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ORIGINAL ARTICLE

A variational approach for soliton solutions of good Boussinesq equation

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Abstract In this paper, we apply He's semi-inverse method to establish a variational theory for the good Boussinesq equation. Based on the obtained variational principle, a solitary wave solution is obtained. Moreover, the results are also compared with He's homotopy perturbation method (HPM). It is observed that the proposed algorithm is easier to implement, user friendly and highly accurate. Moreover, it is observed that the suggested technique is compatible with physical nature of such problems.

1. Introduction

In recent years, searching for solitary wave and soliton solutions of nonlinear wave systems play an important role in the study of nonlinear physical phenomena. The wave phenomena are observed in number of physical problems including elastic media, fluid dynamics, optic fibers, plasma. Several techniques including Adomian's decomposition (Adomian, 1988; Wazwaz, 1988; El-Sayed, 2003), homotopy perturbation method (He, 1999b, 2005b,a, 2004b,c; Öziş and Yildirim, 2007e,f,a,b; Yildirim and Öziş, 2007), exp function and variational iteration (He, 1999a, 2000; He and Wu, 2006; Öziş and Yildirim, 2007c; Öziş et al., in press; Wazwaz, 2007) have been used to search traveling wave solutions for various nonlinear wave equations. It is pertinent to highlight that He (2006a) made a complete on the field. Moreover, variational approach to solitary solutions was first introduced by He (2006b) in his famous review article. The basic motivation of this paper is the implementation of He's variational approach (He, 2006b, 1997, 2004a, 2005; Öziş and Yıldırım, 2007d; Tao, in press) for the good Boussinesq equation (Mohyud-Din et al., 2009; Mohyud-Din and Noor, 2009; Mohyud-Din et al., 2008; Noor et al., 2008) which describes shallow water waves propagating in both directions, has been proposed as a model for propagation of pulses along a transmission line made of a large number of LC-circuits and to describe vibrations of a single one-dimensional dense lattice. In addition, this equation arises in elasticity for longitudinal waves in bars, long water waves, acoustic waves on elastic rods and plasma waves. It is worthmentioning that good Boussinesq equation is a special case of Boussinesq equation:

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$$u_{tt} = u_{xx} + qu_{xxxx} + r(u^2)_{xx}, \quad x \in R, \tag{1}$$

with $q = -1$, and $r = 1$, and has the following form:

$$u_{tt} = u_{xx} - u_{xxxx} + (u^2)_{xx}, \quad x \in R, \tag{2}$$

It is to be highlighted that variational methods (He et al., 2010) always lead to approximate solutions in the required forms. Recently, many authors search for exact solutions without considering their physical understandings, for example, $u = 1$ is the exact solution of kdv equation, but it has no meaning, see He et al. (2010) and the references therein.

2. He's semi-inverse method

In the past few decades, qualitative analysis together with ingenious mathematical techniques for handling various nonlinear problems has been studied. Among them, a variational approach, such as the semi-inverse method (He, 2006b, 1997, 2004a; Liu, 2005; Öziş and Yıldırım, 2007d; Tao, in press) is a powerful and effective method to search for variational principles for physical problems and provides physical insight into the nature of the solution of problem. It should be pointed out that He (2006b) first applied the proposed method to search for solitary solution for KdV equation. In this paper, we consider 'good' Boussinesq equation in the following form:

$$u_{tt} = u_{xx} - u_{xxxx} + (u^2)_{xx}, \quad x \in R, \tag{2}$$

In order to seek its travelling wave solution, we introduce a transformation

$$u(x,t) = u(\xi), \tag{3}$$

$$\xi = x + ct, \tag{4}$$

where c is arbitrary constant. Substituting Eqs. (3) and (4) into Eq. (2) yields

$$u^{(4)} + (c^2 - 1)u'' - (u^2)'' = 0, \tag{5}$$

where the prime expresses the derivative with respect to ξ. Integrating the Eq. (5) twice, we have

$$u'' + (c^2 - 1)u - u^2 + a_1\xi + a_2 = 0, \tag{6}$$

where a_1 and a_2 are the constants of integration. We set $a_1 = 0$ and $a_2 = 0$ for simplicity, Eq. (6) reduces

$$u'' + (c^2 - 1)u - u^2 = 0, \tag{7}$$

By He's semi-inverse method, the following variational formulation is established

$$J = \int_0^\infty \left[\frac{1}{2}(u')^2 + (1 - c^2)\frac{u^2}{2} + \frac{u^3}{3}\right] d\xi \tag{8}$$

By Ritz-like method, we search for a solitary wave solution in the form

$$u = p \sec h^2(q\xi), \tag{9}$$

where p and q are constants to be further determined. Substituting Eq. (9) into Eq. (8) results in

$$J = \int_0^\infty \left[2p^2q^2 \sec h^4(q\xi)\tanh^2(q\xi) + \left(\frac{1-c^2}{2}\right)p^2 \sec h^4(q\xi) + \frac{1}{3}p^3 \sec h^6(q\xi)\right] d\xi \tag{10}$$

$$= \frac{4p^2q}{15} + \frac{(1-c^2)p^2}{3q} + \frac{8p^3}{45q} \tag{11}$$

Making J stationary with respect to p and q results in

$$\frac{\partial J}{\partial p} = \frac{8pq}{15} + \frac{2(1-c^2)p}{3q} + \frac{24p^2}{45q} = 0 \tag{12}$$

$$\frac{\partial J}{\partial q} = \frac{4p^2}{15} - \frac{(1-c^2)p^2}{3q^2} - \frac{8p^3}{45q^2} = 0 \tag{13}$$

From Eqs. (12) and (13), we can easily obtain the following relations:

$$p = \frac{3}{2}(c^2 - 1) \text{ and } q = \frac{\sqrt{1-c^2}}{2} \tag{14}$$

The solitary solution is, therefore, obtained as follows

$$u(x,t) = \frac{3}{2}(c^2 - 1)\sec h^2\left[\frac{\sqrt{1-c^2}}{2}(x + ct)\right] \tag{15}$$

3. Applying homotopy prturbation method (HPM) on good Boussinesq equation

Now, we again consider the good Boussinesq equation

$$u_{tt} = u_{xx} - u_{xxxx} + (u^2)_{xx} \quad x \in R \tag{16}$$

In order to solve Eq. (16) by HPM, we choose the initial approximation;

$$u_0(x,t) = g(x), \tag{17}$$

and construct the following homotopy:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = p\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u_0}{\partial t^2}\right), \tag{18}$$

Assume the solution of Eq. (18) in the form:

$$u(x,t) = u_0(x,t) + pu_1(x,t) + p^2u_2(x,t) + p^3u_3(x,t) + \cdots \tag{19}$$

Substituting Eq. (19) into Eq. (18) and collecting terms of the same power of p gives:

$$p^0 : \frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = 0, \tag{20}$$

$$p^1 : \frac{\partial^2 u_1}{\partial t^2} = \frac{\partial^2 u_0}{\partial x^2} - \frac{\partial^4 u_0}{\partial x^4} + 2\left(\frac{\partial u_0}{\partial x}\right)^2 + 2u_0\frac{\partial^2 u_0}{\partial x^2}, \tag{21}$$

$$p^2 : \frac{\partial^2 u_2}{\partial t^2} = \frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^4 u_1}{\partial x^4} + 4\frac{\partial u_0}{\partial x}\frac{\partial u_1}{\partial x} + 2u_1\frac{\partial^2 u_0}{\partial x^2} + 2u_0\frac{\partial^2 u_1}{\partial x^2}, \tag{22}$$

$$p^3 : \frac{\partial^2 u_3}{\partial t^2} = \frac{\partial^2 u_2}{\partial x^2} - \frac{\partial^4 u_2}{\partial x^4} + 2\left(\left(\frac{\partial u_1}{\partial x}\right)^2 + 2\frac{\partial u_0}{\partial x}\frac{\partial u_2}{\partial x} + u_2\frac{\partial^2 u_0}{\partial x^2} + u_1\frac{\partial^2 u_1}{\partial x^2} + u_0\frac{\partial^2 u_2}{\partial x^2}\right), \tag{23}$$

$$p^4 : \frac{\partial^2 u_4}{\partial t^2} = \frac{\partial^2 u_3}{\partial x^2} - \frac{\partial^4 u_3}{\partial x^4} + 2\left(2\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x} + 2\frac{\partial u_0}{\partial x}\frac{\partial u_3}{\partial x} + u_3\frac{\partial^2 u_0}{\partial x^2} + u_2\frac{\partial^2 u_1}{\partial x^2} + u_1\frac{\partial^2 u_2}{\partial x^2} + u_0\frac{\partial^2 u_3}{\partial x^2}\right), \tag{24}$$

$$\vdots$$

We can start with $u_0(x,t) = g(x)$, and all the linear equations above can be easily solved, we get all the solutions. The solution of Eq. (16) can be obtained by setting $p = 1$ in Eq. (19):

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots \tag{25}$$

Consider the good Boussinesq Eq. (16) with the following constraint

$$u(x,0) = \frac{3(c^2-1)}{2}\sec h^2\left[\frac{\sqrt{1-c^2}x}{2}\right], \tag{26}$$

$$u_t(x,0) = \frac{-3(c^3-c)\sqrt{1-c^2}}{2}\sec h^2\left[\frac{\sqrt{1-c^2}x}{2}\right]\tanh\left[\frac{\sqrt{1-c^2}x}{2}\right]. \tag{27}$$

Using the homotopy perturbation procedure (18)–(24), we obtain the following resulting components:

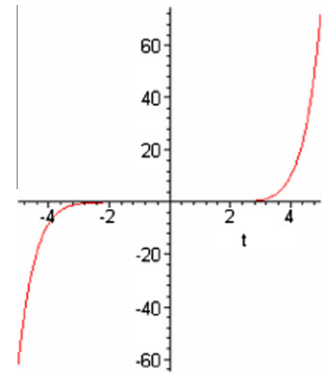

Figure 2 The error between the exact solution $u(x,t)$ and the truncated series solution $\psi(x,t)$ for Boussinesq equation at $c = 0.5$ and $x = 0$. We can say that He's semi-inverse method gives exact solution and HPM gives numerical solution from the figures.

$$\psi(x,t) = \sum_{m=0}^{n} u_m(x,t). \tag{28}$$

$$\begin{aligned}
\psi(x,t) = {} & \tfrac{3}{2}(c^2-1)\sec h^2\left[\tfrac{\sqrt{1-c^2}x}{2}\right] + \tfrac{-3(c^3-c)\sqrt{1-c^2}}{2}\sec h^2\left[\tfrac{\sqrt{1-c^2}x}{2}\right]\tanh\left[\tfrac{\sqrt{1-c^2}x}{2}\right]t \\
& + \tfrac{3}{8}c^4(1-c^2)\left(-2+\cosh\left[\sqrt{1-c^2}x\right]\right)\sec h^4\left[\tfrac{\sqrt{1-c^2}x}{2}\right]t^2 \\
& + \tfrac{1}{16}c^5(1-c^2)^{\frac{3}{2}}\sec h^5\left[\tfrac{\sqrt{1-c^2}x}{2}\right]\left(-11\sinh\left[\tfrac{\sqrt{1-c^2}x}{2}\right]+\sinh\left[\tfrac{3\sqrt{1-c^2}x}{2}\right]\right)t^3 \\
& - \tfrac{1}{128}c^6(-1+c^2)^2\sec h^6\left[\tfrac{\sqrt{1-c^2}x}{2}\right]\left(33-26\cosh\left[\sqrt{1-c^2}x\right]+\cosh\left[2\sqrt{1-c^2}x\right]\right)t^4 \\
& + \tfrac{1}{1280}c^7(1-c^2)^{\frac{5}{2}}\sec h^7\left[\tfrac{\sqrt{1-c^2}x}{2}\right]\left(302\sinh\left[\tfrac{\sqrt{1-c^2}x}{2}\right]-57\sinh\left[\tfrac{3\sqrt{1-c^2}x}{2}\right]+\sinh\left[\tfrac{5\sqrt{1-c^2}x}{2}\right]\right)t^5 \\
& + \tfrac{1}{15360}c^8(-1+c^2)^3\sec h^8\left[\tfrac{\sqrt{1-c^2}x}{2}\right]\begin{pmatrix}-1208+1191\cosh\left[\sqrt{1-c^2}x\right] \\ -120\cosh\left[2\sqrt{1-c^2}x\right]+\cosh\left[3\sqrt{1-c^2}x\right]\end{pmatrix}t^6 \\
& - \tfrac{1}{215040}c^9(1-c^2)^{\frac{7}{2}}\sec h^9\left[\tfrac{\sqrt{1-c^2}x}{2}\right]\begin{pmatrix}-15619\sinh\left[\tfrac{\sqrt{1-c^2}x}{2}\right]+4293\sinh\left[\tfrac{3\sqrt{1-c^2}x}{2}\right] \\ -247\sinh\left[\tfrac{5\sqrt{1-c^2}x}{2}\right]+\sinh\left[\tfrac{7\sqrt{1-c^2}x}{2}\right]\end{pmatrix}t^7 \\
& + \tfrac{1}{3440640}c^{10}(-1+c^2)^4\sec h^{10}\left[\tfrac{\sqrt{1-c^2}x}{2}\right]\begin{pmatrix}-78095-88234\cosh\left[\sqrt{1-c^2}x\right]+14608\cosh\left[2\sqrt{1-c^2}x\right] \\ -502\cosh\left[3\sqrt{c^2-1}x\right]+\cosh\left[4\sqrt{c^2-1}x\right]\end{pmatrix}t^8 \\
& + \tfrac{1}{61931520}c^{11}(1-c^2)^{\frac{9}{2}}\sec h^{11}\left[\tfrac{\sqrt{1-c^2}x}{2}\right]\begin{pmatrix}1310354\sinh\left[\tfrac{\sqrt{1-c^2}x}{2}\right]-455192\sinh\left[\tfrac{3\sqrt{1-c^2}x}{2}\right] \\ +47840\sinh\left[\tfrac{5\sqrt{1-c^2}x}{2}\right]-1013\sinh\left[\tfrac{7\sqrt{1-c^2}x}{2}\right]+\sinh\left[\tfrac{9\sqrt{1-c^2}x}{2}\right]\end{pmatrix}t^9
\end{aligned}$$

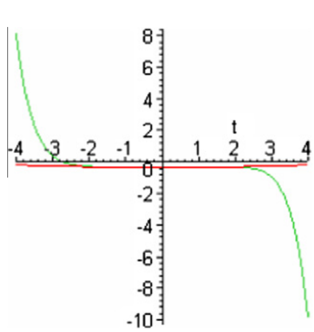

Figure 1 The truncated HPM series solution $\psi(x,t)$ (green) and the exact solution $u(x,t)$ (red) for good Boussinesq equation at $c = 0.5$ and $x = 0$.

Fig. 1 shows HPM truncated series solution compared with the exact solution (He's semi-inverse solution) at $x = 0$. The error between the exact solution and the HPM truncated series solution $u(x, t)$ at $x = 0$ is shown in Fig. 2.

4. Conclusion

In this paper, He's semi-inverse method has been tested by applying it successfully to good Boussinesq equation. The most interesting feature of the method is its simplicity coupled with the accuracy. We have also made the comparison of results with He's homotopy perturbation method (HPM). Hence it may be concluded that He's semi-inverse method can be applied to other nonlinear equations arising in mathematical physics and nonlinear sciences.

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