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Analytical approach to two-dimensional viscous flow with a shrinking sheet via variational iteration algorithm-II

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Abstract The purpose of this paper is to employ an analytical approach to a two-dimensional viscous flow with a shrinking sheet. A comparative study of the variational iteration algorithm-II (VIM-II) and the Adomian decomposition method (ADM) are discussed. Both approaches have been applied to obtain the solution of a two-dimensional viscous flow due to a shrinking sheet. This study outlines the significant features of the two methods. Comparison is made with the ADM to highlight the significant features of the VIM-II and its capability of handling completely integrable equations. Through careful investigation of the iteration formulas of the earlier variational iteration algorithm (VIM), we find unnecessary repeated calculations in each iteration. To overcome this shortcoming, we suggest the VIM-II, which has advantages over other iteration formulas, such as the VIM, and the ADM. Further iterations can produce more accurate results and decrease the error.

1. Introduction

This paper presents a reliable comparison between two recently developed, popular iteration methods, the variational iteration algorithm-II (VIM-II) developed by He et al. (2009) and the Adomian decomposition method (ADM) introduced by Adomian (1988). The use of both methods is common in the literature. The most extensive work carried out on the variational iteration algorithm (VIM) generally is that of He (2000, 2004, 2007, 2008) and He and Lee (2009). Although many researchers have compared the VIM with the ADM, to the best of the authors' knowledge, no comparison of the VIM-II and the ADM has appeared in the literature thus far. The objectives of this paper are threefold: first, to introduce the advantages of the VIM-II, which primarily lie in its ability to avoid the unnecessary calculations of other iteration methods, namely, the VIM and ADM; second, to illustrate through this comparison that, unlike the widely used ADM, the VIM-II does not require the calculation of Adomian polynomials (Adomian, 1988) for the nonlinear terms that appear

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in differential equations, as a solution can be obtained without the incorporation of these polynomials; and third, to apply the VIM-II to a fluid mechanics problem, namely, a viscous flow for a shrinking sheet (Fang et al., 2009). An analytical approach is followed to find the numerical value of $f''(0)$, which Wazwaz (2007) has calculated to solve the Blasius equation. This goal is achieved by employing the reliable VIM-II developed by He (2007) and the ADM (Adomian, 1988). For numerical approximation, the resulting series is best manipulated by Padé approximants (Baker, 1975).

2. Padé approximants

Padé approximants constitute the best approximation of a function by a rational function of a given order. Developed by Henri Padé, Padé approximants often provide better approximation of a function than does truncating its Taylor Series, and they may still work in cases in which the Taylor series does not converge. For these reasons, Padé approximants are used extensively in computer calculations, and it is now well known that these approximants have the advantage of being able to manipulate polynomial approximation into the rational functions of polynomials. Through such manipulation, we can gain more information about the mathematical behavior of the solution. In addition, power series are not useful for large values of a variable, say $\eta \to \infty$, which can be attributed to the possibility of the radius of convergence not being sufficiently large to contain the boundaries of the domain. To provide an effective tool that can handle boundary value problems on an infinite or semi-infinite domain, it is therefore essential to combine the series solution, which is obtained by the iteration method or any other series solution method, with the Padé approximants.

3. Formulation

In this paper, we consider the two basic equations of fluid mechanics in Cartesian coordinates. The continuity equation and momentum equations for viscous flow are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right), \tag{2}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right), \tag{3}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right), \tag{4}$$

where $\upsilon=\frac{\mu}{\rho}$ is the kinematic viscosity.

The boundary conditions applicable to the present flow are

$$\begin{gathered} u=-ax,\ \ v=-a(m-1)y,\ \ w=-W \text{ at } y=0,\\ u\to 0 \quad \text{as} \quad y\to\infty. \end{gathered} \tag{5}$$

For shrinking phenomena, $a > 0$, a is a shrinking constant, and W is the suction velocity. $m = 1$ when the sheet shrinks in the x-direction alone, and $m = 2$ when it shrinks axisymmetrically. We introduce the following similarity transformations.

$$u=axf'(\eta),\quad v=a(m-1)yf'(\eta),\quad \eta=\sqrt{\frac{a}{\upsilon}}z. \tag{6}$$

Eq. (1) is identically satisfied, and Eq. (4) can be integrated to give

$$\frac{p}{\rho}=\upsilon\frac{\partial w}{\partial z}-\frac{w^2}{2}+\text{Constant}. \tag{7}$$

Eqs. (2), (3), and (5) are reduced to the boundary value problem,

$$f'''-(f')^2+mff''=0, \tag{8}$$

and the corresponding boundary conditions take the form

$$\begin{gathered} f=s,\quad f'=-1 \text{ at } \eta=0,\\ f'\to 0 \text{ as } \eta\to\infty, \end{gathered} \tag{9}$$

where $s=W/m\sqrt{a\upsilon}$.

4. Methods

4.1. He's variational method

To illustrate the basic concept of He's VIM, we consider the following general differential equation

$$Lf+Nf=g(x),$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is the source term. According to the VIM, we can construct a correction functional as follows

$$f_{n+1}(x)=f_n(x)+\int_0^x \lambda(Lf_n(s)+N\tilde{f}(s)-g(s))ds, \tag{10}$$

where λ is a Lagrange multiplier that can be identified through a variational iteration method. The subscript n denotes the nth approximation, and $\tilde{f}_n$ is considered to be a restricted variation, i.e., $\delta\tilde{f}_n=0$. The solution of linear problems can be achieved in a single iteration step due to the exact identification of the Lagrange multiplier. This method requires that the Lagrange multiplier λ is first determined optimally. The successive approximation, f_{n+1}, $n \geqslant 0$, of the solution f can then be readily obtained by using the Lagrange multiplier determined and any selective function f_0; consequently, the solution is given by $f=\lim_{n\to\infty} f_n$. According to the variational iteration method, we can construct a correction functional of Eq. (8) as follows

$$f_{n+1}(\eta)=f_n(\eta)+\int_0^\eta \lambda\left(\frac{\partial^3 f_n}{\partial\xi^3}-\left(\frac{\partial f_n}{\partial\xi}\right)^2+mf_n\frac{\partial^2 f_n}{\partial\xi^2}\right)d\xi, \tag{11}$$

with $\lambda=-\frac{(\xi-\eta)^2}{2}$, and the initial approximation is $f_0=s-\eta+\frac{\alpha\eta^2}{2}$, where $f''(0)=\alpha$, $m=2$, $s=2$.

However, according to the VIM-II, the general form of the algorithm takes the following form

$$f_{n+1}(\eta)=f_0(\eta)+\int_0^\eta \lambda\left(-\left(\frac{\partial f_n}{\partial\xi}\right)^2+2f_n\frac{\partial^2 f_n}{\partial\xi^2}\right)d\xi, \tag{12}$$

$$f_{n+1}(\eta)=f_0(\eta)+\int_0^\eta \frac{(\xi-\eta)^2}{2}\left[\left(\frac{\partial f_n}{\partial\xi}\right)^2-2f_n\frac{\partial^2 f_n}{\partial\xi^2}\right]d\xi, \text{ and} \tag{13}$$

consequently, the following approximants are obtained.

$$f_0 = 2 - \eta + \frac{\alpha\eta^2}{2}, \tag{14}$$

$$f_1 = 2 - \eta + \frac{\alpha\eta^2}{2} - \frac{\eta^3}{6} + \frac{2\alpha\eta^3}{3}, \tag{15}$$

$$f_2 = 2 - \eta + \frac{\alpha\eta^2}{2} + \left(\begin{array}{c} \frac{(-1+4\alpha)\eta^3}{6} + \frac{(-2+5\alpha)\eta^4}{12} + \frac{(4-16\alpha+9\alpha^2)\eta^5}{60} \\ + \frac{4\alpha(-1+4\alpha)\eta^6}{45} + \frac{4(1-4\alpha)^2\eta^7}{315} \end{array} \right), \tag{16}$$

$$\begin{aligned} f_3 = {} & 2 - \eta + \frac{\alpha\eta^2}{2} \\ & + \left(\begin{array}{l} \frac{(-1+4\alpha)\eta^3}{6} + \frac{(-1+4\alpha)\eta^4}{6} + \frac{(-7+16\alpha)\eta^5}{60} + \frac{(24-85\alpha+40\alpha^2)\eta^6}{360} \\ + \frac{(-23-52\alpha+504\alpha^2)\eta^7}{2520} + \frac{(74-425\alpha+516\alpha^2+54\alpha^3)\eta^8}{10080} \\ \frac{(-164+1432\alpha-3433\alpha^2+1676\alpha^3)\eta^9}{90720} + \frac{(-56+556\alpha-2222\alpha^2+3387\alpha^3)\eta^{10}}{151200} \\ \frac{(-80+2304\alpha-12744\alpha^2+19232\alpha^3+567\alpha^4)\eta^{11}}{10080} + \\ \frac{(-26+217\alpha-488\alpha^2+18\alpha^3+504\alpha^4)\eta^{12}}{311850} + \frac{(1-4\alpha)^2(108-432\alpha-691\alpha^2)\eta^{13}}{4054050} \\ + \frac{8\alpha(-1+4\alpha)^3\eta^{14}}{257985} + \frac{8(1-4\alpha)^4\eta^{15}}{3869775} \cdots \end{array} \right) \end{aligned} \tag{17}$$

$\vdots$

It is quite clear that only three iterations are needed to obtain approximation $f_3(\eta)$. Other methods require many more iterations to obtain this result. Fewer calculations are also involved with the VIM-II.

4.2. Adomian decomposition method

A detailed description of the ADM is given in Adomian (1988). Here, we provide only the basic steps as a reminder. Writing Eq. (8) in operator form, we have

$$Lf(\eta) + Rf(\eta) + Nf(\eta) = 0, \tag{18}$$

where L is the linear operator, i.e., $Lf = \frac{d^3f}{d\eta^3}$, which is the highest order derivative; $R = 0$; and

$$Nf = \left(\frac{df}{d\eta}\right)^2 - mf\frac{d^2f}{df^2}. \tag{19}$$

The linear operator is also the inverse operator, i.e.,

$$L^{-1}(\cdot) = \int_0^\eta \int_0^\eta \int_0^\eta (\cdot)d\eta\, d\eta\, d\eta. \tag{20}$$

The ADM defines the unknown function, $f(\eta)$, by an infinite series:

$$f(\eta) = \sum_{\eta=0}^{\infty} f_n(\eta), \tag{21}$$

where the components $f_n(\eta)$ are usually determined recurrently. The nonlinear operator, $G(f)$, can be decomposed into infinite polynomials given by

$$G(f) = \overset{\infty}{\underset{n=0}{m}} A_n, \tag{22}$$

where A_n are the so-called Adomian polynomials of $f_0, f_1, f_2, \ldots, f_n$ defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[G\left(\sum_{i=0}^{n}\lambda^i f_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots, \tag{23}$$

or, equivalently,

$$\begin{aligned} A_0 &= G(f_0), \\ A_1 &= f_1 G'(f_0), \\ A_2 &= f_2 G'(f_0) + \frac{1}{2} f_1^2 G''(f_0), \\ A_3 &= f_3 G'(f_0) + f_1 f_2 G''(f_0) + \frac{1}{3} f_1^3 G'''(f_0), \\ A_4 &= f_4 G'(f_0) + \left(f_1 f_3 + \frac{1}{2} f_2^2\right) G''(f_0) \\ &\quad + \frac{1}{2} f_1^2 f_2 G'''(f_0) + \frac{1}{24} f_1^4 G^{iv}(f_0). \end{aligned} \tag{24}$$

$\vdots$

It is now well known that these polynomials can be generated for all classes of nonlinearity according to specific algorithms.

Write the general algorithm of Eq. (8) with the initial approximation mentioned in Eq. (14)

$$f_{n+1}(\eta) = \int_0^\eta \int_0^\eta \int_0^\eta \left(\sum_{n=0}^{\infty} A_n - 2\sum_{n=0}^{\infty} B_n\right) d\eta\, d\eta\, d\eta \quad n = 0, 1, 2, \ldots \tag{25}$$

where

$$\begin{aligned} A_0 &= (f_0')^2, \\ A_1 &= 2f_0' f_1', \\ A_2 &= 2f_0' f_2' + f_1' f_1', \\ A_2 &= 2f_0' f_3' + f_1' f_2'. \end{aligned} \tag{26}$$

$\vdots$

Similarly,

$$\begin{aligned} B_0 &= f_0 f_0'', \\ B_1 &= f_0 f_1'' + f_1 f_0'', \\ B_2 &= f_0 f_2'' + f_1 f_2'' + f_2 f_1'', \\ B_3 &= f_0 f_3'' + f_1 f_2'' + f_2 f_1'' + f_3 f_0''. \end{aligned} \tag{27}$$

$\vdots$

After successive iterations, we obtain the following results

$$f_0 = 2 - \eta + \frac{\alpha\eta^2}{2}, \tag{28}$$

$$f_1 = \frac{\eta^3}{6} - \frac{2\alpha\eta^3}{3}, \tag{29}$$

$$f_2 = -\frac{\eta^4}{6} + \frac{2\alpha\eta^4}{3} + \frac{\eta^5}{60} - \frac{\alpha\eta^5}{15} + \frac{\alpha\eta^6}{360} - \frac{\alpha^2\eta^6}{90}, \tag{30}$$

$$f_3 = \frac{2\eta^5}{15} - \frac{8\alpha\eta^5}{15} - \frac{\eta^6}{30} + \frac{2\alpha\eta^6}{15} + \frac{\eta^7}{504} - \frac{\alpha\eta^7}{315} - \frac{2\alpha^2\eta^7}{105}$$

$$- \frac{\alpha\eta^8}{5040} + \frac{\alpha^2\eta^8}{1260} - \frac{\alpha^2\eta^9}{9072} + \frac{\alpha^3\eta^9}{2268}, \tag{31}$$

$$f_4 = -\frac{4\eta^6}{45} + \frac{16\alpha\eta^6}{45} + \frac{4\eta^7}{105} - \frac{16\alpha\eta^7}{105} - \frac{\eta^8}{360} - \frac{\alpha\eta^8}{126} + \frac{8\alpha^2\eta^8}{105} - \frac{\eta^9}{90720} + \frac{37\alpha\eta^9}{11340} - \frac{73\alpha^2\eta^9}{5670} - \frac{47\alpha\eta^{10}}{30400} + \frac{37\alpha^2\eta^{10}}{56700} - \frac{\alpha^3\eta^{10}}{8100} - \frac{\alpha^2\eta^{11}}{178200} + \frac{\alpha^3\eta^{11}}{44550} + \frac{\alpha^3\eta^{12}}{213840} - \frac{\alpha^4\eta^{12}}{53460}, \quad (32)$$

$$\vdots$$

and so on. In this manner, the remainder of the terms in the decomposition series (21) can be calculated.

The series solution is given by

$$f = f_0 + f_1 + f_2 + f_3 + f_4 + \dots, \quad (33)$$

Substituting Eqs. (27)–(32) into Eq. (33), we obtain the following series solution

$$f(\eta) = 2 - \eta + \frac{\alpha\eta^2}{2} + \frac{\eta^3}{6} - \frac{2\alpha\eta^3}{3} - \frac{\eta^4}{6} + \frac{2\alpha\eta^4}{3} + \frac{3\eta^5}{20} - \frac{3\alpha\eta^5}{5} - \frac{11\eta^6}{90} + \frac{59\alpha\eta^6}{120} - \frac{\alpha^2\eta^6}{90} + \frac{101\eta^7}{2520} - \frac{7\alpha\eta^7}{45} - \frac{2\alpha^2\eta^7}{105} - \frac{\eta^8}{360} - \frac{41\alpha\eta^8}{5040} + \frac{97\alpha^2\eta^8}{1260} - \frac{\eta^9}{90720} + \frac{37\alpha\eta^9}{11340} - \frac{589\alpha^2\eta^9}{45360} + \frac{\alpha^3\eta^9}{2268} - \frac{47\alpha\eta^{10}}{302400} + \frac{37\alpha^2\eta^{10}}{56700} - \frac{\alpha^3\eta^{10}}{8100} - \frac{\alpha^2\eta^{11}}{178200} + \frac{\alpha^3\eta^{11}}{44550} + \frac{\alpha^3\eta^{12}}{213840} - \frac{\alpha^4\eta^{12}}{53460} + \dots \quad (34)$$

Table 1 The numerical values for $f''(0) = \alpha$ using Padé approximation.

Padé approximation	$f''(0) = \alpha$ for ADM	$f''(0) = \alpha$ for AVIM
[1/1]	0.292893	0.224748
[2/2]	Complex number	Complex number
[3/3]	Complex number	0.294748
[4/4]	0.247723	0.308086
[5/5]	0.24893	0.249556

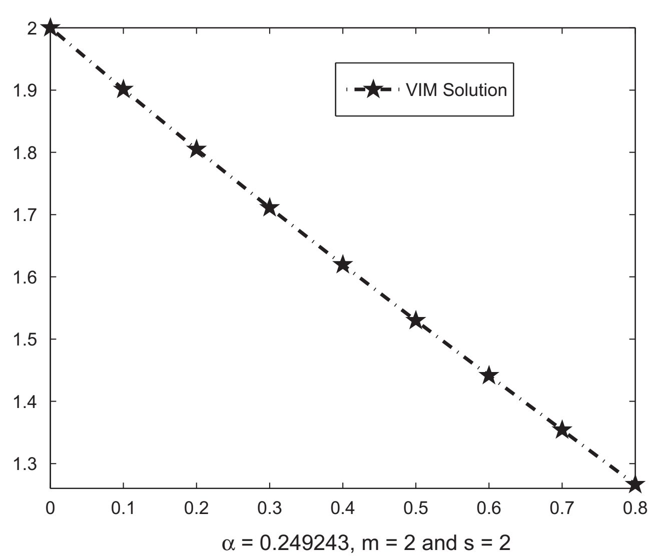

Figure 1 Graphical presentation of VIM solution.

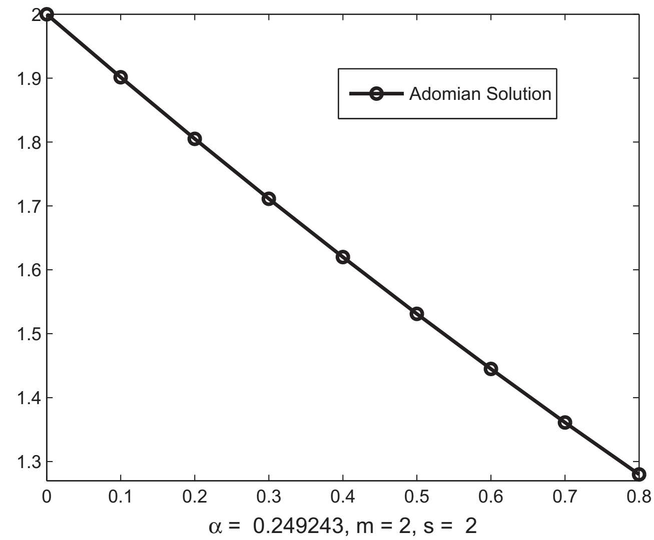

Figure 2 Graphical presentation of Adomian solution.

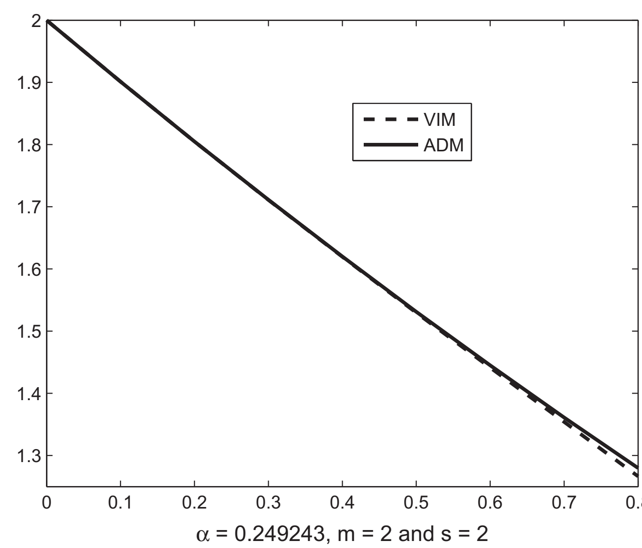

Figure 3 Comparison of the VIM solution and the Adomian solution.

Software packages such as Mathematica or Maple can be used to solve the polynomial $f'(\eta)$ to calculate the value of α with the help of boundary condition $f'(\eta) \to 0$ for $\eta \to \infty$. By using the table above, we can choose the value of $\alpha = f'(\eta) \to 0 = 0.249243$ for both solutions, which is an average value of [5/5] Padé approximation (Table 1).

The result of VIM and ADM solutions are depicted in Figs. 1 and 2. Fig. 3 compares the two solutions.

5. Conclusion

This paper presents a variational iteration method, the VIM-II, that can be employed to solve nonlinear differential equations. The method is applied here in a direct manner without the use of linearization, transformation, discretization, perturbation, or restrictive assumptions. The proposed algorithm's ability to solve nonlinear problems without the use of Adomian polynomials is evidence of its clear advantage over the decomposition method. This study has considered only an axisymmetrically shrinking sheet by taking $m = 2$.

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