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Existence and uniqueness results for a fractional Riemann–Liouville nonlocal thermistor problem on arbitrary time scales

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ABSTRACT

Using a fixed point theorem in a proper Banach space, we prove existence and uniqueness results of positive solutions for a fractional Riemann–Liouville nonlocal thermistor problem on arbitrary nonempty closed subsets of the real numbers.

1. Introduction

The calculus on time scales is a recent area of research introduced by Aulbach and Hilger (1990), unifying and extending the theories of difference and differential equations into a single theory. A time scale is a model of time, and the theory has found important applications in several contexts that require simultaneous modeling of discrete and continuous data. It is under strong current research in areas as diverse as the calculus of variations, optimal control, economics, biology, quantum calculus, communication networks and robotic control. The interested reader is refereed to Agarwal and Bohner (1999), Agarwal et al. (2002), Bohner and Peterson (2001a,b), Martins and Torres (2009), Ortigueira et al. (2016) and references therein.

On the other hand, many phenomena in engineering, physics and other sciences, can be successfully modeled by using mathematical tools inspired by the fractional calculus, that is, the theory of derivatives and integrals of noninteger order. See, for example, Gaul et al. (1991), Hilfer (2000), Kilbas et al. (2006), Sabatier et al. (2007), Samko et al. (1993) and Srivastava and Saxena (2001). This allows one to describe physical phenomena more accurately. In this line of thought, fractional differential equations have emerged in recent years as an interdisciplinary area of research (Abbas et al., 2012). The nonlocal nature of fractional derivatives can be utilized to simulate accurately diversified natural phenomena containing long memory (Debbouche and Torres, 2015; Machado et al., 2011).

A thermistor is a thermally sensitive resistor whose electrical conductivity changes drastically by orders of magnitude, as the temperature reaches a certain threshold. Thermistors are used as temperature control elements in a wide variety of military and industrial equipment, ranging from space vehicles to air conditioning controllers. They are also used in the medical field, for localized and general body temperature measurement; in meteorology, for

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weather forecasting; as well as in chemical industries, as process temperature sensors (Kwok, 1995; Maclen, 1979).

Throughout the reminder of the paper, we denote by $\mathbb{T}$ a time scale, which is a nonempty closed subset of $\mathbb{R}$ with its inherited topology. For convenience, we make the blanket assumption that t_0 and T are points in $\mathbb{T}$. Our main concern is to prove existence and uniqueness of solution to a fractional order nonlocal thermistor problem of the form

$$\begin{aligned} {}_{t_0}^{\mathbb{T}}D_t^{2\alpha}u(t) &= \frac{\lambda f(u)}{\left(\int_{t_0}^{T} f(u)\,\triangle x\right)^2}, \quad t\in(t_0,T),\\ {}_{t_0}^{\mathbb{T}}I_t^{\beta}u(t_0) &= 0, \quad \forall\beta\in(0,1), \end{aligned} \tag{1}$$

under suitable conditions on f as described below. We assume that $\alpha\in(0,1)$ is a parameter describing the order of the fractional derivative; ${}_{t_0}^{\mathbb{T}}D_t^{2\alpha}$ is the left Riemann–Liouville fractional derivative operator of order 2α on ${}_{t_0}^{\mathbb{T}}I_t^{\beta}$ is the left Riemann–Liouville fractional integral operator of order β defined on $\mathbb{T}$ by Benkhettou et al. (2016b). By u, we denote the temperature inside the conductor; $f(u)$ is the electrical conductivity of the material.

In the literature, many existence results for dynamic equations on time scales are available (Dogan, 2013a; Dogan, 2013b). In recent years, there has been also significant interest in the use of fractional differential equations in mathematical modeling (Aghababa, 2015; Ma et al., 2016; Yu et al., 2016). However, much of the work published to date has been concerned separately, either by the time-scale community or by the fractional one. Results on fractional dynamic equations on time scales are scarce (Ahmadkhanlu and Jahanshahi, 2012).

In contrast with our previous works, which make use of fixed point theorems like the Krasnoselskii fixed point theorem, the fixed point index theory, and the Leggett–Williams fixed point theorem, to obtain several results of existence of positive solutions to linear and nonlinear dynamic equations on time scales, and recently also to fractional differential equations (Sidi Ammi et al., 2012; Sidi Ammi and Torres, 2012b, 2013; Souahi et al., 2016); here we prove new existence and uniqueness results for the fractional order nonlocal thermistor problem on time scales (1), putting together time scale and fractional domains. This seems to be quite appropriate from the point of view of practical applications (Machado et al., 2015; Nwaeze and Torres, 2017; Ortigueira et al., 2016).

The rest of the article is arranged as follows. In Section 2, we state preliminary definitions, notations, propositions and properties of the fractional operators on time scales needed in the sequel. Our main aim is to prove existence of solutions for (1) using a fixed point theorem and, consequently, uniqueness. This is done in Section 3: see Theorems 3.2 and 3.6.

2. Preliminaries

In this section, we recall fundamental definitions, hypotheses and preliminary facts that are used through the paper. For more details, see the seminal paper Benkhettou et al. (2016b). From physical considerations, we assume that the electrical conductivity is bounded (Antontsev and Chipot, 1994). Precisely, we consider the following assumption:

(H1). Function $f:\mathbb{R}^+\to\mathbb{R}^+$ of problem (1) is Lipschitz continuous with Lipschitz constant L_f such that $c_1\leqslant f(u)\leqslant c_2$, with c_1 and c_2 two positive constants.

We deal with the notions of left Riemann–Liouville fractional integral and derivative on time scales, as proposed in Benkhettou et al. (2016b), the so called BHT fractional calculus on time scales (Nwaeze and Torres, 2017). The corresponding right operators are obtained by changing the limits of integrals from a to t into t to b. For local approaches to fractional calculus on arbitrary time scales we refer the reader to Benkhettou et al. (2015, 2016a). Here we are interested in nonlocal operators, which are the ones who make sense with respect to the thermistor problem (Sidi Ammi and Torres, 2008, 2012a). Although we restrict ourselves to the delta approach on time scales, similar results are trivially obtained for the nabla fractional case (Girejko and Torres, 2012).

Definition 2.1 (*Riemann–Liouville fractional integral on time scales (Benkhettou et al., 2016b)*). Let $\mathbb{T}$ be a time scale and $[a,b]$ an interval of $\mathbb{T}$. Then the left fractional integral on time scales of order $0<\alpha<1$ of a function $g:\mathbb{T}\to\mathbb{R}$ is defined by

$${}_{a}^{\mathbb{T}}I_t^{\alpha}g(t)=\int_a^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)\,\triangle s,$$

where Γ is the Euler gamma function.

The left Riemann–Liouville fractional derivative operator of order α on time scales is then defined using Definition 2.1 of fractional integral.

Definition 2.2 (*Riemann–Liouville fractional derivative on time scales (Benkhettou et al., 2016b)*). Let $\mathbb{T}$ be a time scale, $[a,b]$ an interval of $\mathbb{T}$, and $0<\alpha<1$. Then the left Riemann–Liouville fractional derivative on time scales of order α of a function $g:\mathbb{T}\to\mathbb{R}$ is defined by

$${}_{a}^{\mathbb{T}}D_t^{\alpha}g(t)=\left(\int_a^t\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}g(s)\,\triangle s\right)^{\triangle}.$$

Remark 2.3. If $\mathbb{T}=\mathbb{R}$, then we obtain from Definitions 2.1 and 2.2, respectively, the usual left Rieman–Liouville fractional integral and derivative.

Proposition 2.4 (*See* Benkhettou et al. (2016b)). *Let $\mathbb{T}$ be a time scale, $g:\mathbb{T}\to\mathbb{R}$ and $0<\alpha<1$. Then*

$${}_{a}^{\mathbb{T}}D_t^{\alpha}g=\triangle\circ{}_{a}^{\mathbb{T}}I_t^{1-\alpha}g.$$

Proposition 2.5 (*See* Benkhettou et al. (2016b)). If $\alpha>0$ and $g\in C([a,b])$, then

$${}_{a}^{\mathbb{T}}D_t^{\alpha}\circ{}_{a}^{\mathbb{T}}I_t^{\alpha}g=g.$$

Proposition 2.6 (*See* Benkhettou et al. (2016b)). Let $g\in C([a,b])$ and $0<\alpha<1$. If ${}_{a}^{\mathbb{T}}I_t^{1-\alpha}u(a)=0$, then

$${}_{a}^{\mathbb{T}}I_t^{\alpha}\circ{}_{a}^{\mathbb{T}}D_t^{\alpha}g=g.$$

Theorem 2.7 (*See* Benkhettou et al. (2016b)). *Let $g\in C([a,b])$, $\alpha>0$, and ${}_{a}^{\mathbb{T}}I_t^{\alpha}([a,b])$ be the space of functions that can be represented by the Riemann–Liouville $\triangle$-integral of order α of some $C([a,b])$-function. Then,*

$$g\in{}_{a}^{\mathbb{T}}I_t^{\alpha}([a,b])$$

if and only if

$${}_{a}^{\mathbb{T}}I_t^{1-\alpha}g\in C^1([a,b])$$

and

$${}_{a}^{\mathbb{T}}I_t^{1-\alpha}g(a)=0.$$

The following result of the calculus on time scales is also useful.

Proposition 2.8 (*See Ahmadkhanlu and Jahanshahi (2012)*). *Let $\mathbb{T}$ be a time scale and g an increasing continuous function on the time-scale interval $[a,b]$. If G is the extension of g to the real interval $[a,b]$ defined by*

$$G(s) := \begin{cases} g(s) & \text{if } s \in \mathbb{T}, \\ g(t) & \text{if } s \in (t,\sigma(t)) \notin \mathbb{T}, \end{cases}$$

then

$$\int_a^b g(t)\triangle t \leqslant \int_a^b G(t)dt,$$

where $\sigma : \mathbb{T} \to \mathbb{T}$ is the forward jump operator of $\mathbb{T}$ defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$.

Along the paper, by $C([0,T])$ we denote the space of all continuous functions on $[0,T]$ endowed with the norm $\|x\|_\infty = \sup_{t\in[0,T]}\{|x(t)|\}$. Then, $X = (C([0,T]), \|\cdot\|)$ is a Banach space.

3. Main Results

We begin by giving an integral representation to our problem (1). Due to physical considerations, the only relevant case is the one with $0 < \alpha < \frac{1}{2}$. Note that this is coherent with our fractional operators with $2\alpha - 1 < 0$.

Lemma 3.1. *Let $0 < \alpha < \frac{1}{2}$. Problem (1) is equivalent to*

$$u(t) = \frac{\lambda}{\Gamma(2\alpha)} \int_{t_0}^t (t-s)^{2\alpha-1} \frac{f(u(s))}{\left(\int_{t_0}^T f(u)\triangle x\right)^2} \triangle s. \tag{2}$$

Proof. We have

$$\begin{aligned} {}_{t_0}^{\mathbb{T}}D_t^{2\alpha} u(t) &= \frac{\lambda}{\Gamma(2\alpha)} \left(\int_{t_0}^t (t-s)^{2\alpha-1} \frac{f(u(s))}{\left(\int_{t_0}^T f(u)\triangle x\right)^2} \triangle s \right)^{\triangle} \\ &= \left({}_{t_0}^{\mathbb{T}}I_t^{1-2\alpha} u(t) \right)^{\triangle} = \left(\triangle \circ {}_{t_0}^{\mathbb{T}}I_t^{1-2\alpha} \right) u(t). \end{aligned}$$

The result follows from Proposition 2.6: ${}_{t_0}^{\mathbb{T}}I_t^{2\alpha} \circ ({}_{t_0}^{\mathbb{T}}D_t^{2\alpha}(u)) = u$. □

For the sake of simplicity, we take $t_0 = 0$. It is easy to see that (1) has a solution $u = u(t)$ if and only if u is a fixed point of the operator $K : X \to X$defined by

$$Ku(t) = \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{f(u(s))}{\left(\int_0^T f(u)\triangle x\right)^2} \triangle s. \tag{3}$$

Follows our first main result.

Theorem 3.2 (*Existence of solution*). *Let $0 < \alpha < \frac{1}{2}$ and f satisfies hypothesis (H1). Then there exists a solution $u \in X$ of (1) for all $\lambda > 0$.*

3.1. Proof of Existence

In this subsection we prove Theorem 3.2. For that, firstly we prove that the operator K defined by (3) verifies the conditions of Schauder's fixed point theorem (Cronin, 1994).

Lemma 3.3. *The operator K is continuous.*

Proof. Let us consider a sequence u_n converging to u in X. Then,

$$\begin{aligned} |Ku_n(t) - Ku(t)| &\leqslant \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \left| \frac{f(u_n(s))}{\left(\int_0^T f(u_n)\triangle x\right)^2} - \frac{f(u(s))}{\left(\int_0^T f(u)\triangle x\right)^2} \right| \triangle s \\ &\leqslant \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \left| \frac{1}{\left(\int_0^T f(u_n)\triangle x\right)^2} (f(u_n(s)) - f(u(s))) \right. \\ &\quad \left. + f(u(s)) \left(\frac{1}{\left(\int_0^T f(u_n)\triangle x\right)^2} - \frac{1}{\left(\int_0^T f(u)\triangle x\right)^2} \right) \right| \\ &\leqslant \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{1}{\left(\int_0^T f(u_n)\triangle x\right)^2} |f(u_n(s)) - f(u(s))| \triangle s \\ &\quad + \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |f(u(s))| \left| \frac{1}{\left(\int_0^T f(u_n)\triangle x\right)^2} - \frac{1}{\left(\int_0^T f(u)\triangle x\right)^2} \right| \leqslant I_1 + I_2. \end{aligned} \tag{4}$$

We estimate both right-hand terms separately. By hypothesis (H1) and Proposition 2.8, we have

$$\begin{aligned} I_1 &\leqslant \frac{\lambda}{(c_1T)^2\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |f(u_n(s)) - f(u(s))| \triangle s \\ &\leqslant \frac{\lambda L_f}{(c_1T)^2\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |u_n(s) - u(s)| \triangle s \\ &\leqslant \frac{\lambda L_f}{(c_1T)^2\Gamma(2\alpha)} \|u_n - u\|_\infty \int_0^t (t-s)^{2\alpha-1} \triangle s \\ &\leqslant \frac{\lambda L_f}{(c_1T)^2\Gamma(2\alpha)} \|u_n - u\|_\infty \int_0^t (t-s)^{2\alpha-1} ds, \end{aligned}$$

since $(t-s)^{2\alpha-1}$ is nondecreasing. Then,

$$I_1 \leqslant \frac{\lambda T^{2\alpha} L_f}{(c_1T)^2\Gamma(2\alpha+1)} \|u_n - u\|_\infty. \tag{5}$$

Once again, since $(t-s)^{2\alpha-1}$ is nondecreasing, we have

$$\begin{aligned} I_2 &\leqslant \frac{\lambda}{\Gamma(2\alpha)} \int_0^t \frac{(t-s)^{2\alpha-1}|f(u(s))| \left| \left(\int_0^T f(u_n)\triangle x\right)^2 - \left(\int_0^T f(u)\triangle x\right)^2 \right|}{\left(\int_0^T f(u_n)\triangle x\right)^2 \left(\int_0^T f(u)\triangle x\right)^2} \triangle s \\ &\leqslant \frac{\lambda c_2}{(c_1T)^4\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \left| \left(\int_0^T f(u_n)\triangle x\right)^2 - \left(\int_0^T f(u)\triangle x\right)^2 \right| \triangle s \\ &\leqslant \frac{\lambda c_2}{(c_1T)^4\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \left| \left(\int_0^T (f(u_n) - f(u))\triangle x\right) \left(\int_0^T (f(u_n) + f(u))\triangle x\right) \right| \triangle s \\ &\leqslant \frac{2\lambda c_2^2 T}{(c_1T)^4\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \left(\int_0^T |f(u_n) - f(u)| \triangle x\right) \triangle s \\ &\leqslant \frac{2\lambda c_2^2 T L_f}{(c_1T)^4\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \left(\int_0^T |u_n(x) - u(x)| \triangle x\right) \triangle s \\ &\leqslant \frac{2\lambda c_2^2 T^2 L_f}{(c_1T)^4\Gamma(2\alpha)} \|u_n - u\|_\infty \int_0^t (t-s)^{2\alpha-1} \triangle s \\ &\leqslant \frac{2\lambda c_2^2 T^2 L_f}{(c_1T)^4\Gamma(2\alpha)} \|u_n - u\|_\infty \int_0^t (t-s)^{2\alpha-1} ds \\ &\leqslant \frac{2\lambda c_2^2 T^{2(\alpha+1)} L_f}{(c_1T)^4\Gamma(2\alpha+1)} \|u_n - u\|_\infty. \end{aligned}$$

It follows that

$$I_2 \leqslant \frac{2\lambda c_2^2 T^{2(\alpha+1)} L_f}{(c_1T)^4\Gamma(2\alpha+1)} \|u_n - u\|_\infty. \tag{6}$$

Bringing inequalities (5) and (6) into (4), we have

$$\begin{aligned} |Ku_n(t) - Ku(t)| &\leqslant I_1 + I_2 \\ &\leqslant \left(\frac{\lambda T^{2\alpha} L_f}{(c_1T)^2\Gamma(2\alpha+1)} + \frac{2\lambda c_2^2 T^{2(\alpha+1)} L_f}{(c_1T)^4\Gamma(2\alpha+1)} \right) \|u_n - u\|_\infty. \end{aligned}$$

Then

$$\|Ku_n - Ku\|_\infty \leqslant \left(\frac{\lambda T^{2\alpha} L_f}{(c_1T)^2\Gamma(2\alpha+1)} + \frac{2\lambda c_2^2 T^{2(\alpha+1)} L_f}{(c_1T)^4\Gamma(2\alpha+1)} \right) \|u_n - u\|_\infty. \tag{7}$$

Hence, independently of λ, the right-hand side of the above inequality converges to zero as $u_n \to u$. Therefore, $Ku_n \to Ku$. This proves the continuity of K. □

Lemma 3.4. *The operator K sends bounded sets into bounded sets on $\mathbb{C}([0,T],\mathbb{R})$.*

Proof. Let $I=[0,T]$. We need to prove that for all $r>0$ there exists $l>0$ such that for all $u \in B_r = \{u \in \mathbb{C}(I,\mathbb{R}), \|u\|_\infty \leqslant r\}$ we have $\|K(u)\|_\infty \leqslant l$. Let $t \in I$ and $u \in B_r$. Then,

$$\begin{aligned}|K(u(t))| &\leqslant \frac{\lambda}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}\frac{|f(u(s))|}{\left(\int_0^T f(u)\,\triangle x\right)^2}\triangle s\\ &\leqslant \frac{\lambda M}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}\triangle s\\ &\leqslant \frac{\lambda M}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}ds\\ &\leqslant \frac{\lambda M T^{2\alpha}}{\Gamma(2\alpha+1)},\end{aligned}$$

where $M = \frac{\sup_{B_r} f}{(c_1T)^2}$. Hence, taking the supremum over t, it follows that

$$\|K(u)\|_\infty \leqslant \frac{\lambda M T^{2\alpha}}{\Gamma(2\alpha+1)},$$

that is, $K(u)$ is bounded. □

Now, we shall prove that $K(B_r)$ is an equicontinuous set in X. This ends the proof of our Theorem 3.2.

Lemma 3.5. *The operator K sends bounded sets into equicontinuous sets of $\mathbb{C}(I,\mathbb{R})$.*

Proof. Let $t_1,t_2 \in I$ such that $0 \leqslant t_1 < t_2 \leqslant T$, B_r is a bounded set of $\mathbb{C}(I,\mathbb{R})$ and $u \in B_r$. Then,

$$|K(u(t_2)) - K(u(t_1))| \leqslant \frac{\lambda}{\Gamma(2\alpha)}\left|\int_0^{t_2}(t_2-s)^{2\alpha-1}\frac{f(u(s))}{\left(\int_0^T f(u)\,\triangle x\right)^2}\triangle s - \int_0^{t_1}(t_1-s)^{2\alpha-1}\frac{f(u(s))}{\left(\int_0^T f(u)\,\triangle x\right)^2}\triangle s\right|$$

$$\leqslant \frac{\lambda}{\Gamma(2\alpha)}\left|\int_0^{t_2}\left((t_2-s)^{2\alpha-1}-(t_1-s)^{2\alpha-1}+(t_1-s)^{2\alpha-1}\right)\frac{f(u(s))}{\left(\int_0^T f(u)\,\triangle x\right)^2}\triangle s\right.$$

$$\left.-\int_0^{t_1}(t_1-s)^{2\alpha-1}\frac{f(u(s))}{\left(\int_0^T f(u)\,\triangle x\right)^2}\triangle s\right| \leqslant \frac{\lambda}{\Gamma(2\alpha)}\left|\int_0^{t_2}\left((t_2-s)^{2\alpha-1}-(t_1-s)^{2\alpha-1}\right)\frac{f(u(s))}{\left(\int_0^T f(u)\,\triangle x\right)^2}\triangle s\right.$$

$$\left.+\int_{t_1}^{t_2}(t_1-s)^{2\alpha-1}\frac{f(u(s))}{\left(\int_0^T f(u)\,\triangle x\right)^2}\triangle s\right|$$

$$\begin{aligned}&\leqslant \frac{\lambda c_2}{(c_1T)^2\Gamma(2\alpha)}\left|\int_0^{t_2}\left((t_2-s)^{2\alpha-1}-(t_1-s)^{2\alpha-1}\right)ds + \int_{t_1}^{t_2}(t_1-s)^{2\alpha-1}ds\right|\\ &\leqslant \frac{\lambda c_2}{(c_1T)^2\Gamma(2\alpha+1)}\left|t_1^{2\alpha}-t_2^{2\alpha}+(t_1-t_2)^{2\alpha}-(t_1-t_2)^{2\alpha}\right|\\ &\leqslant \frac{\lambda c_2}{(c_1T)^2\Gamma(2\alpha+1)}\left|t_2^{2\alpha}-t_1^{2\alpha}\right|.\end{aligned}$$

Because the right-hand side of the above inequality does not depend on u and tends to zero when $t_2 \to t_1$, we conclude that $K(\overline{B_r})$ is relatively compact. Hence, B is compact by the Arzela–Ascoli theorem. Consequently, since K is continuous, it follows by Schauder's fixed point theorem (Cronin, 1994) that problem (1) has a solution on I. This ends the proof of Theorem 3.2. □

3.2. Uniqueness

We now derive uniqueness of solution to problem (1).

Theorem 3.6 (*Uniqueness of solution*). *Let $0<\alpha<\frac{1}{2}$ and f satisfies hypothesis* (H1). *If*

$$0 < \lambda < \left(\frac{T^{2\alpha}L_f}{(c_1T)^2\Gamma(2\alpha+1)} + \frac{2c_2^2T^{2(\alpha+1)}L_f}{(c_1T)^4\Gamma(2\alpha+1)}\right)^{-1},$$

then the solution predicted by Theorem 3.2 is unique.

Proof. Let u and v be two solutions of (1). Then, from (7), one has

$$\|Kv - Ku\|_\infty \leqslant \left(\frac{\lambda T^{2\alpha}L_f}{(c_1T)^2\Gamma(2\alpha+1)} + \frac{2\lambda c_2^2T^{2(\alpha+1)}L_f}{(c_1T)^4\Gamma(2\alpha+1)}\right)\|v-u\|_\infty.$$

Choosing λ such that $0<\lambda<\left(\frac{T^{2\alpha}L_f}{(c_1T)^2\Gamma(2\alpha+1)} + \frac{2c_2^2T^{2(\alpha+1)}L_f}{(c_1T)^4\Gamma(2\alpha+1)}\right)^{-1}$, the map $K : X \to X$is a contraction. It follows by the Banach principle that it has a fixed point $u = Fu$. Hence, there exists a unique $u \in X$ that is solution of (2). □

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