}^{\theta_2} \phi(z)dz. \tag{2}$$

Definition 2.5. (Ahmadkhanlu and Jahanshahi, 2012) Let $\phi : [a, b] \to \mathbb{R}$ is an integrable function, then delta fractional integral of ϕ is given by

$$^{\Delta}I_{a^+}^q \phi(\theta) = \int_a^\theta \frac{(\theta - s)^{q-1}}{\Gamma(q)} \phi(z)\Delta z, \tag{3}$$

where $\Gamma(q)$ denotes the usual Euler Gamma function.

Definition 2.6. (Ahmadkhanlu and Jahanshahi, 2012) Let $\phi : \mathbb{T} \to \mathbb{R}$. The Caputo delta fractional derivative of ϕ is denoted by ${}^c\Delta_{a^+}^q \phi(\theta)$ and defined by

$${}^c\Delta_{a^+}^q \phi(\theta) = \int_a^\theta \frac{(\theta - z)^{n-q-1}}{\Gamma(n - q)} \phi^{\Delta^n}(z)\Delta z, \tag{4}$$

where $n = [q] + 1$ and $[q]$ denotes the integer part of q.

Theorem 2.7. *(Ahmadkhanlu and Jahanshahi, 2012) Let $q \in (0, 1)$ and $f : I \times \mathbb{R} \to \mathbb{R}$ be a given function then the function $u(\theta)$ is a solution of*

$${}^c\Delta^q u(\theta) = f(\theta, u(\theta)),\ u(0) = u_0,$$

if and only if $u(\theta)$ is the solution of the following integral equation :

$$u(\theta) = u_0 + \frac{1}{\Gamma(q)} \int_0^\theta (\theta - z)^{q-1} f(z, u(z))\Delta z.$$

Lemma 2.8. Let $g : I \to \mathbb{R}$ be a right dense continuous function. Then, for any $k = 1, 2, \dots, p$, the solution of the following problem

$${}^c\Delta^q u(\theta) = g(\theta), \quad \theta \in \cup_{k=0}^p (\eta_k, \theta_{k+1}]_{\mathbb{T}}, \tag{5}$$

$$u(\theta) = \frac{1}{\Gamma(q)} \int_{\theta_k}^\theta (\theta - z)^{q-1} g_k(z, u(\theta_k^-))\Delta z, \quad \theta \in (\theta_k, \eta_k],\ k = 1, 2, \dots, p, \tag{6}$$

$$u(0) = u(T), \tag{7}$$

is given by the following integral equation

$$\begin{aligned} u(\theta) &= \frac{1}{\Gamma(q)} \int_{\theta_p}^{\eta_p} (\eta_p - z)^{q-1} g_p(z, u(\theta_p^-))\Delta z + \frac{1}{\Gamma(q)} \int_{\eta_p}^{T} (T - z)^{q-1} g(z)\Delta z \\ &\quad + \frac{1}{\Gamma(q)} \int_0^\theta (\theta - z)^{q-1} g(z)\Delta z, \quad \theta \in [0, \theta_1], \\ u(\theta) &= \frac{1}{\Gamma(q)} \int_{\theta_k}^\theta (\theta - z)^{q-1} g_k(z, u(\theta_k^-))\Delta z, \quad \forall\ \theta \in (\theta_k, \eta_k], \\ u(\theta) &= \frac{1}{\Gamma(q)} \int_{\theta_k}^{\eta_k} (\eta_k - z)^{q-1} g_k(z, u(\theta_k^-))\Delta z \\ &\quad + \frac{1}{\Gamma(q)} \int_{\eta_k}^\theta (\theta - z)^{q-1} g(z)\Delta z \quad \theta \in (\eta_k, \theta_{k+1}]. \end{aligned}$$

Proof. The proof is divided into three cases:

Case 1 : When $\theta \in (\eta_k, \theta_{k+1}]$, then from Theorem 2.7, we have

$$u(\theta) = u(\eta_k) + \frac{1}{\Gamma(q)} \int_{\eta_k}^\theta (\theta - z)^{q-1} g(z)\Delta z. \tag{8}$$

Therefore, from Eq. (6) and (8), we have

$$u(\theta)=\frac{1}{\Gamma(q)}\int_{\theta_k}^{\eta_k}(\eta_k-z)^{q-1}g_k(z,u(\theta_k^-))\Delta z + \frac{1}{\Gamma(q)}\int_{\eta_k}^{\theta}(\theta-z)^{q-1}g(z)\Delta z,\quad \theta\in(\eta_k,\theta_{k+1}]. \tag{9}$$

Case 2 : Similarly, when $\theta\in[0,\theta_1]$ we have

$$u(\theta)=u(0)+\frac{1}{\Gamma(q)}\int_0^{\theta}(\theta-z)^{q-1}g(z)\Delta z. \tag{10}$$

Now, at $\theta=T$ Eq. (9) becomes

$$u(T)=\frac{1}{\Gamma(q)}\int_{\theta_p}^{\eta_p}(\eta_p-z)^{q-1}g_p(z,u(\theta_p^-))\Delta z + \frac{1}{\Gamma(q)}\int_{\eta_p}^{T}(T-z)^{q-1}g(z)\Delta z. \tag{11}$$

Subsequently, using the Eqs. (7), (10) and (11) we get:

$$u(t)=\frac{1}{\Gamma(q)}\int_{\theta_p}^{\eta_p}(\eta_p-z)^{q-1}g_p(z,u(\theta_p^-))\Delta z + \frac{1}{\Gamma(q)}\int_{\eta_p}^{T}(T-z)^{q-1}g(z)\Delta z+\frac{1}{\Gamma(q)}\int_0^{\theta}(\theta-z)^{q-1}g(z)\Delta z.$$

Case 3 : When $\theta\in(\theta_k,\eta_k]$, it is given that

$$u(\theta)=\frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta-z)^{q-1}g_k(z,u(\theta_k^-))\Delta z.$$

Hence, the results follows. □

For $\epsilon>0,\psi\geqslant 0$, and nondecresing $\varphi\in PC(I,\mathbb{R}^+)$, consider the following inequalities

$$\begin{cases} \left|{}^c\Delta^q v(\theta)-\mathscr{M}(\theta,v(\theta),\mathscr{N}(v(\theta)))\right|\leqslant\epsilon, & \theta\in\cup_{k=0}^{p}(\eta_k,\theta_{k+1}]. \\ \left|v(\theta)-\frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta-z)^{q-1}g_k(z,v(\theta_k^-))\Delta z\right|\leqslant\epsilon, & \theta\in(\theta_k,\eta_k],\ k=1,2,\ldots,p. \end{cases} \tag{12}$$

$$\begin{cases} \left|{}^c\Delta^q v(\theta)-\mathscr{M}(\theta,v(\theta),\mathscr{N}(v(\theta)))\right|\leqslant\epsilon\varphi(\theta), & \theta\in\cup_{k=0}^{p}(\eta_k,\theta_{k+1}]. \\ \left|v(\theta)-\frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta-z)^{q-1}g_k(z,v(\theta_k^-))\Delta z\right|\leqslant\epsilon\psi, & \theta\in(\theta_k,\eta_k],\ k=1,2,\ldots,p. \end{cases} \tag{13}$$

Definition 2.9. (Wang et al., 2012) Equation (1) is said to be Ulam-Hyer's stable if there exist a positive constant $H_{(L_1,L_2,L_h,L_g)}$ such that for $\epsilon>0$ and for each solution v of inequality (12), there exist a unique solution u of equation (1) satisfies the following inequality

$$|v(\theta)-u(\theta)|\leqslant H_{(L_1,L_2,L_h,L_g)}\epsilon,\quad \forall\ \theta\in I.$$

Definition 2.10. (Wang et al., 2012) Equation (1) is said to be generalized Ulam-Hyer's stable if there exist $\mathscr{H}_{(L_1,L_2,L_h,L_g)}\in C(\mathbb{R}^+,\mathbb{R}^+)$,$\mathscr{H}_{(L_1,L_2,L_h,L_g)}(0)=0$ such that for each solution v of inequalities (12), there exists a solution u of equation (1) satisfies the following inequality

$$|v(\theta)-u(\theta)|\leqslant \mathscr{H}_{(L_1,L_2,L_h,L_g)}(\epsilon),\quad \forall\ \theta\in I.$$

Remark 2.11. Definition 2.9 ⇒Definition 2.10.

Definition 2.12. (Wang et al., 2012) Equation (1) is said to be Ulam-Hyers-Rassias stable with respect to (φ,ψ), if there exists $H_{(L_1,L_2,L_h,L_g,\varphi)}$ such that for $\epsilon>0$ and for each solution v of inequality (13), there exist a unique solution u of equation (1) satisfies the following inequality

$$|v(\theta)-u(\theta)|\leqslant H_{(L_1,L_2,L_h,L_g,\varphi)}\epsilon(\varphi(\theta),\psi),\quad \forall\ \theta\in I.$$

Lemma 2.13. A function $v\in PC^1(I,\mathbb{R})$ is a solution of inequality (12) if and only if there is $\mathsf{G},\mathsf{G}_k\in PC(I,\mathbb{R}),\ k=1,2,\ldots,p$ such that

(a) $|\mathsf{G}(\theta)|\leqslant\epsilon,\forall\ \theta\in\cup_{k=0}^{p}(\eta_k,\theta_{k+1}]$ and $|\mathsf{G}_k(\theta)|\leqslant\epsilon,\ \forall\ \theta\in(\theta_k,\eta_k], k=1,2,\ldots,p.$
(b) ${}^c\Delta^q v(\theta)=\mathscr{M}(\theta,v(\theta),\mathscr{N}(v(\theta)))+\mathsf{G}(\theta),\ \theta\in(\eta_k,\theta_{k+1}],\ k=0,1,\ldots,p.$
(c) $v(\theta)=\frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta-z)^{q-1}g_k(z,v(\theta_k^-))\Delta z+\mathsf{G}_k(\theta),\quad \theta\in(\theta_k,\eta_k], k=1,2,\ldots,p.$

Proof. Firstly, we suppose that $v\in PC^1(I,\mathbb{R})$ is the solution of inequality (12). We need to show that $(a),(b)$, and (c) are holds. For this, we set

$$\mathsf{G}(\theta)={}^c\Delta^q v(\theta)-\mathscr{M}(\theta,v(\theta),\mathscr{N}(v(\theta))),\ \ \theta\in(\eta_k,\theta_{k+1}],\ k=0,1,\ldots,p$$

and

$$\mathsf{G}_k(\theta)=v(\theta)-\frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta-z)^{q-1}g_k(z,v(\theta_k^-))\Delta z,\quad \theta\in(\theta_k,\eta_k],\ k=1,2,\ldots,p.$$

Consequently, one can easily see that $(a),(b)$ and (c) are satisfied. Conversely, from (b) we have

$$|{}^c\Delta^q v(\theta)-\mathscr{M}(\theta,v(\theta),\mathscr{N}(v(\theta)))|=|\mathsf{G}(\theta)|,\ \ \theta\in(\eta_k,\theta_{k+1}],\ k=0,1,\ldots,p.$$

Now, using (a) in the above equation, we get:

$$|{}^c\Delta^q v(\theta)-\mathscr{M}(\theta,v(\theta),\mathscr{N}(v(\theta)))|\leqslant\epsilon,\ \ \theta\in(\eta_k,\theta_{k+1}],\ \ k=0,1,\ldots,p.$$

Similarly, from (a) and (c), we get

$$|v(\theta)-\frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta-z)^{q-1}g_k(z,v(\theta_k^-))\Delta z|\leqslant\epsilon,\quad \theta\in(\theta_k,\eta_k],\ k=1,2,\ldots,p.$$ □

We have similar lemma for the inequality (13).
From the Lemma 2.13, we have

$$\begin{cases} {}^c\Delta^q v(\theta))=\mathscr{M}(\theta,v(\theta),\mathscr{N}(v(\theta)))+\mathsf{G}(\theta),\ \theta\in(\eta_k,\theta_{k+1}],\ k=0,1,\ldots,p. \\ v(\theta)=\frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta-z)^{q-1}g_k(z,v(\theta_k^-))\Delta z+\mathsf{G}_k(\theta),\quad \theta\in(\theta_k,\eta_k],\ k=1,2,\ldots,p. \end{cases}$$

Also, by Lemma 2.8, one can find that the solution v with $v(0)=v(T)$of the above equation is given by

$$v(\theta)=\frac{1}{\Gamma(q)}\int_{\theta_p}^{\eta_p}(\eta_p-z)^{q-1}g_p(z,v(\theta_p^-))\Delta z + \frac{1}{\Gamma(q)}\int_{\eta_p}^{T}(T-z)^{q-1}(\mathscr{M}(z,v(z),\mathscr{N}(v(z)))+\mathsf{G}(z))\Delta z + \mathsf{G}_p(\theta)+\frac{1}{\Gamma(q)}\int_0^{\theta}(\theta-z)^{q-1}(\mathscr{M}(z,v(z),\mathscr{N}(v(z))) + \mathsf{G}(z))\Delta z,\quad \forall\ \theta\in[0,\theta_1],$$

$$v(\theta)=\frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta-z)^{q-1}g_k(z,v(\theta_k^-))\Delta z+\mathsf{G}_k(\theta),\quad \theta\in(\theta_k,\eta_k],\ k=1,2,\ldots,p,$$

$$v(\theta)=\frac{1}{\Gamma(q)}\int_{\theta_k}^{\eta_k}(\eta_k-z)^{q-1}g_k(z,v(\theta_k^-))\Delta z + \frac{1}{\Gamma(q)}\int_{\eta_k}^{\theta}(\theta-z)^{q-1}(\mathscr{M}(z,v(z),\mathscr{N}(v(z)))+\mathsf{G}(z))\Delta z + \mathsf{G}_k(\theta),\qquad \forall\ \theta\in(\eta_k,\theta_{k+1}],\ k=1,2,\ldots,p.$$

Therefore, for $\theta \in (\eta_k, \theta_{k+1}],\ k = 1, 2, \ldots, p$, we have

$$\begin{aligned}&|v(\theta) - \frac{1}{\Gamma(q)}\int_{\theta_k}^{\eta_k}(\eta_k - z)^{q-1}g_k(z, v(\theta_k^-))\Delta z\\&-\frac{1}{\Gamma(q)}\int_{\eta_k}^{\theta}(\theta - z)^{q-1}\mathscr{M}(z, v(z), \mathscr{N}(v(z)))\Delta z|\\&\leqslant |\mathsf{G}_k(\theta)| + \frac{1}{\Gamma(q)}\int_{\eta_k}^{\theta}(\theta - z)^{q-1}|\mathsf{G}(z)|\Delta z\\&\leqslant \epsilon\left(1 + \frac{T^q}{\Gamma(q+1)}\right).\end{aligned}$$

Also, for $\theta \in [0, \theta_1]$,

$$\begin{aligned}&\left|v(\theta) - \frac{1}{\Gamma(q)}\int_{\theta_p}^{\eta_p}(\eta_p - z)^{q-1}\right.\\&\left.+\frac{1}{\Gamma(q)}\int_0^{\theta}(\theta - z)^{q-1}\mathscr{M}(z, v(z), \mathscr{N}(v(z)))\Delta z\right|\\&\leqslant |\mathsf{G}_p(\theta)| + \frac{1}{\Gamma(q)}\int_{\eta_p}^{T}(T - z)^{q-1}\mathsf{G}(z)\Delta z + \frac{1}{\Gamma(q)}\int_0^{\theta}(\theta - z)^{q-1}\mathsf{G}(z)\Delta z\\&\leqslant \epsilon\left(1 + \frac{2T^q}{\Gamma(q+1)}\right).\end{aligned}$$

Similarly, when $\theta \in (\theta_k, \eta_k],\ k = 1, 2, \ldots, p$,

$$|v(\theta) - \frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta - z)^{q-1}g_k(z, v(\theta_k^-))\Delta z| \leqslant \epsilon. \tag{14}$$

To prove our main results, we consider the following assumptions:

(H1): Function $\mathscr{M} : J_1 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ J_1 = \cup_{k=0}^{p}[\eta_k, \theta_{k+1}]$ is continuous and $\exists$ positive constants $L_1,\ L_2, C_1,\ M_1$ and N_1 such that
1. $|\mathscr{M}(\theta, u_1, u_2) - \mathscr{M}(\theta, v_1, v_2)| \leqslant L_1|u_1 - v_1| + L_2|u_2 - v_2|$, $\forall\ \theta \in I,\ u_j, v_j \in \mathbb{R},\ j = 1, 2.$
2. $|\mathscr{M}(\theta, u, v)| \leqslant C_1 + M_1|u| + N_1|v|, \quad \forall\ \theta \in I,\ u, v \in \mathbb{R}.$

(H2): $h : \mathscr{Q} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\exists$ positive constants $L_h, C_2,\ M_2$ such that
1. $|h(\theta, s, u) - h(\theta, z, v)| \leqslant L_h|u - v|, \quad \forall\ \theta, s \in \mathscr{Q},\ u, v \in \mathbb{R}.$
2. $|h(\theta, s, u)| \leqslant C_2 + M_2|u|, \quad \forall\ \theta, s \in \mathscr{Q},\ u \in \mathbb{R}.$

(H3): The functions $g_k : I_k \times \mathbb{R} \to \mathbb{R},\ I_k = [\theta_k, \eta_k],\ k = 1, 2, \ldots, p$, are continuous and $\exists$ a positive constants $L_g,\ M_g$ such that
1. $|g_k(\theta, u) - g_k(\theta, v)| \leqslant L_g|u - v|, \quad \forall\ u, v \in \mathbb{R}$, $\theta \in I_k,\ k = 1, 2, \ldots, p.$
2. $|g_k(\theta, u)| \leqslant M_g, \forall\ \theta \in I_k$ and $u \in \mathbb{R}$.

(H4): $\frac{2T^q(M_1+N_1M_2T)}{\Gamma(q+1)} < 1.$

3. Existence and uniqueness of solutions

In this section, we establish our main results for the Eq. (1). These results are carried out using the Banach contraction theorem.

Theorem 3.1. *If the assumptions* **(H1)–(H4)** *are satisfied, then Eq. (1) has a unique solution provided*

$$\frac{T^q}{\Gamma(q+1)}\left(L_g + 2(L_1 + L_2L_hT)\right) < 1.$$

Proof. Consider $\mathscr{B} \subseteq PC(I, \mathbb{R})$ such that

$$\mathscr{B} = \{u \in PC(I, \mathbb{R}) : \|u\|_0 \leqslant \beta\},$$

where

$$\beta = \frac{T^q(M_g + 2(C_1 + N_1C_2T))}{\Gamma(q+1) - 2T^q(M_1 + M_2T)}.$$

Now, define an operator $F : \mathscr{B} \to \mathscr{B}$ given by

$$\begin{aligned}(\Xi u)(\theta) &= \frac{1}{\Gamma(q)}\int_{\theta_p}^{\eta_p}(\eta_p - z)^{q-1}g_p(z, u(\theta_p^-))\Delta z\\&\quad+\frac{1}{\Gamma(q)}\int_{\eta_p}^{T}(T - z)^{q-1}\mathscr{M}(z, u(z), \mathscr{N}(u(z))))\Delta z\\&\quad+\frac{1}{\Gamma(q)}\int_0^{\theta}(\theta - z)^{q-1}\mathscr{M}(z, u(z), \mathscr{N}(u(z)))\Delta z, \quad \forall\ \theta \in [0, \theta_1],\\(\Xi u)(\theta) &= \frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta - z)^{q-1}g_k(z, u(\theta_k^-))\Delta z,\\&\quad\forall\ \theta \in (\theta_k, \eta_k],\ k = 1, 2, \ldots, p,\\(\Xi u)(\theta) &= \frac{1}{\Gamma(q)}\int_{\theta_k}^{\eta_k}(\eta_k - z)^{q-1}g_k(z, u(\theta_k^-))\Delta z\\&\quad+\frac{1}{\Gamma(q)}\int_{\eta_k}^{\theta}(\theta - z)^{q-1}\mathscr{M}(z, u(z), \mathscr{N}(u(z)))\Delta z,\\&\quad\forall\ \theta \in (\eta_k, \theta_{k+1}],\ k = 1, 2, \ldots, p.\end{aligned}$$

To use the Banach fixed point theorem, we have to show that $\Xi : \mathscr{B} \to \mathscr{B}$. For $\theta \in (\eta_k, \theta_{k+1}], k = 1, 2, \ldots, p$ and $u \in \mathscr{B}$, we have

$$\begin{aligned}|(\Xi u)(\theta)| &\leqslant \frac{1}{\Gamma(q)}\int_{\theta_k}^{\eta_k}(\eta_k - z)^{q-1}|g_k(z, u(\theta_k^-))|\Delta z\\&\quad+\frac{1}{\Gamma(q)}\int_{\eta_k}^{\theta}(\theta - z)^{q-1}|\mathscr{M}(z, u(z), \mathscr{N}(u(z)))|\Delta z\\&\leqslant \frac{M_g}{\Gamma(q)}\int_{\theta_k}^{\eta_k}(\eta_k - z)^{q-1}\Delta z + \frac{1}{\Gamma(q)}\int_{\eta_k}^{\theta}(\theta - z)^{q-1}(C_1 + M_1|u(z)|\\&\quad+N_1|\mathscr{N}(u(z))|)\Delta z\\&\leqslant \frac{M_g(\eta_k - \theta_k)^q}{\Gamma(q+1)} + \frac{(C_1 + M_1\beta)(\theta_{k+1} - \eta_k)^q}{\Gamma(q+1)}\\&\quad+\frac{N_1(C_2 + M_2\beta)\theta_{k+1}(\theta_{k+1} - \eta_k)^q}{\Gamma(q+1)}\\&\leqslant \frac{T^q}{\Gamma(q+1)}\left(M_g + (C_1 + M_1\beta) + N_1(C_2 + M_2\beta)T\right).\end{aligned}$$

Hence,

$$\|\Xi u\|_0 \leqslant \frac{T^q}{\Gamma(q+1)}\left(M_g + (C_1 + N_1C_2T) + (M_1 + N_1M_2T)\beta\right). \tag{15}$$

Also, for $\theta \in [0, \theta_1]$ and $u \in \mathscr{B}$, we have

$$\begin{aligned}|(\Xi u)(\theta)| &\leqslant \frac{1}{\Gamma(q)}\int_{\theta_p}^{\eta_p}(\eta_p - z)^{q-1}|g_p(z, u(\theta_p^-))|\Delta z\\&\quad+\frac{1}{\Gamma(q)}\int_{\eta_p}^{T}(T - z)^{q-1}|\mathscr{M}(z, u(z), \mathscr{N}(u(z))))|\Delta z\\&\quad+\frac{1}{\Gamma(q)}\int_0^{\theta}(\theta - z)^{q-1}|\mathscr{M}(z, u(z), \mathscr{N}(u(z)))|\Delta z\\&\leqslant \frac{M_g}{\Gamma(q)}\int_{\theta_p}^{\eta_p}(\eta_p - z)^{q-1}\Delta z\\&\quad+\frac{1}{\Gamma(q)}\int_{\eta_p}^{T}(T - z)^{q-1}(C_1 + M_1|u(z)| + N_1|\mathscr{N}(u(z))|)\Delta z\\&\quad+\frac{1}{\Gamma(q)}\int_0^{\theta}(\theta - z)^{q-1}(C_1 + M_1|u(z)| + N_1|\mathscr{N}(u(z))|)\Delta z\\&\leqslant \frac{M_g(\eta_p - \theta_p)^q}{\Gamma(q+1)} + \frac{(C_1 + M_1\beta)(T - \eta_p)^q}{\Gamma(q+1)}\\&\quad+\frac{N_1(C_2 + M_2\beta)T(T - \eta_p)^q}{\Gamma(q+1)}\\&\quad+\frac{(C_1 + M_1\beta)(\theta_1)^q}{\Gamma(q+1)} + \frac{N_1(C_2 + M_2\beta)(\theta_1)^{q+1}}{\Gamma(q+1)}.\end{aligned}$$

Hence,

$$\|\Xi u\|_0 \leqslant \frac{T^q}{\Gamma(q+1)}\left(M_g + 2(C_1 + N_1C_2T) + 2(M_1 + N_1M_2T)\beta\right). \tag{16}$$

Similarly, for $\theta \in (\theta_k, \eta_k],\ k = 1, 2, \ldots, p$and $u \in \mathscr{B}$, we have

$$\|\Xi u\|_0 = \frac{M_g T^q}{\Gamma(q+1)}. \tag{17}$$

From the inequalities (15), (16) and (17), we get:

$$\|\Xi u\|_0 \leqslant \beta.$$

Therefore, $\Xi : \mathscr{B} \to \mathscr{B}$. Now, for any $u, v \in \mathscr{B}$, $\theta \in (\eta_k, \theta_{k+1}], k = 1, 2, \ldots, p$, we have

$$\begin{aligned}
|(\Xi u)(\theta) - (\Xi v)(\theta)| &\leqslant \frac{1}{\Gamma(q)}\int_{\theta_k}^{\eta_k}(\eta_k - z)^{q-1}|g_k(z, u(\theta_k^-)) \\
&\quad - g_k(z, v(\theta_k^-))|\Delta z \\
&\quad + \frac{1}{\Gamma(q)}\int_{\eta_k}^{\theta}(\theta - z)^{q-1}|\mathscr{M}(z, u(z), \mathscr{N}(u(z))) \\
&\quad - \mathscr{M}(z, v(z), \mathscr{N}(v(z)))|\Delta z \\
&\leqslant \frac{L_g\|u - v\|_0}{\Gamma(q)}\int_{\theta_k}^{\eta_k}(\eta_k - z)^{q-1}\Delta z + \frac{L_1}{\Gamma(q)}\|u - v\|_0\int_{\eta_k}^{\theta}(\theta - z)^{q-1}\Delta z \\
&\quad + \frac{L_2}{\Gamma(q)}\int_{\eta_k}^{\theta}(\theta - z)^{q-1}|\mathscr{N}(u(z)) - \mathscr{N}(v(z))|\Delta z \\
&\leqslant \frac{L_g(\eta_k - \theta_k)^q\|u - v\|_0}{\Gamma(q+1)} + \frac{L_1\|u - v\|_0(\theta_{k+1} - \eta_k)^q}{\Gamma(q+1)} \\
&\quad + \frac{L_2 L_h \theta_{k+1}(\theta_{k+1} - \eta_k)^q\|u - v\|_0}{\Gamma(q+1)}.
\end{aligned}$$

Thus,

$$\|\Xi u - \Xi v\|_0 \leqslant \frac{T^q}{\Gamma(q+1)}[L_g + L_1 + L_2 L_h T]\|u - v\|_0. \tag{18}$$

Also, for any $u, v \in \mathscr{B}, \ \theta \in [0, \theta_1]$, we get:

$$\begin{aligned}
&|(\Xi u)(\theta) - (\Xi v)(\theta)| \\
&\leqslant \frac{1}{\Gamma(q)}\int_{\theta_p}^{\eta_p}(\eta_p - z)^{q-1}|g_p(z, u(\theta_p^-)) - g_p(z, v(\theta_p^-))|\Delta z \\
&\quad + \frac{1}{\Gamma(q)}\int_{\eta_p}^{T}(T - z)^{q-1}|\mathscr{M}(z, u(z), \mathscr{N}(u(z))) - \mathscr{M}(z, v(z), \mathscr{N}(v(z))))|\Delta z \\
&\quad + \frac{1}{\Gamma(q)}\int_{0}^{\theta}(\theta - z)^{q-1}|\mathscr{M}(z, u(z), \mathscr{N}(u(z))) - \mathscr{M}(z, v(z), \mathscr{N}(v(z))))|\Delta z \\
&\leqslant \frac{L_g|u(\theta_p^-) - v(\theta_p^-)|}{\Gamma(q)}\int_{\theta_p}^{\eta_p}(\eta_p - z)^{q-1}\Delta z + \frac{L_1}{\Gamma(q)}\int_0^{\theta}(\theta - z)^{q-1}|u(z) - v(z)|\Delta z \\
&\quad + \frac{L_2}{\Gamma(q)}\int_0^{\theta}(\theta - z)^{q-1}|\mathscr{N}(u(z)) - \mathscr{N}(v(z))|\Delta z \\
&\quad + \frac{L_2}{\Gamma(q)}\int_{\eta_p}^{T}(T - z)^{q-1}|\mathscr{N}(u(z)) - \mathscr{N}(v(z))|\Delta z \\
&\quad + \frac{L_1}{\Gamma(q)}\int_{\eta_p}^{T}(T - z)^{q-1}|u(z) - v(z)|\Delta z \\
&\leqslant \frac{L_g(\eta_p - \theta_p)^q\|u - v\|_0}{\Gamma(q+1)} + \frac{L_1(T - \eta_p)^q\|u - v\|_0}{\Gamma(q+1)} + \frac{L_1\theta_1^{q+1}\|u - v\|_0}{\Gamma(q+1)} \\
&\quad + \frac{L_2 L_h T(T - \eta_p)^q\|u - v\|_0}{\Gamma(q+1)} + \frac{L_2 L_h \theta_1^{q+1}\|u - v\|_0}{\Gamma(q+1)}.
\end{aligned}$$

Therefore,

$$\|\Xi u - \Xi v\|_0 \leqslant \frac{T^q}{\Gamma(q+1)}[L_g + 2(L_1 + L_2 L_h T)]\|u - v\|_0. \tag{19}$$

Similarly, for $\theta \in (\theta_k, \eta_k], \ k = 1, 2, \ldots, p$ and $u \in \mathscr{B}$, we have

$$\|\Xi u - \Xi v\|_0 \leqslant \frac{L_g T^q}{\Gamma(q+1)}\|u - v\|_0. \tag{20}$$

From the above inequalities (18, 19, 24), we get:

$$\|\Xi u - \Xi v\|_0 \leqslant L_\Xi\|u - v\|_0,$$

where

$$L_\Xi = \frac{T^q}{\Gamma(q+1)}[L_g + 2(L_1 + L_2 L_h T)].$$

Hence, Ξ is a strict contraction mapping. Therefore, Ξ has a unique fixed point which is the solution of the Eq. (1). □

Let us consider a special case when $\mathscr{M}(\theta, u(\theta), \mathscr{N}(u(\theta))) = \mathscr{P}(\theta, u) + \int_0^\theta h(\theta, z, u(z))\Delta z$ then the Eq. (1) becomes:

$$\begin{aligned}
&{}^c\Delta^q u(\theta) = \mathscr{P}(\theta, u) + \int_0^\theta h(\theta, z, u(z))\Delta z, \quad \theta \in \cup_{k=0}^{p}(\eta_k, \theta_{k+1}]_{\mathbb{T}}, \\
&u(\theta) = \frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta - z)^{q-1}g_k(z, u(\theta_k^-))\Delta, \quad \theta \in (\theta_k, \eta_k]_{\mathbb{T}}, \\
&k = 1, 2, \ldots, p, \\
&u(0) = u(T),
\end{aligned} \tag{21}$$

(H5): The non-linear function $\mathscr{P} : J_1 \times \mathbb{R} \to \mathbb{R}$ is continuous and $\exists$ positive constants $L_{\mathscr{P}}, C_{\mathscr{P}}$ and $M_{\mathscr{P}}$ such that
 (a) $|\mathscr{P}(\theta, u) - \mathscr{P}(\theta, v)| \leqslant L_{\mathscr{P}}|u - v|, \quad \forall\ \theta \in I, \ u, v \in \mathbb{R}$.
 (b) $|\mathscr{P}(\theta, u)| \leqslant C_{\mathscr{P}} + M_{\mathscr{P}}|u|, \quad \forall\ \theta \in I, \ u \in \mathbb{R}$.

(H6): $\frac{2T^q(M_{\mathscr{P}} + M_2 T)}{\Gamma(q+1)} < 1$.

Corollary 1. If the assumptions **(H2)**, **(H3)**, **(H5)** and **(H6)** are satisfied, then the equation (21) has a unique solution provided

$$\frac{T^q}{\Gamma(q+1)}(L_g + 2(L_{\mathscr{P}} + L_h T)) < 1.$$

4. Hyer-Ulam's stability

Theorem 4.1. *If the assumptions of the Theorem 3.1 are satisfied, then the equation (1) is Ulam-Hyer's stable.*

Proof. Let $v \in PC^1(I, \mathbb{R})$ be the solution of inequality (12) and u be a unique solution of the equation (1). Therefore, for $\theta \in (\eta_k, \theta_{k+1}], \ k = 1, 2, \ldots, p,$we have

$$\begin{aligned}
|v(\theta) - u(\theta)| &\leqslant |v(\theta) - \frac{1}{\Gamma(q)}\int_{\theta_k}^{\eta_k}(\eta_k - z)^{q-1}g_k(z, u(\theta_k^-))\Delta z \\
&\quad - \frac{1}{\Gamma(q)}\int_{\eta_k}^{\theta}(\theta - z)^{q-1}\mathscr{M}(z, u(z), \mathscr{N}(u(z)))\Delta z| \\
&\leqslant \epsilon\left(1 + \frac{T^q}{\Gamma(q+1)}\right) + \frac{L_g\|v - u\|_0}{\Gamma(q)}\int_{\theta_k}^{\eta_k}(\eta_k - z)^{q-1}\Delta z \\
&\quad + \frac{L_1}{\Gamma(q)}\|v - u\|_0\int_{\eta_k}^{\theta}(\theta - z)^{q-1}\Delta z \\
&\quad + \frac{L_2}{\Gamma(q)}\int_{\eta_k}^{\theta}(\theta - z)^{q-1}|\mathscr{N}(u(z)) - \mathscr{N}(v(z))|\Delta z \\
&\leqslant \epsilon\left(1 + \frac{T^q}{\Gamma(q+1)}\right) + \frac{L_g(\eta_k - \theta_k)^q\|v - u\|_0}{\Gamma(q+1)} + \frac{L_1\|v - u\|_0(\theta_{k+1} - \eta_k)^q}{\Gamma(q+1)} \\
&\quad + \frac{L_2 L_h \theta_{k+1}(\theta_{k+1} - \eta_k)^q\|v - u\|_0}{\Gamma(q+1)}.
\end{aligned}$$

Thus,

$$\|v - u\|_0 \leqslant \epsilon\left(1 + \frac{T^q}{\Gamma(q+1)}\right) + \frac{T^q}{\Gamma(q+1)}[L_g + L_1 + L_2 L_h T]\|v - u\|_0. \tag{22}$$

Also, for $\theta \in [0, \theta_1]$, we have

$$
\begin{aligned}
|v(\theta)-u(\theta)| \leqslant |&\frac{1}{\Gamma(q)}\int_{\theta_p}^{\eta_p}(\eta_p-z)^{q-1}g_p(z,u(\theta_p^-))\Delta z\\
&-\frac{1}{\Gamma(q)}\int_{\eta_p}^{T}(T-z)^{q-1}\mathscr{M}(z,u(z),\mathscr{N}(u(z))))\Delta z\\
&-\frac{1}{\Gamma(q)}\int_{0}^{\theta}(\theta-z)^{q-1}\mathscr{M}(z,u(z),\mathscr{N}(u(z)))\Delta z|\\
&\leqslant \epsilon\left(1+\frac{2T^q}{\Gamma(q+1)}\right)+\frac{L_g(\eta_p-\theta_p)^q\|v-u\|_0}{\Gamma(q+1)}\\
&+\frac{L_1(T-\eta_p)^q\|v-u\|_0}{\Gamma(q+1)}\\
&+\frac{L_2L_hT(T-\eta_p)^q\|v-u\|_0}{\Gamma(q+1)}+\frac{L_1\theta_1^{q+1}\|v-u\|_0}{\Gamma(q+1)}\\
&+\frac{L_2L_h\theta_1^{q+1}\|v-u\|_0}{\Gamma(q+1)}.
\end{aligned}
$$

Therefore,

$$\|v-u\|_0 \leqslant \epsilon\left(1+\frac{2T^q}{\Gamma(q+1)}\right)+\frac{T^q}{\Gamma(q+1)}\left[L_g+2(L_1+L_2L_hT)\right]\|v-u\|_0. \tag{23}$$

Similarly, for $\theta \in (\theta_k,\eta_k],\ k=1,2,\ldots,p$, we can easily find

$$|v(\theta)-u(\theta)| \leqslant \epsilon+\frac{L_g(\eta_k-\theta_k)^q\|v-u\|_0}{\Gamma(q+1)}.$$

Therefore,

$$\|v-u\|_0 \leqslant \epsilon+\frac{L_gT^q}{\Gamma(q+1)}\|v-u\|_0. \tag{24}$$

From the above inequalities (22), (23) and (24), we get:

$$\|v-u\|_0 \leqslant \epsilon\left(1+\frac{2T^q}{\Gamma(q+1)}\right)+\frac{T^q}{\Gamma(q+1)}\left[L_g+2(L_1+L_2L_hT)\right]\|v-u\|_0,\ \forall\,\theta\in I.$$

Thus,

$$\|v-u\|_0 \leqslant H_{(L_1,L_2,L_h,L_g)}\epsilon, \quad \theta\in I,$$

where $H_{(L_1,L_2,L_h,L_g)}=\frac{1}{1-L_\Xi}\left(1+\frac{2T^q}{\Gamma(q+1)}\right)>0$. Thus, the Eq. (1) is Ulam-Hyer's stable. Moreover, if we put $\mathscr{H}_{(L_1,L_2,L_h,L_g)}(\epsilon)=H_{(L_1,L_2,L_h,L_g)}\epsilon$, $\mathscr{H}_{(L_1,L_2,L_h,L_g)}(0)=0$, then the Eq. (1) is generalized Ulam-Hyer's stable. □

In order to prove our next result, we need the following assumption:

(H7): There exists a $\lambda_\varphi>0$ such that ${}^{\Delta}I^q\varphi(\theta)\leqslant\lambda_\varphi\varphi(\theta),\ \forall\,\theta\in I.$

The following theorem is the consequence of the Theorem 4.1.

Theorem 4.2. *If the assumptions of Theorem 3.1 and* **(H7)** *are satisfied, then the equation (1) is Ulam-Hyers-Rassias stable.*

5. An example

Example 5.1. Consider the following equation with impulses on the general time scale $\mathbb{T}, (0, 15/21, 20/21, 1 \in \mathbb{T})$

$$
\begin{aligned}
&{}^c\Delta^q u(\theta)=\frac{3+|u(\theta)|}{40e^{\theta^2+3}(1+|u(\theta)|)}+\frac{1}{20}\int_0^\theta\frac{\theta s^2\sin(u(s))}{e^{s^2+5}}\Delta s,\\
&\quad\theta\in I'=[0,1]_{\mathbb{T}}\setminus(\theta_1,\eta_1],\\
&\quad u(\theta)=\frac{1}{\Gamma(q)}\int_{\theta_1}^{\theta}\frac{(\theta-z)^{q-1}(1+z\sin(u(\theta_1^-)))}{25}\Delta z,\quad \theta\in(\theta_1,\eta_1],\\
&\quad u(0)=u(1).
\end{aligned} \tag{25}
$$

Set,

$$\mathscr{M}(\theta,u,v)=\frac{3+|u(\theta)|}{40e^{\theta^2+3}(1+|u(\theta)|)}+\frac{1}{20}v,\ \ \theta\in I',\ \ u,v\in\mathbb{R},$$

$$h(\theta,s,u)=\frac{\theta s^2\sin(u(s))}{e^{s^2+5}},\ \forall\ \theta,s\in I',u\in\mathbb{R}$$

and

$$g_1(\theta,u)=\frac{1+\theta\sin(u(\theta_1^-))}{25},\ \ \theta\in(\theta_1,\eta_1],\ \ u\in\mathbb{R}.$$

Then, the assumptions (H1)-(H4) are holds with $L_1=\frac{1}{40e^3}, L_2=\frac{1}{20}$, $C_1=\frac{3}{40e^3}$, $M_1=\frac{1}{40e^3}$, $N_1=\frac{1}{20}$, $L_h=\frac{1}{e^5}$, $C_2=\frac{1}{e^5}$, $M_2=\frac{1}{e^5}, L_g=\frac{1}{25}$, $M_g=\frac{2}{25}$. Also, for $p=1,\ \theta_1=15/21,\ \eta_1=20/21,\ \ T=1,\ q=1/2$ the condition

$$\frac{T^q}{\Gamma(q+1)}(L_g+2(L_1+L_2L_hT))=\frac{1}{\Gamma(3/2)}\left(\frac{1}{25}+2\left(\frac{1}{40e^3}+\frac{1}{20e^5}\right)\right)<1$$

holds. Therefore, the coditions of the Theorem 3.1 is satisfied. Hence, Eq. (25) has a unique solution which is Ulam Hyer's stable.

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