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Numerical solution for Fredholm–Volterra integral equation of the second kind by using collocation and Galerkin methods

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Abstract The Fredholm–Volterra integral equation of the second kind with continuous kernels with respect to position and time, is solved numerically, using the Collocation and Galerkin methods. Also the error, in each case, is estimated.

1. Introduction

Many problems of mathematical physics, engineering and contact problems in the theory of elasticity lead to integral equations. The following references Muskhelishvili et al. (1953), Green (1969), Atkinson et al. (1976) and Delves and Mohamed (1985), contain many different methods to solve the integral equations analytically. At the same time the numerical methods take an important place in solving the integral equations numerically. The references Linz et al. (1985), Kanwal et al. (1996), Atkinson et al. (1997) and Abdou and Mohamed (2002) contain many different methods for solving the integral equations numerically. The discussion of the Fredholm–Volterra integral equations numerically and analytically can be found in the works of Abdou and co-workers, see (Schiavane and Constanda, 2002; Abdou et al., 2003; Abdou and Salama, 2004), when the Fredholm integral term is considered in position and Volterra integral term in time. In all work of Abdou in Fredholm–Volterra integral equation when the kernel of position is continuous have not been solved.

Therefore, in this paper, we consider the Fredholm–Volterra integral equations of the second kind with continuous kernels with respect to position and time. The

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existence and uniqueness of the solution, under certain conditions, will be proved in the space $L_2[a,b]\times C[0,T]$, $0\leqslant t\leqslant T\leqslant\infty$.

A numerical method is used to represent the Fredholm–Volterra equation in the form of a linear system of Fredholm integral equations where the existence and uniqueness of the system are discussed. Also we used the Collocation and Galerkin methods to obtain a linear system of algebraic equations, which is also solved numerically. Moreover the error estimate, in each case, is calculating.

2. The existence and uniqueness of the solution

Consider the Fredholm–Volterra integral equation

$$\mu\phi(x,t)-\lambda\int_a^b k(x,y)\phi(y,t)dy-\lambda\int_0^t F(t,\tau)\phi(x,\tau)d\tau=f(x,t) \tag{2.1}$$

where $k(x,y)$ and $F(t,\tau)$ are continuous functions which represent the kernel of Fredholm and Volterra integral terms, respectively. The known function $f(x,t)$ is called the free term of the integral equation, while $\phi(x,t)$ is unknown and called the potential function. Here the Fredholm is considered in position, while Volterra in time. The constant μ defines the kind of integral equation while the constant λ, may be complex, has a physical meaning.

In order to guarantee the existence of a unique solution of (2.1). We assume the following:

(i) The kernel of position satisfies $|k(x,y)|\leqslant N_1$ for all, $a\leqslant x,y\leqslant b$, where N_1 is a constant.
(ii) The positive continuous function $F(t,\tau)\in C([0,T]\times[0,T])$ for all $0\leqslant t,\tau\leqslant T\prec\infty$ and satisfies $|F(t,\tau)|\leqslant N_2$ where N_2 is a constant.
(iii) The given function and its norm is defined as: $f(x,t)\in L_2[a,b]\times C[0,T]$,

$$\|f(x,t)\|=\max_{0\leqslant t\leqslant T}\int_0^t\int_a^b\{f^2(x,\tau)\}^{\frac{1}{2}}dxd\tau=N_3$$

where N_3 is a constant.
(iv) The unknown function $\phi(x,t)$ satisfy the Lipschitz condition with respect toposition $|\phi(x_1,t)-\phi(x_2,t)|\leqslant A(t)\ |x_1-x_2|$ and Hölder condition with respect to time $|\phi(x,t_1)-\phi(x,t_2)|\leqslant B(x)\ |t_1-t_2|^{\alpha},0\leqslant\alpha\leqslant 1$, and its norm is defined as

$$\|\phi(x,t)\|=\max_{0\leqslant t\leqslant T}\int_0^t\int_a^b\{\phi^2(x,\tau)\}^{\frac{1}{2}}dxd\tau=N_4$$

3. The system of Fredholm integral equations

For representing (2.1) as a system of Fredholm integral equations we use the following numerical method, see Delves and Mohamed (1985) and Atkinson et al. (1997).

Divide the interval [0, T] as

$0=t_0<t_1<\ldots<t_k<\ldots<t_N=T,\ \ i.e.\,t=t_k,$
$k=0,1,2,\ldots,N.$

Then, using the quadrature formula, Te Volterra term in (2.1) becomes

$$\int_0^{t_k}\phi(x,\tau)\ F(t_k,\tau)d\tau=\sum_{j=0}^{k}u_jF(t_k,t_j)\phi(x,t_j)+O(\hbar_i^{\tilde{p}+1}),\qquad(\hbar_k\to 0,\tilde{p}>0) \tag{3.1}$$

where $\hbar_k=\max_{0\leqslant j\leqslant k}h_j$, $h_j=t_{j+1}-t_j$

The values of k and the constant $\tilde{p}$ depend on the number of derivatives of $F(t,\tau)$, for all $\tau\in[0,\ T]$, w.r.t. t, and $u_0=\frac{1}{2}h_0,u_k=\frac{1}{2}\ h_k,u_i=h_i,(i\neq 0,k)$.

Using (3.1) in (2.1) after letting $t=t_k,k=1,2,\ldots,N$, We have

$$\mu\phi(x,t_k)=f(x,t_k)+\lambda\int_a^b k(x,y)\,\phi(y,t_k)dy+\lambda\sum_{j=0}^{k}u_jF(t_i,t_j)\,\phi(x,t_k) \tag{3.2}$$

or,

$$\mu\phi_k(x)=f_k(x)+\lambda\int_a^b k(x,y)\ \phi_k(y)dy+\lambda\sum_{j=0}^{k}u_jF_{kj}\phi_j(x)\phi_k(x)$$
$$=\phi(x,t_k),f_k(x)=f(x,t_k),F_{k},j=F(t_k,t_j), \tag{3.3}$$

the formula (3.3) become

$$\mu_n\phi_n(x)=G_n(x)+\lambda\int_a^b k(x,y)\phi_n(y)dy \tag{3.4}$$

where $\mu_n=\mu-\lambda u_nF_{n,n};\ \lambda_n=\lambda u_nF_{n,n}$,

$$G_n(x)=f_n(x)+\lambda\sum_{j=0}^{n-1}u_jF_{n,j}\phi_j(x),\quad n=0,1,\ldots,N$$

The formula (3.4) represents a linear system of Fredholm integral equations of the second kind, where $\lambda u_nF_{n,n}\neq\mu$.

Now, we will solve the linear system (3.4) using the Collocation method and Galerkin method.

3.1. Collocation method

Collocation method is based on approximating the solution $\phi(x,\ t)$ by a partial sum:

$$S(x,t_i)=\sum_{k=1}^{N}c_k(t_i)\psi_k(x) \tag{3.5}$$

of N linearly independent functions $\psi_1(x)$, $\psi_2(x)$, …, $\psi_N(x)$ on the interval (a,b). Therefore we have

$$\mu S_i(x)-\lambda\int_a^b k(x,y)S_i(y)dy$$
$$\approx f(x,t_i)+\lambda\sum_{j=0}^{i-1}w_jF_{i,j}S_j(x)+\varepsilon(x,c_1(t),c_2(t),..,c_N(t)+R(\hbar_i^{p+1}) \tag{3.6}$$

Of course, if the approximate solution (3.5) is to be substituted into (3.4) for $\phi(x,t)$, there will be an error $\varepsilon(x,c_1(t),c_2(t),\ldots,c_N(t))$. This error depends on x, t and the way for which the coefficients $c_1(t),c_2(t),\ldots,c_N(t)$ are chosen in the formula (3.6). Let $t=t_i\ ,i-0,1,2,\ldots,N$. Then using the quadrature formula, we have

$$\mu_iS_i(x_m)-\lambda\int_a^b k(x_m,y)S_i(y)dy$$
$$\approx f(x_m,t_i)+\lambda\sum_{j=0}^{i-1}w_jF_{i,j}S_j(x_m),i,m=0,1,2,\ldots,N \tag{3.7}$$

For determining the coefficients $c_1(t_i),c_2(t_i),\ldots,c_N(t_i)$ of the approximate solution $S_N(x_N)$, from (3.5), in terms of the given N linearly independent functions $\psi_1(x)$, $\psi_2(x)$, …,

$\psi_N(x)$, perform the integration, then substitute $x = x_1, x_2, \ldots, x_N$ for which the error $\varepsilon(x, c_1(t), c_2(t), \ldots, c_N(t))$ vanishes.

Substituting from (3.5) in (3.6), we get:

$$\mu_i \sum_{k=1}^{N} c_k(t_i)\psi_k(x_m) - \lambda \sum_{k=1}^{N} c_k(t_i) \int_a^b k(x_m, y)\psi_k(y)dy = f_{mi} + \lambda \sum_{j=0}^{i-1} \sum_{k=1}^{N} w_j F_{i,j} c_k(t_j)\psi_k(x_m) \tag{3.8}$$

3.2. Galerkin method

This method establishes the N conditions necessary for the determination of the N coefficients in Eq. (3.5):

By making the error $\varepsilon(x, c_1(t), c_2(t), \ldots, c_N(t))$ in (3.6) orthogonal to N given linearly independent functions $\psi_1(x), \psi_2(x), \ldots, \psi_N(x)$ on the interval (a, b), i.e.

$$\int_a^b \psi_j(x)\varepsilon(x, c_1(t), c_2(t), \ldots, c_N(t))dx = 0 \tag{3.9}$$

Then from (3.6), we have

$$\int_a^b \psi_j(x)[\mu_i S_i(x_m) - f(x_m, y) - \lambda \int_a^b k(x_m, y)S_i(y)dy - \lambda \sum_{j=0}^{i-1} w_j F_{i,j} S_j(x) - R(\hbar_i^{p+1})]dx = 0 \tag{3.10}$$

Eq. (3.10) can be written in the form:

$$\int_a^b \psi_j(x)[\mu_i S_i(x_m) - \lambda \int_a^b [k(x_m, y)S_i(y)dy - \lambda \sum_{j=0}^{i-1} w_j F_{i,j} S_j(x_m) - R(\hbar_i^{p+1})]dx = \int_a^b \psi_j(x).f(x_m, t_i)dx \ (i, m = 0, 1, \ldots, N), \tag{3.11}$$

where $R(\hbar_i^{p+1})$ is the error from dividing the time and $\hbar_i = \max_{0 \leqslant j \leqslant i} h_j, h_j = t_{j+1} - t_j$. The values of i and the constant p depend on the derivatives of $F(t, \tau)$, for all $\tau \in [0, T]$, with respect to t. Substituting from (3.5) into (3.11) we get

$$\int_a^b \psi_j(x)[\mu_i \sum_{k=1}^{N} c_k(t_i)\psi_k(x_m) - \lambda \int_a^b k(x_m, y) \cdot \sum_{k=1}^{N} c_k(t_i)\psi_k(y)dy - \lambda \sum_{j=0}^{i-1} w_j F_{i,j} \sum_{k=1}^{N} c_k(t_j)\psi_k(x_m)]dx = \int_a^b \psi_{j\cdot}(x)f(x_m, t_i)dx \tag{3.12}$$

4. Examples

Example 4.1. Consider the integral equation:

$$\phi(x, t) = f(x, t) + \lambda \int_0^t \tau^2 \phi(x, \tau)d\tau + \lambda \int_a^b e^{x+y}\phi(y, t)dy$$

where the exact solution $\phi(x, t) = t^2 e^x$.

4.1. Using collocation method

In Eq. (4.1) we shall take $N = 2$, $a = 0$, $b = 1$,

$$f(x, t) = \frac{3}{2}t^2 e^x - \frac{1}{5}e^x t^5 - \frac{1}{2}t^2 e^x e^2$$

Let the approximate solution has the form of Eq. (3.5), the three independent functions are $\psi_1(x) = 1, \psi_2(x) = x, \psi_3(x) = x^2$. Substituting these values in Eq. (3.8), then solving the equations formulas when $x = 0, 0.5, 1.0$, in this case $R = 0$. We get:

$$\begin{aligned} &c_1(t_0) = 0, \quad c_2(t_0) = 0, \quad c_3(t_0) = 0. \\ &c_1(t_1) = 0.0005130967957, \quad c_2(t_1) = 0.000432839493, \\ &c_3(t_1) = 0.000415594769. \end{aligned} \tag{4.1}$$

$$\begin{aligned} &c_1(t_2) = 0.002104378751, \quad c_2(t_2) = 0.001731361786, \\ &c_3(t_2) = 0.001662382707. \end{aligned} \tag{4.2}$$

So, the solution, for $t \in [0, 0.03]$, takes the form (see Table 1):

$$\begin{aligned} S(x, t_0) &= 0 \\ S(x, t_1) &= 0.0005130967957 + 0.000432839493x \\ &\quad - 0.000415594769x^2 \\ S(x, t_2) &= 0.002104378751 + 0.001731361786x \\ &\quad + 0.001662382707x^2 \end{aligned} \tag{4.3}$$

4.2. Using Galerkin method

As in collocation method, using (3.9) in (3.12), Choose three independent functions $\psi_1(x) = 1, \psi_2(x) = x, \psi_3(x) = x^2$ and three points $x = 0, 0.5, 1.00$, when we assume $t \in [0, 0.03]$, then we have:

$$\begin{aligned} &c_1(t_0) = 0, \quad c_2(t_0) = 0, \quad c_3(t_0) = 0. \\ &c_1(t_1) = 0.00051949274, \quad c_2(t_1) = 0.00042025907, \\ &c_3(t_1) = 0.00041436290. \\ &c_1(t_2) = 0.00212991168, \quad c_2(t_2) = 0.0016810402, \\ &c_3(t_2) = 0.0016574556. \end{aligned} \tag{4.4}$$

Table 1 Values of the error E^C, E^G using collocation and Galerkin methods.

x	$\phi(x, t)$	E^C	E^G
$t = 0$			
0	0	0	0
0.5	0	0	0
1.00	0	0	0
$t = 0.01500000000$			
0	0.000225	0.0002880967957	0.00029449274
0.5	0.0003709622860	0.0004624529484	0.0004622507140
1.00	0.0006116134113	0.0007499176467	0.0007425012987
$t = 0.030000000$			
0	0.0009	0.001204378714	0.00122991170
0.5	0.001483849144	0.001901806122	0.001900946656
1.00	0.002446453645	0.003051669518	0.003021953955

$$
\begin{aligned}
&S(x,t_0) = 0,\\
&S(x,t_1) = 0.00051949274 + 0.00042025907x\\
&\qquad + 0.00041436290x^2,\\
&S(x,t_2) = 0.00212991168 + 0.0016810402x\\
&\qquad + 0.0016574556x^2.
\end{aligned}
\tag{4.5}
$$

So, the solution is taken from:

Example 4.2. Consider the integral equation:

$$\phi(x,t) = f(x,t) + \lambda \int_0^t t\tau\phi(x,\tau)d\tau + \lambda \int_a^b e^{-y}\phi(y,t)dy, \tag{4.6}$$

where the exact solution $\phi(x,t) = te^{-x}$.

4.3. Using collocation method

In Eq. (4.6) we shall take $N = 2$, $a = 0$, $b = 1$,

$f(x,t) = -0.00432t + e^{-x}t - 0.333e^{-x}t^4$.

Let the approximate solution in the form of Eq. (3.5), then choose three independent functions $\psi_1(x) = 1, \psi_2(x) = x, \psi_3(x) = x^2$. Substituting these values in Eq. (3.8), then solving the equations formulas, when $x = 0, 0.5, 1.00$, in this case $R = 0$, we get:

$$
\begin{aligned}
&c_1(t_0) = 0, \quad c_2(t_0) = 0, \quad c_3(t_0) = 0.\\
&c_1(t_1) = 0.008541939634, \quad c_2(t_1) = -.01412633660,\\
&c_3(t_1) = 0.004644538564. \quad c_1(t_2) = 0.01708389624,\\
&c_2(t_2) = -0.02825268914, \quad c_3(t_2) = 0.009289082405
\end{aligned}
\tag{4.7}
$$

So, the solution, for $t \in [0, 0.03]$ takes the form:

$$
\begin{aligned}
&S(x,t_0) = 0,\\
&S(x,t_1) = 0.008541939634 - 0.01412633660x + 0.004644538564x^2,\\
&S(x,t_2) = 0.01708389624 - \quad 0.02825268914x + 0.009289082405x^2.
\end{aligned}
\tag{4.8}
$$

4.4. Using Galerkin method

We choose three independent functions $\psi_1(x) = 1, \psi_2(x) = x$, $\psi_3(x) = x^2$ and three points $x = 0, 0.5, 1.00$, when we assume $t \in [0, 0.03]$, we have:

$$
\begin{aligned}
&c_1(t_0) = 0, \quad c_2(t_0) = 0, \quad c_3(t_0) = 0.\\
&c_1(t_1) = 0.008459192683, c_2(t_1) = -0.01395821152,\\
&c_3(t_1) = 0.00463077286.\\
&c_1(t_2) = 0.01691840427, c_2(t_2) = -0.02791644232,\\
&c_3(t_2) = 0.00926155092.
\end{aligned}
\tag{4.9}
$$

So, the solution is taken the form (see Table 2):

$$
\begin{aligned}
&S(x,t_0) = 0,\\
&S(x,t_1) = 0.008459192683 - 0.01395821152x\\
&\qquad + 0.00463077286x^2,\\
&S(x,t_2) = 0.01691840427 - 0.02791644232x\\
&\qquad + 0.00926155092x^2.
\end{aligned}
\tag{4.10}
$$

Table 2 Values of the error E^C, E^G using collocation and Galerkin methods.

x	$\phi(x,t)$	E^C	E^G
$t = 0$			
0	0	0	0
0.5	0	0	0
1.00	0	0	0
$t = 0.01500000000$			
0	0.015	0.006458060366	0.006540807317
0.5	0.009097959896	0.006458053921	0.006460179758
1.00	0.005518191618	0.006458050020	0.006386437595
$t = 0.030000000$			
0	0.03	0.01291610376	0.01308159573
0.5	0.01819591979	0.01291609752	0.01292034895
1.00	0.01103638324	0.01291609374	0.01277287037

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