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A Legendre-homotopy method for the solutions of higher order boundary value problems

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ABSTRACT

In this paper, the Legendre-homotopy analysis method is proposed using orthogonal Legendre polynomials for the approximate solutions of linear and nonlinear higher order boundary value problems. The deformation equations obtained in this case are easily integrable and the calculations involved in the algorithm are much simpler than the standard homotopy analysis method. The method is numerically illustrated by application on linear and nonlinear higher order boundary value problems. The absolute errors in the approximate solution values are calculated and compared with the results available in literature. The approximate solutions are also compared with the exact solutions through graphical illustrations. The numerical and graphical comparisons reveal that the presented method gives highly accurate results.

1. Introduction

Orthogonal polynomials have been of great interest of research due to their application for computation and approximation purposes in different problems of mathematics and physics. These polynomials have many applications to ordinary differential equations, boundary value problems and computational fluid dynamics (Butcher, 1992; Gottlieb and Orszag, 1977; Canuto et al., 1989; Voigt et al., 1984). Doha and Bhrawyb (2008) presented spectral-Galerkin algorithms for solutions of fourth order differential equations using Jacobi polynomials. Yalçinbas et al. (2009) obtained Legendre polynomial solutions of high order Fredholm integro-differential equations using Legendre collocation matrix method. Parand et al. (2010) approximated the solutions to nonlinear Lane-Emden type equations using a collocation method which involved Hermite functions to convert the problem into a system of algebraic equations. Odibat (2011) proposed algorithms for variational iteration method and homotopy analysis method using Legendre polynomials for the solutions of fractional differential equations. Bhrawy and Al-Shomrani (2012) proposed shifted Legendre tau method for the solutions of multi-order fractional differential equations with constant coefficients. Sweilam et al. (2012) used Legendre collocation method for the solutions of Fredholm-Hammerstein integral equations. Liu (2013) used Legendre polynomials to obtain the solutions to Volterra integral equations of second kind. Khader et al. (2014) used Legendre polynomials in an integral collocation approach for solving Riccati, logistic and delay differential equations. Xu and Zhou (2015) approximated the solutions to eighth order initial and boundary value problems using the second kind Chebyshev wavelets. Orthogonal polynomials have also been used to approximate the solutions to Volterra equations using Galerkin method (Mamadu and Njoseh, 2016).

Homotopy analysis method is an effective and reliable mathematical tool to determine the solutions of linear and nonlinear differential equations. It is an analytical approximate solution technique which enables to evaluate the solution to a problem in the form of a convergent series (Liao, 1992). The homotopy analysis is not only an efficient method to solve nonlinear differential equation problems but also allows great freedom to choose the initial approximation and is highly flexible in many respects so that it might overcome restrictions of numerical techniques, perturbation techniques and other non-perturbation methods, such as variational iteration method, homotopy perturbation method, finite element method and collocation method *etc* (Sadighi and Ganji, 2007; Jalaal and Ganji, 2010; Sheikholeslami et al., 2012, 2014, 2016, 2017; Sheikholeslami and Ganji, 2013; Malvandi and Ganji, 2014;

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Hosseini et al., 2018; Sheikholeslami and Ganji, 2018). The efficiency and practical usefulness of the homotopy analysis method has caught the attention of many researchers in recent years. It has been successfully implemented to investigate a wide range of problems arising in the study of nonlinear dynamics, micropolar fluids, heat transfer problems and many other areas of science (Ziabakhsh and Domairry, 2009; Sheikholeslami and Ganji, 2017; Sheikholeslami et al., 2018; Shah et al., 2017, 2018; Khan et al., 2018; Muhammad et al., 2018; Dawar et al., 2018; Khan et al., 2017; Tahir et al., 2017; Fiza et al., 2018; Gul, 2018; Alshomrani and Gul, 2017).

In this paper, a modification of homotopy analysis method is introduced for the analytical approximate solutions of higher order boundary value problems using the orthogonal Legendre polynomials. The applications of higher order boundary value problems are reported in various fields of science and engineering. Ninth order boundary value problems arise in astorphysics and aerodynamics, hydrodynamic and hydromagnetic stability (Lyshevski and Dunipace, 1997; Mohyud-Din and Yildirim, 2010; Mohyud-Din and Yildirim, 2010; Chandrasekhar, 1961). Eighth and tenth order boundary value problems also appear in the problems of hydrodynamic and hydromagnetic stability (Chandrasekhar, 1961). The exact solutions of many higher order boundary value problems arising in mathematical models of many real life phenomena cannot be obtained with any existing mathematical technique. Due to this fact, the approximate solutions are often investigated to understand the phenomena modeled, which provide either the numerical or the analytical approximations to the exact solutions. In general, the analytical approximate solutions provide a better account of qualitative behaviour and character of the solutions than the numerical solutions. In this regard, the proposed Legndre-homotopy analysis method can play a significant role to obtain the analytical approximate solutions of various higher order boundary value problems arising in physics and engineering.

2. Preliminaries for Legendre polynomials

Legendre polynomials form a class of functions which are encountered in finding the solutions to many physical problems. For example, Legendre and Associate Legendre polynomials are employed to determine the wave functions of electrons in the orbits of an atom and potential functions in the spherically symmetric geometry,*etc.* Legendre polynomials also play an important role in the nuclear reactor physics (Anli and Gungor, 2007).

Legendre polynomials are solutions to a very important differential equation, known as Legendre equation, which can be stated, as

$$(1-\xi^2)\frac{d^2v}{d\xi^2}-2\xi\frac{dv}{d\xi}+j(j+1)v=0.$$

This ordinary differential equation is frequently encountered in physics and other technical fields. In particular, it occurs when solving Laplace's equation (and related partial differential equations) in spherical coordinates.

Legendre polynomials can be denoted by $L_j(\xi)$, where j is the degree of the polynomial. These polynomials are defined over the interval $[-1,1]$ and satisfy the recurrence relation

$$L_{j+1}(\xi)=\frac{2j+1}{j+1}\xi L_j(\xi)-\frac{j}{j+1}L_{j-1}(\xi),\quad j=1,2,3,\ldots, \tag{2.1}$$

where $L_0(\xi)=1$ and $L_1(\xi)=\xi$. The shifted Legendre polynomials over the interval $[0,1]$ are defined as

$$L_j^*(\xi)=L_j(2\xi-1)$$

and satisfy the recurrence relation

$$L_{j+1}^*(\xi)=\frac{2j+1}{j+1}(2\xi-1)L_j^*(\xi)-\frac{j}{j+1}L_{j-1}^*(\xi),\quad j=1,2,3,\ldots, \tag{2.2}$$

where $L_0^*(\xi)=1$ and $L_1^*(\xi)=2\xi-1$. Since Legendre polynomials are orthogonal, therefore any function $g(\xi)$ can be expressed in terms of shifted Legendre polynomials, as

$$g(\xi)=\sum_{k=0}^{\infty}c_kL_k^*(\xi), \tag{2.3}$$

where c_k's can be calculated, as

$$c_k=(2k+1)\int_0^1 g(\xi)L_k^*(\xi)d\xi. \tag{2.4}$$

3. The Legendre-homotopy analysis method

Homotopy analysis method is found to be easily applicable in many problems but sometimes the higher order deformation equations lead to complicated integrals and tedious calculations. To overcome these difficulties, a modification of homotopy analysis method is proposed in this section using the well known Legendre polynomials.

The nonlinear differential equation is considered, as

$$\mathcal{N}[v(\xi)]+q(\xi)=0,\quad \xi\in\Theta, \tag{3.5}$$

where $\mathcal{N}$ is a nonlinear operator, ξ is an independent variable, $v(\xi)$ is an unknown function, Θ is the interval of domain and $q(\xi)$ is a continuous function. In view of the homotopy analysis method, a homotopy $V(\xi,p)$ can be defined using an embedding parameter $p\in[0,1]$ by

$$(1-p)\mathcal{L}[V(\xi,p)-v_0(\xi)]-phH(\xi)(\mathcal{N}[V(\xi,p)]+q(\xi))=0,\quad \xi\in\Theta, \tag{3.6}$$

where h is auxiliary parameter, $H(\xi)$ is an auxiliary function and $\mathcal{L}$ is auxiliary linear operator.

An initial approximation $v_0(\xi)$ can be obtained, as

$$v_0(\xi)=\sum_{j=0}^{m}a_{0j}L_j^*(\xi), \tag{3.7}$$

where a_{0j}'s can be calculated, as

$$a_{0j}=(2j+1)\int_0^1 V_0(\xi)L_j^*(\xi)d\xi. \tag{3.8}$$

Here $V_0(\xi)$ is the solution of the equation $\mathcal{L}[v(\xi)]=0$, subject to the given boundary conditions and m is some fixed natural number.

The first order deformation equations can be expressed using Legendre polynomials, as

$$\mathcal{L}[v_1(\xi)]-hH(\xi)\left(\mathcal{N}\left(\sum_{j=0}^{m}a_{0j}L_j^*(\xi)\right)+\sum_{j=0}^{m}b_jL_j^*(\xi)\right)=0. \tag{3.9}$$

For $i=1,2,3,\ldots$, the deformation equations can be expressed, as

$$\mathcal{L}[v_{i+1}(\xi)]-\mathcal{L}[v_i(\xi)]-hH(\xi)\left(\mathcal{N}\left(\sum_{j=0}^{m}a_{ij}L_j^*(\xi)\right)\right)=0. \tag{3.10}$$

Moreover,

$$a_{ij}=(2j+1)\int_0^1 v_i(\xi)L_j^*(\xi)d\xi,\quad j=0,1,2,\ldots\quad m, i=1,2,3,\ldots \tag{3.11}$$

and

$$b_j=(2j+1)\int_0^1 q(\xi)L_j^*(\xi)d\xi,\qquad j=0,1,2,\ldots m. \tag{3.12}$$

Finally, the solution of Eq. (3.5) can be written, as

$$v(\xi) = \sum_{i=0}^{\infty} v_i(\xi) \tag{3.13}$$

and the n-th order analytical approximate solution can be calculated, as

$$v_{n^*}(\xi) = \sum_{i=0}^{n} v_i(\xi). \tag{3.14}$$

4. Convergence of the solution

In this section, convergence of the solution series using the proposed technique is discussed.

Theorem. *If the series $v(\xi) = \sum_{i=0}^{\infty} v_i(\xi)$ is convergent, where $v_i(\xi)$ is governed by Eqs. (3.9) and (3.10), it must be an exact solution of problem (3.5).*

Proof Convergence of the series $\sum_{i=0}^{\infty} v_i(\xi)$ implies

$$\lim_{i\to\infty} v_i(\xi) = 0. \tag{4.15}$$

Consider the series $\sum_{i=0}^{\infty} hH(\xi)(\mathcal{N}(\sum_{j=0}^{\infty} a_{ij}L_j^*(\xi)) + (1-\chi_{i+1})\sum_{j=0}^{\infty} b_j L_j^*(\xi))$.

Using Eqs. (4.15), (3.9) and (3.10) yields

$$\begin{aligned}
&\sum_{i=0}^{\infty} hH(\xi)\left(\mathcal{N}\left(\sum_{j=0}^{\infty} a_{ij}L_j^*(\xi)\right) + (1-\chi_{i+1})\sum_{j=0}^{\infty} b_j L_j^*(\xi)\right)\\
&= \mathcal{L}\sum_{i=0}^{\infty}(v_{i+1}(\xi) - \chi_{i+1} v_i(\xi))\\
&= \mathcal{L}(\lim_{i\to\infty} v_{i+1}(x))\\
&= 0,
\end{aligned} \tag{4.16}$$

where

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1 \end{cases} \tag{4.17}$$

and the linearity of the operator $\mathcal{L}$ is used.

Since $h \neq 0, H(x) \neq 0$, it can be expressed, as

$$\sum_{i=0}^{\infty}\left(\mathcal{N}\left(\sum_{j=0}^{\infty} a_{ij}L_j^*(\xi)\right) + (1-\chi_{i+1})\sum_{j=0}^{\infty} b_j L_j^*(\xi)\right) = 0. \tag{4.18}$$

Moreover,

$$\begin{aligned}
&\sum_{i=0}^{\infty}\left(\mathcal{N}(\sum_{j=0}^{\infty} a_{ij}L_j^*(\xi)) + (1-\chi_{i+1})\sum_{j=0}^{\infty} b_j L_j^*(\xi)\right)\\
&= \sum_{i=0}^{\infty}(\mathcal{N}[v_{i+1}(\xi)] + (1-\chi_{i+1})q(\xi))\\
&= \sum_{i=0}^{\infty}(\mathcal{N}[v_{i+1}(\xi)] + q(\xi).
\end{aligned} \tag{4.19}$$

From Eqs. (4.18) and (4.19), it can be written as

$$\sum_{i=0}^{\infty}(\mathcal{N}[v_{i+1}(\xi)] + q(\xi)) = 0, \tag{4.20}$$

which shows that the series solution satisfies the differential Eq. (3.5). This completes the proof.

5. Procedure of the Legendre-homotopy method

Step 1 The solution $V_0(\xi)$ of the differential equation $\mathcal{L}[v(\xi)] = 0$, subject to the given boundary conditions, is obtained.

Step 2 $V_0(\xi)$ is expressed in terms of Legendre polynomials to get the initial approximate solution, as

$$v_0(\xi) = \sum_{j=0}^{m} a_{0j}L_j^*(\xi), \tag{5.21}$$

where a_{0j}'s are calculated using Eq. (3.8).

Step 3 Using Eq. (3.9), $v_1(\xi)$ can be calculated, as

$$v_1(\xi) = \mathcal{L}^{-1}hH(\xi)\left(\mathcal{N}\left(\sum_{j=0}^{m} a_{0j}L_j^*(\xi)\right) + \sum_{j=0}^{m} b_j L_j^*(\xi)\right). \tag{5.22}$$

Step 4 Using Eq. (3.10), $v_i(\xi)$'s for i = 2,3,4,. . ., can be calculated, as

$$v_i(\xi) = v_{i-1}(\xi) + \mathcal{L}^{-1}hH(\xi)\left(\mathcal{N}\left(\sum_{j=0}^{m} a_{i-1j}L_j^*(\xi)\right)\right) = 0, \tag{5.23}$$

where a_{i-1j}'s can be calculated using Eq. (3.11).

Step 5 The n-th order approximate solution is calculated, as

$$v_{n^*}(\xi) = \sum_{i=0}^{n} v_i(\xi). \tag{5.24}$$

When the auxiliary linear operator $\mathcal{L}$ is a differential operator, its inverse $\mathcal{L}^{-1}$ gives rise to integrals in Eqs. (5.22) and (5.23). Due to the application of Legendre polynomials, Eqs. (5.22) and (5.23) involve only polynomial functions. Since, polynomials are easier to integrate and use in arithmetic operations, therefore the proposed modification simplifies the calculations to a great extent.

A flow chart of the procedure is shown in Fig. 1. In the next section, the method is applied on different linear and nonlinear higher order boundary value problems. Steps 1–5 are followed to solve the problems, taking $m = 10$. The calculations are performed using Mathematica 8.0.

6. Numerical examples

Example 1

The following ninth order linear boundary value problem is considered:

$$\left.\begin{aligned}
&v^{(9)}(\xi) - v(\xi) = -9e^{\xi}, \quad \xi \in [0,1]\\
&v(0) = 1,\ v(1) = 0,\\
&v'(0) = 0,\ v'(1) = -e,\\
&v''(0) = -1,\ v''(1) = -2e,\\
&v'''(0) = -2,\ v'''(1) = -3e,\\
&v^{(4)}(0) = -3.
\end{aligned}\right\} \tag{6.25}$$

The exact solution of this problem is

$$v(\xi) = (1-\xi)e^{\xi}.$$

The value of $V_0(\xi)$ is obtained, as

$$\begin{aligned}
v_0(\xi) = {}& 1 - \frac{\xi^2}{2} - \frac{\xi^3}{3} - \frac{\xi^4}{8} + \frac{1}{24}(-1012 + 372e)\xi^5 + \frac{1}{24}\\
&\times (2642 - 972e)\xi^6 + \frac{1}{24}(-2316 + 852e)\xi^7 + \frac{1}{24}\\
&\times (685 - 252e)\xi^8.
\end{aligned} \tag{6.26}$$

The zeroth order approximation is calculated, as

$$v_0(\xi) = \sum_{j=0}^{10} a_{0j}L_j^*(\xi). \tag{6.27}$$

The homogeneous part of the nonlinear operator $\mathcal{N}$ is taken as the linear differential operator $\mathcal{L}$. Using the first and second order deformation equations, $v_1(\xi)$ and $v_2(\xi)$ are obtained, where

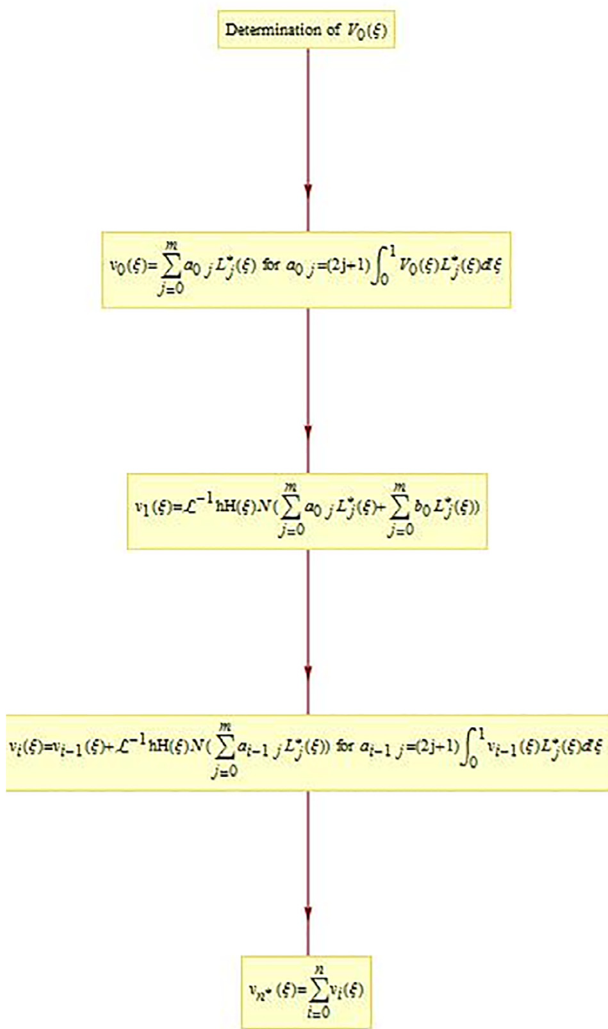

Fig. 1. Flow chart for the proposed Legendre-homotopy method.

$$
\begin{aligned}
v_1(\xi) = \frac{1}{379399923302400}(&338998227872682409162299h\xi^5 \\
&-1247104786279049057551 80eh\xi^5 - 9148961081628804584446 20h\xi^6 \\
&+3365714690008612368738 36eh\xi^6 - 1210027460836135683636 18h\xi^7 \\
&+4451442260948879941598 4eh\xi^7 + 1887818747352851715800526h\xi^8 \\
&-694489705809164249335005eh\xi^8 + 533639212591661507030 70h\xi^9 \\
&-196314895315335039843 60eh\xi^9 - 2702512316820915383304520h\xi^{10} \\
&+994198720871018535272400eh\xi^{10} - 533767161771607674288 56h\xi^{11} \\
&+196361965188206033623 68eh\xi^{11} + 2661968899514615252276988h\xi^{12} \\
&-979283631169195957795008eh\xi^{12} + 355903877032116676175 64h\xi^{13} \\
&-130929719393324808532 80eh\xi^{13} - 1755339637376751422090400h\xi^{14} \\
&+645753364864241522176200eh\xi^{14} - 152548817925160335770 16h\xi^{15} \\
&+561195738896720876606 4eh\xi^{15} + 746082842071026976993876h\xi^{16} \\
&-274468539008690861139669eh\xi^{16} + 381407587824775091100 5h\xi^{17} \\
&-140312010267526068795 6eh\xi^{17} - 185314074172190097874740h\xi^{18} \\
&+681732380476685060523 00eh\xi^{18} - 423817395166793988360h\xi^{19} \\
&+155913706492696512360eh\xi^{19} + 20483196329392601606802h\xi^{20} \\
&-7535346819061888881054eh\xi^{20}).
\end{aligned}
\tag{6.28}
$$

The value of $v_2(\xi)$ is not expressed here because it is very bulky and not fit to write. The second order approximation to the exact solution can be determined, as

$$v(\xi) = v_0(\xi) + v_1(\xi) + v_2(\xi), \tag{6.29}$$

where the auxiliary function $H(\xi)$ is chosen as $H(\xi) = 1$ and the value of h is taken as $h = -0.01$. The numerical results are summarized in Table 1.

Example 2. For $x \in [0, 1]$, the ninth order nonlinear boundary value problem is considered:

$$
\left.
\begin{array}{l}
v^{(9)}(\xi) - v(\xi)v'(\xi) = e^{-2\xi}(2 + e^{\xi}(\xi - 10) - 3\xi + \xi^2), \\
v(0) = 1, \ \ v(1) = 0, \\
v'(0) = -2, \ \ v'(1) = -e^{-1}, \\
v''(0) = 3, \ \ v''(1) = 2e^{-1}, \\
v'''(0) = -4, \ \ v'''(1) = -3e^{-1}, \\
v^{(4)}(0) = 5.
\end{array}
\right\}
\tag{6.30}
$$

The exact solution of this problem is

$$v(\xi) = (1 - \xi)e^{-\xi}.$$

The value of $V_0(\xi)$ is obtained, as

$$
\begin{aligned}
V_0(\xi) = &1 - 2\xi + \frac{3\xi^2}{2} - \frac{2\xi^3}{3} + \frac{5\xi^4}{24} + \frac{(660 - 244e)\xi^5}{24e} \\
&+ \frac{(-1788 + 658e)\xi^6}{24\,e} + \frac{(1620 - 596e)\xi^7}{24e} + \frac{(-492 + 181e)\xi^8}{24e}.
\end{aligned}
\tag{6.31}
$$

The zeroth order approximation is calculated, as

$$v_0(\xi) = \sum_{j=0}^{10} a_{0j} L_j^*(\xi). \tag{6.32}$$

The homogeneous part of the nonlinear operator $\mathcal{N}$ is taken as the linear differential operator $\mathcal{L}$. Moreover, the auxiliary function $H(\xi)$ is taken as $H(\xi) = 1$ and h is chosen as $h = -1$. The second order approximation to the exact solution is calculated using the proposed method and the numerical results are summarized in Table 3.

Example 3. The following eighth order linear boundary value problem is considered:

$$
\left.
\begin{array}{l}
v^{(8)}(\xi) - v(\xi) = 8e^{\xi}, \quad \xi \in [0, 1], \\
v(0) = 1, \ \ v^{(4)}(0) = -3, \\
v'(0) = 0, \ \ v^{(5)}(0) = -4, \\
v''(0) = -1, \ \ v'(1) = -e, \\
v'''(0) = -2, \ \ v''(1) = -2e.
\end{array}
\right\}
\tag{6.33}
$$

The exact solution of the problem is

$$v(\xi) = (1 - \xi)e^{\xi}.$$

Table 1
Absolute errors for Example 1.

ξ	Exact value of $v(\xi)$	Approximate value of $v(\xi)$	Absolute error
0.0	1.000000	1.000000	0.000000
0.1	0.994654	0.994654	2.289423×10^{-10}
0.2	0.977122	0.977122	4.623567×10^{-9}
0.3	0.944901	0.944901	2.081388×10^{-8}
0.4	0.895095	0.895095	4.783428×10^{-8}
0.5	0.824361	0.824361	7.122960×10^{-8}
0.6	0.728848	0.728848	7.339381×10^{-8}
0.7	0.604126	0.604126	4.961130×10^{-8}
0.8	0.445108	0.445108	1.629848×10^{-8}
0.9	0.245960	0.245960	1.814813×10^{-9}
1.0	0.000000	-6.940873×10^{-9}	6.940873×10^{-9}

The value of $V_0(\xi)$ is obtained, as

$$V_0(\xi) = 1 - \frac{\xi^2}{2} - \frac{\xi^3}{3} - \frac{\xi^4}{8} - \frac{\xi^5}{30} + \frac{(4550 - 1680e)\xi^6}{2520} + \frac{(-2940 + 1080e)\xi^7}{2520},$$

The zeroth order approximation is calculated, as

$$v_0(\xi) = \sum_{j=0}^{10} a_{0j} L_j^*(\xi). \tag{6.34}$$

The homogeneous part of the nonlinear operator $\mathcal{N}$ is taken as the linear differential operator $\mathcal{L}$ and the function $H(\xi)$ is chosen as $H(\xi) = 1$. Value of h is chosen as $h = -0.69$. A second order approximation to the exact solution is calculated using the proposed method and the results are summarized in Table 4.

Example 4. The following tenth order linear boundary value problem is considered:

$$\left.\begin{array}{l} v^{(10)}(\xi)) - \xi v(\xi) + e^{\xi}(89 + 21\xi + \xi^2 - \xi^3) = 0, \qquad \xi \in [-1, 1], \\ v(-1) = 0, \;\; v(1) = 0, \\ v'(-1) = \frac{2}{e}, \;\; v'(1) = -2e, \\ v''(-1) = \frac{2}{e}, \;\; v''(1) = -6e, \\ v'''(-1) = 0, \;\; v''(1) = -12e, \\ v^{(4)}(-1) = -\frac{4}{e}, \;\; v^{(4)}(1) = -20e \end{array}\right\} \tag{6.35}$$

The exact solution of this problem is

$$v(\xi) = (1 - \xi^2)e^{\xi}.$$

The value of $V_0(\xi)$ is calculated, as

$$\begin{aligned} V_0(\xi) = \frac{1}{96e}(&91 + 23e^2 - 175\xi + 59e^2\xi - 160\xi^2 + 4e^2\xi^2 \\ &+ 418\xi^3 - 86e^2\xi^3 + 102\xi^4 - 30e^2\xi^4 - 396\xi^5 \\ &+ 48e^2\xi^5 - 40\xi^6 + 4e^2\xi^6 + 190\xi^7 - 26e^2\xi^7 + 7\xi^8 \\ &- e^2\xi^8 - 37\xi^9 + 5e^2\xi^9). \end{aligned} \tag{6.36}$$

The zeroth order approximation is calculated, as

$$v_0(\xi) = \sum_{j=0}^{10} a_{0j} L_j^*(\xi). \tag{6.37}$$

The homogeneous part of the nonlinear operator $\mathcal{N}$ is taken as the linear operator $\mathcal{N}$ the auxiliary function $H(\xi)$ is taken as $H(\xi) = 1$. Value of h is taken as $h = -1$. A second order approximation to the exact solution is calculated using the presented method. The numerical results are summarized in Table 6.

Figs. 2–5 show the comparison of exact and approximate solutions curves for Examples 1–4 respectively. Solid lines show exact solutions and dashed lines show approximate solutions.

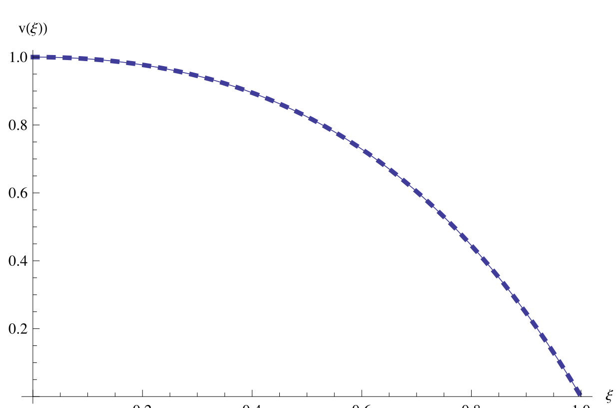

Fig. 2. Comparison of exact and approximate solution curves for Example 1.

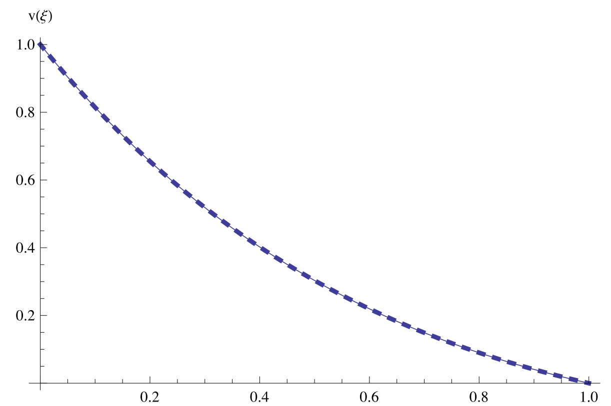

Fig. 3. Comparison of exact and approximate solution curves for Example 2.

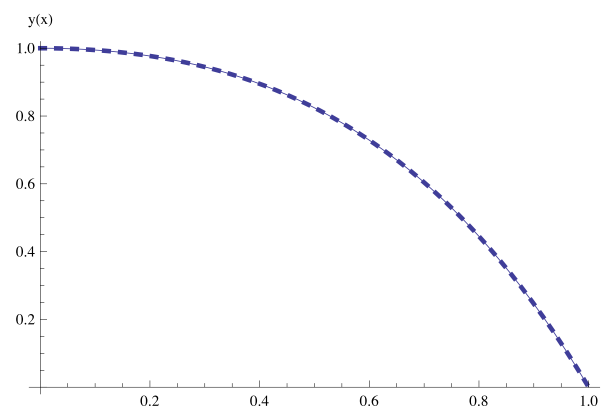

Fig. 4. Comparison of exact and approximate solution curves for Example 3.

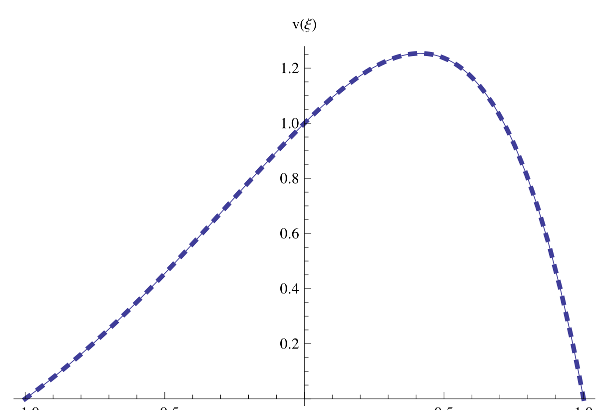

Fig. 5. Comparison of exact and approximate solution curves for Example 4.

Table 2
Comparison of maximum absolute error with DTM (Hassan and Erturk, 2009).

DTM (Hassan and Erturk, 2009)	Present method
3.0×10^{-7}	7.339381×10^{-8}

Table 3
Absolute errors for Example 2.

ξ	Exact value of $v(\xi)$	Approximate value of $v(\xi)$	Absolute error
0.0	1.000000	1.000000	0.000000
0.1	0.814354	0.814354	3.260858×10^{-11}
0.2	0.654985	0.654985	6.342264×10^{-10}
0.3	0.518573	0.518573	2.721959×10^{-9}
0.4	0.402192	0.402192	5.962918×10^{-9}
0.5	0.303265	0.303265	9.179561×10^{-9}
0.6	0.219525	0.219525	8.321407×10^{-9}
0.7	0.148976	0.148976	8.904865×10^{-9}
0.8	0.089866	0.089866	1.372047×10^{-8}
0.9	0.040657	0.040657	5.576126×10^{-9}
1.0	0.000000	7.517892×10^{-8}	7.517892×10^{-8}

Table 4
Absolute errors for Example 3.

ξ	Exact value of $v(\xi)$	Approximate value of $v(\xi)$	Absolute error
0.0	1.000000	1.000000	0.000000
0.1	0.994654	0.994654	1.066014×10^{-11}
0.2	0.977122	0.977122	4.818624×10^{-10}
0.3	0.944901	0.944901	3.701736×10^{-9}
0.4	0.895095	0.895095	1.293098×10^{-8}
0.5	0.824361	0.824361	2.847439×10^{-8}
0.6	0.728848	0.728848	4.114151×10^{-8}
0.7	0.604126	0.604126	2.726661×10^{-8}
0.8	0.445108	0.445108	1.487792×10^{-8}
0.9	0.245960	0.245960	6.976558×10^{-8}
1.0	0.000000	-8.648293×10^{-8}	8.648293×10^{-8}

Table 5
Comparison of absolute errors with other methods.

ξ	Present method	Golbabai and Javidi ($N = 8$) (Golbabai and Javidi, 2007)	Torvattanabun and Koonprasert (Torvattanabun and Koonprasert, 2010)
0.25	1.5137×10^{-9}	2.1630×10^{-9}	3.8922×10^{-10}
0.50	2.8474×10^{-8}	1.1571×10^{-7}	1.1571×10^{-7}
0.75	2.8074×10^{-8}	1.0479×10^{-6}	1.0479×10^{-6}
1.00	8.6423×10^{-8}	4.2188×10^{-6}	4.2188×10^{-6}

Table 6
Absolute errors for Example 4.

ξ	Exact value of $v(\xi)$	Approximate value of $v(\xi)$	Absolute error
−1.0	1.00000	1.00000	8.85958×10^{-14}
−0.8	1.09412	1.09412	8.65974×10^{-14}
−0.6	1.172550	1.172550	4.91607×10^{-13}
−0.4	1.228370	1.228370	5.80425×10^{-13}
−0.2	1.253130	1.253130	5.32907×10^{-15}
0	1.236540	1.236540	9.34564×10^{-12}
0.2	1.166160	1.166160	1.18881×10^{-10}
0.4	1.027010	1.027010	4.71345×10^{-11}
0.6	0.801195	0.801195	1.16518×10^{-9}
0.8	0.467325	0.467325	2.18991×10^{-9}
1.0	0.000000	-6.33957×10^{-9}	6.33957×10^{-9}

Table 7
Comparison of maximum absolute errors with other methods.

Present method	Siddiqi et al. (2009)	Siddiqi and Akram (2007)	Lamnii et al. (2008)
3.29×10^{-9}	1.97×10^{-6}	3.28×10^{-6}	1.86×10^{-8}

7. Conclusion

In this paper, the Legendre-homotopy analysis method is proposed using Legendre polynomials to approximate the solutions of linear and nonlinear higher order boundary value problems arising in mathematics and physics. The proposed scheme is a modification of the well-known homotopy analysis method which uses the orthogonal property of the well known Legendre polynomials to simplify the computations involved in each iteration. The resulting higher order deformation equations involve only polynomials which overcomes the difficulty arising in the calculation of integrals. The proposed scheme is effectively applied on different higher order linear and nonlinear boundary value problems. The absolute errors in the approximate solution values are calculated and summarized in Tables 1–7. Tables 2, 5 and 7 show the comparison of the proposed scheme with different methods available in literature revealing that the proposed method provides better approximations to the exact solutions. Figs. 2–5 show that the approximate solution curves match favorably well with the exact solution curves. The numerical computations and graphical illustrations are performed using Mathematica 8.0. The numerical and graphical results depict the efficiency and accuracy of the proposed method.

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