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Extension of the operational Tau method for solving 1-D nonlinear transient heat conduction equations

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Abstract In this paper, we consider a class of nonlinear transient heat conduction equations with some supplementary conditions. We apply the operational Tau method with arbitrary polynomial bases to approximate the solution of these equations. In addition, some theoretical results are given to simplify and reduce the computational cost. Finally some numerical examples are given to clarify the efficiency and accuracy of the proposed method.

1. Introduction

A parabolic partial differential equation has extensive applications in engineering and applied sciences such as phenomena of dispersion, diffusion, conduction, convection, reaction and dissipation (Su et al., 2009; Helal, 2012; Khan et al., 2012; Ozisik, 1993; Rahman, 2002; Wazwaz, 2009). Most of these equations are usually difficult to solve analytically, therefore approximate or numerical techniques must be used. We are interested in presenting an approximate scheme based on the operational Tau method to solve nonlinear transient heat conduction equations with variable thermo-physical properties which can involve heat generation terms.

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The Tau method can be described as a spectral method with arbitrary polynomial bases. In fact, the Tau method is a modification of the spectral Galerkin method that is applicable to problems with non-periodic boundary conditions. The main difference between them are the test functions which are not required individually to satisfy the boundary conditions in the Tau method (Canuto et al., 2006; Gottlieb and Orszag, 1977).

Ortiz and Samara (1981) proposed an operational approach to the Tau method as an approximation technique for solving nonlinear ordinary differential equations with supplementary conditions. The advantages of this technique are using simple operational matrices which reduce the computational costs remarkably, since only non zero elements of these matrices are needed to save.

During recent years, much work has been done for solving various types of ordinary differential equations, partial differential equations, and integral and integro-differential equations (Ebadi et al., 2007; Ghoreishi and Hadizadeh, 2009; Hosseini, 2009; Hosseini Aliabadi and Shahmorad, 2002; Liu and Ortiz, 1989; Liu and Pan, 1999) by the Tau method.

In this work, we state the required preliminaries of the operational Tau method to apply on nonlinear second-order partial differential equations. Then, we present a numerical

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scheme for solving 1-D nonlinear transient heat conduction equation of the form

$$\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right)+f(x,t)=\rho c_p\frac{\partial u}{\partial t},\quad x\in I=[a,b],\ t>0, \tag{1}$$

with the supplementary conditions

$$B_{c_1}u(x,t)=\varphi_1(t), \tag{2}$$

$$B_{c_2}u(x,t)=\varphi_2(t), \tag{3}$$

$$I_c u(x,t)=\psi(x), \tag{4}$$

where $f(x,t)$, $\varphi_1(t)$, $\varphi_2(t)$ and $\psi(x)$ are given smooth real valued functions and the linear operators B_{c_1}, B_{c_2} and I_c define the left-hand side of the supplementary conditions. Note that (2) and (3) denote the boundary conditions and (4) denotes the initial condition. For example, these conditions can be expressed as follows:

$$c_{11}u(a,t)+c_{12}\frac{\partial u}{\partial x}(a,t)=\varphi_1(t), \tag{5}$$

$$c_{21}u(b,t)+c_{22}\frac{\partial u}{\partial x}(b,t)=\varphi_2(t), \tag{6}$$

$$u(x,t_0)=\psi(x), \tag{7}$$

where c_{ij} are constants.

The parameters ρ and c_p in Eq. (1) denote density and specific heat, respectively. Also, we assume that the thermal conductivity varies with temperature in the form

$$k(u)=k_0(1+\alpha_1 u+\alpha_2 u^2), \tag{8}$$

where k_0 is the value of $k(u)$ at reference temperature and α_1, α_2 are known coefficients.

This paper is organized as follows: In the next section, we briefly review some preliminaries of the Tau method. Also, we present some theorems and lemmas to formulate nonlinear transient heat conduction equation with some given supplementary conditions. In Section 3, we recall an efficient Tau error estimator. Numerical results of some problems are given in Section 4 to clarify the efficiency of the method. Finally, Section 5 contains the conclusion.

2. Some theoretical results

The operational approach to the Tau method proposed by Ortiz and Samara (1981) based on the use of following simple matrices

$$\mu=[\mu_{ij}]_{i,j=0}^{\infty},\quad \mu_{ij}=\delta_{i+1,j},$$

$$\eta=[\eta_{ij}]_{i,j=0}^{\infty},\quad \eta_{ij}=(j+1)\delta_{i,j+1}$$

having the following properties:

Lemma 2.1 Ortiz and Samara, 1981. *If $y_N(x)=\underline{a}_N X_x$, where $\underline{a}_N=(a_0,a_1,\ldots,a_N,0,\ldots)$ and $X_x=(1,x,\ldots,x^N,\ldots)^T$, then*

(a) $xy_N(x)=\underline{a}_N\mu X_x$;
(b) $\frac{d}{dx}y_N(x)=\underline{a}_N\eta X_x$.

Corollary 2.2 Ortiz and Samara, 1981. *Generally, under assumptions of* Lemma 2.1, *we have*

(a) $x^i X_x=\mu^i X_x$;
(b) $\frac{d^r}{dx^r}X_x=\eta^r X_x$.

Let $\underline{\phi}_x=\{\phi_i(x): i\in\mathcal{N}=\{0,1,\ldots,N\}\}$ be a polynomial basis given by $\underline{\phi}_x=\widehat{\Phi}\underline{X}_x$, where $\widehat{\Phi}$ is a nonsingular lower triangular coefficients matrix and $\underline{X}_x=(1,x,\ldots,x^N)^T$ is the standard basis. In this work, we assume the approximate solution has the truncated series form

$$u_N(x,t)=\sum_{i=0}^{N}\sum_{j=0}^{N}u_{i(N+1)+j}\phi_i(x)\phi_j(t)=\underline{u}\underline{\phi}_{x,t}=\underline{u}\Phi\underline{X}_{x,t}, \tag{9}$$

where $\underline{u}=(u_0,u_1,\ldots,u_{(N+1)^2-1})$ is the vector of unknown coefficients, $\underline{\phi}_{x,t}=\underline{\phi}_x\otimes\underline{\phi}_t$ is a basis for the space of bivariate orthogonal polynomials with the lower triangular coefficients matrix $\Phi=\widehat{\Phi}\otimes\widehat{\Phi}$ and $\underline{X}_{x,t}=\underline{X}_x\otimes\underline{X}_t$ is a standard basis for bivariate polynomials. Note that $\otimes$ denotes the kronecker product.

In the remaining part of this paper, we assume that μ and η are $(N+1)\times(N+1)$ matrices.

We proceed to convert Eq. (1) with the supplementary conditions (2)–(4) to the corresponding nonlinear system of algebraic equations. To this end, we state some useful lemmas and theorems.

Theorem 2.3. *If $u_N(x,t)=\underline{u}\underline{\phi}_{x,t}$,* then

(a) *$xu_N(x,t)=\underline{u}\hat{\mu}_x\underline{\phi}_{x,t}$, where $\hat{\mu}_x=\Phi\tilde{\mu}_x\Phi^{-1}$, $\tilde{\mu}_x=\mu\otimes I$ with the elements $(\tilde{\mu}_x)_{ij}=\delta_{i+N+1,j}, i=0,1,\ldots,N(N+1)-1$, I is an $(N+1)\times(N+1)$ identity matrix and δ denote the kronecker delta;*
(b) *$tu_N(x,t)=\underline{u}\hat{\mu}_t\underline{\phi}_{x,t}$, where $\hat{\mu}_t=\Phi\tilde{\mu}_t\Phi^{-1}$ and $\tilde{\mu}_t=I\otimes\mu$ with the elements $(\tilde{\mu}_t)_{ii}=\mu, i=0,1,\ldots,N$;*
(c) *$\frac{\partial^p}{\partial x^p}u_N(x,t)=\underline{u}\hat{\eta}_x^p\underline{\phi}_{x,t}$ for $p\in\mathbb{N}$, where $\hat{\eta}_x=\Phi\tilde{\eta}_x\Phi^{-1}$ and $\tilde{\eta}_x=\eta\otimes I$ with the elements $(\tilde{\eta}_x)_{ij}=(j+1)\delta_{i,j+1}I, j=0,1,\ldots,N-1$;*
(d) *$\frac{\partial^q}{\partial t^q}u_N(x,t)=\underline{u}\hat{\eta}_t^q\underline{\phi}_{x,t}$ for $q\in\mathbb{N}$, where $\hat{\eta}_t=\Phi\tilde{\eta}_t\Phi^{-1}$ and $\tilde{\eta}_t=I\otimes\eta$ with the elements $(\tilde{\eta}_t)_{ii}=\eta, i=0,1,\ldots,N$.*

Proof.

(a) By Corollary 2.2 and kronecker product properties, we can write

$$\begin{aligned}xu(x,t)&=x(\underline{u}\underline{\phi}_{x,t})=\underline{u}\Phi x(\underline{X}_x\otimes\underline{X}_t)=\underline{u}\Phi(x\underline{X}_x)\otimes\underline{X}_t\\&=\underline{u}\Phi\tilde{\mu}_x\Phi^{-1}\underline{\phi}_{x,t}=\underline{u}\hat{\mu}_x\underline{\phi}_{x,t},\end{aligned}$$

where $\tilde{\mu}_x=\mu\otimes I$ and so

$$(\tilde{\mu}_x)_{ij}=\begin{cases}1, & j=i+N+1, i=0,1,\ldots,N(N+1)-1,\\0, & \text{otherwise.}\end{cases}$$

which can be written as $(\tilde{\mu}_x)_{ij}=\delta_{i+N+1,j}, i=0,1,\ldots,N(N+1)-1$.

(b) The proof is similar to the proof of part (**a**).
(c) By Corollary 2.2 and kronecker product properties, we have

$$\begin{aligned}\frac{\partial^p}{\partial x^p}u_N(x,t)&=\frac{\partial^p}{\partial x^p}(\underline{u}\underline{\phi}_{x,t})=\underline{u}\Phi\frac{\partial^p}{\partial x^p}(\underline{X}_x\otimes\underline{X}_t)\\&=\underline{u}\Phi\left(\frac{\partial^p}{\partial x^p}\underline{X}_x\right)\otimes\underline{X}_t=\underline{u}\Phi\tilde{\eta}_x^p\Phi^{-1}\underline{\phi}_{x,t}=\underline{u}\hat{\eta}_x^p\underline{\phi}_{x,t},\end{aligned}$$

where $(\tilde{\eta}_x)_{ij} = (j+1)\delta_{i,j+1}I, j = 0,1,\ldots,N-1$.

(d) The proof is similar to the proof of part **(c)**. □

Remark 2.4. The effect of $x^p t^q, p, q \in \mathbb{N}$ on the coefficients of $u_N(x,t) = \underline{u}\underline{\phi}_{x,t}$ is equivalent to post multiplication of $\underline{u}$ by $\mu_x^p\mu_t^q$, i.e.

$$x^p t^q u_N(x,t) = \underline{u}\widehat{\mu_x^p\mu_t^q}\,\underline{\phi}_{x,t} \ \ or\ \ x^p t^q \underline{\phi}_{x,t} = \widehat{\mu_x^p\mu_t^q}\,\underline{\phi}_{x,t},$$

where $\widehat{\mu_x^p\mu_t^q} = \Phi\tilde{\mu}_x^p\tilde{\mu}_t^q\Phi^{-1}$ and the elements of the matrices $\tilde{\mu}_x^p$ and $\tilde{\mu}_t^q$ are determined by

$$(\tilde{\mu}_x^p)_{ij} = \begin{cases} 1, & j = i+p(N+1), i = 0,1,\ldots,(N-p+1)(N+1)-1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$(\tilde{\mu}_t^q)_{ij} = \begin{cases} \mu^q, & i = j, i = 0,1,\ldots,N, \\ 0, & \text{otherwise} \end{cases}$$

with $(\mu^q)_{ij} = \delta_{i+q,j}, i = 0,1,\ldots,N-q$.

In addition, the matrix $\tilde{\mu}_x^p\tilde{\mu}_t^q$ has the following simple form

$$\tilde{\mu}_x^p\tilde{\mu}_t^q = \tilde{\mu}_t^q\tilde{\mu}_x^p = \begin{pmatrix} \bar{0} & B \\ \bar{0} & \bar{0} \end{pmatrix},$$

where $\bar{0}$ is an $p \times p$ zero matrix and B is an $(N-p+1)\times(N-p+1)$ diagonal matrix with the elements μ^q.

Remark 2.5. The matrices $\tilde{\eta}_x^p$ and $\tilde{\eta}_t^q$ have the following simple forms

$$\tilde{\eta}_x^p = \begin{pmatrix} \bar{0} & \bar{0} \\ C & \bar{0} \end{pmatrix},$$

where

$$C = diag((p(p-1)\times\cdots\times 1)I, ((p+1)p\times\cdots\times 2)I, \ldots, (N(N-1)\times\cdots\times(N-P+1))I),$$

and

$$\tilde{\eta}_t^q = diag(\eta^q, \eta^q, \ldots, \eta^q).$$

To obtain a matrix representation for the nonlinear part of Eq. (1), we state the following lemma and corollaries.

Lemma 2.6. *Let* $u_N(x,t) = \underline{u}\underline{\phi}_{x,t} = \underline{u}\Phi\underline{X}_{x,t}$ *and* $v_N(x,t) = \underline{v}\underline{\phi}_{x,t} = \underline{v}\Phi\underline{X}_{x,t}$, *where* $\underline{v}$ *is a vector similar to* $\underline{u}$ *with elements* v_i. *Then*

$$u_N(x,t)v_N(x,t) = \underline{u}\widehat{V}\underline{\phi}_{x,t} = \underline{u}\Phi V\underline{X}_{x,t},$$

where $\widehat{V} = \Phi V\Phi^{-1}$ and Φ_j's are the columns of matrix Φ with

$$V = \begin{pmatrix} \underline{v}\Phi_0 & \underline{v}\Phi_1 & \cdots & \underline{v}\Phi_{(N+1)^2-1} \\ 0 & \underline{v}\Phi_0 & \cdots & \underline{v}\Phi_{(N+1)^2-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \underline{v}\Phi_0 \end{pmatrix}.$$

Proof. By assumptions of lemma, we have

$$u_N(x,t)v_N(x,t) = \underline{u}\Phi(\underline{X}_{x,t} \times (\underline{v}\Phi\underline{X}_{x,t})),$$

thus it suffices to show $\underline{X}_{x,t} \times \underline{v}\Phi\underline{X}_{x,t}) = V\underline{X}_{x,t}$. By a simple computation, we can write

$$\begin{aligned}\underline{X}_{x,t}\underline{v}\Phi\underline{X}_{x,t} &= [(1,t,\ldots,t^N),(1,t,\ldots,t^N)x,\ldots,(1,t,\ldots,t^N)x^N]^T \\ &(v_0,v_1,\ldots,v_{(N+1)^2-1})(\Phi_0|\Phi_1|\ldots|\Phi_{(N+1)^2-1})\underline{X}_{x,t} \\ &= \left[(1,t,\ldots,t^N)\underline{v},(1,t,\ldots,t^N)x\underline{v},\ldots,(1,t,\ldots,t^N)x^N\underline{v}\right]^T \\ &\times(\Phi_0|\Phi_1|\ldots|\Phi_{(N+1)^2-1})\underline{X}_{x,t} \\ &= \left((\underline{v}\Phi_0)\underline{X}_{x,t}|(\underline{v}\Phi_1)\underline{X}_{x,t}|\ldots|(\underline{v}\Phi_{(N+1)^2-1})\underline{X}_{x,t}\right)^T \\ \underline{X}_{x,t} &= \begin{pmatrix} \underline{v}\Phi_0 & \underline{v}\Phi_1 & \underline{v}\Phi_2 & \cdots & \underline{v}\Phi_{(N+1)^2-1} \\ 0 & \underline{v}\Phi_0 & \underline{v}\Phi_1 & \cdots & \underline{v}\Phi_{(N+1)^2-2} \\ 0 & 0 & \underline{v}\Phi_0 & \cdots & \underline{v}\Phi_{(N+1)^2-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \underline{v}\Phi_0 \end{pmatrix}\underline{X}_{x,t}.\end{aligned}$$

□

Corollary 2.7. *If* $u_N(x,t) = \underline{u}\underline{\phi}_{x,t}$, then

$$u_N^k(x,t) = \underline{u}\widehat{U}^{k-1}\underline{\phi}_{x,t},\ k \in \mathbb{N},$$

where $\widehat{U} = \Phi U^{k-1}\Phi^{-1}$ *and U is an upper triangular matrix with elements*

$$U_{ij} = \begin{cases} \displaystyle\sum_{r=0}^{(N+1)^2-1} u_r\phi_{r,j-i}, & i \leqslant j, i = 0,1,\ldots,(N+1)^2-1, \\ 0, & i > j. \end{cases}$$

Corollary 2.8. *If* $u_N(x,t) = \underline{u}\underline{\phi}_{x,t}$, *then*

(a) $\left(\frac{\partial^p}{\partial x^p}u_N(x,t)\right)^r = \underline{u}\Psi^p M_p^{r-1}\Phi^{-1}\underline{\phi}_{x,t}$ *for* $p, r \in \mathbb{N}$, *where* $\Psi^p = \Phi\tilde{\eta}_x^p$ *and*

$$M_p = \begin{pmatrix} \underline{u}\Psi_0^p & \underline{u}\Psi_1^p & \cdots & \underline{u}\Psi_{(N+1)^2-1}^p \\ 0 & \underline{u}\Psi_0^p & \cdots & \underline{u}\Psi_{(N+1)^2-2}^p \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \underline{u}\Psi_0^p \end{pmatrix}.$$

(b) $\left(\frac{\partial^q}{\partial t^q}u_N(x,t)\right)^s = \underline{u}\Gamma^q N_q^{s-1}\Phi^{-1}\underline{\phi}_{x,t}$ *for* $q, s \in \mathbb{N}$, *where* $\Gamma^q = \Phi\tilde{\eta}_t^q$ *and*

$$N_q = \begin{pmatrix} \underline{u}\Gamma_0^q & \underline{u}\Gamma_1^q & \cdots & \underline{u}\Gamma_{(N+1)^2-1}^q \\ 0 & \underline{u}\Gamma_0^q & \cdots & \underline{u}\Gamma_{(N+1)^2-2}^q \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \underline{u}\Gamma_0^q \end{pmatrix}.$$

Corollary 2.9. *If* $u_N(x,t) = \underline{u}\underline{\phi}_{x,t}$, *then*

(a) $u_N^k(x,t)\frac{\partial^p}{\partial x^p}u_N(x,t) = \underline{u}\widehat{U}^{k-1}\widehat{M}_p\underline{\phi}_{x,t}, k, p \in \mathbb{N}$;

(b) $u_N^k(x,t)\frac{\partial^q}{\partial t^q}u_N(x,t) = \underline{u}\widehat{U}^{k-1}\widehat{N}_q\underline{\phi}_{x,t}, k, q \in \mathbb{N}$,

where $\widehat{U}^{k-1} = \Phi U^{k-1}\Phi^{-1}, \widehat{M}_p = \Phi M_p\Phi^{-1}$ *and* $\widehat{N}_q = \Phi N_q\Phi^{-1}$.

To convert Eq. (1) to a matrix form, we assume that the right-hand side of (1) has the following form

$$f(x,t)=\sum_{i=0}^{N}\sum_{j=0}^{N}f_{i(N+1)+j}\phi_i(x)\phi_j(t)=\underline{f}\underline{\phi}_{x,t}=\underline{f}\Phi\underline{X}_{x,t}, \tag{10}$$

where $\underline{f}=(f_0,f_1,\ldots,f_{(N+1)^2-1})$.

Using the above results, we have provided all requirements to convert Eq. (1) and supplementary conditions (2)–(4) to the corresponding matrix representation.

For simplicity, we write Eq. (1) in the operator form

$$Du(x,t)=f(x,t), \tag{11}$$

where

$$D=\rho c_p\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\left(k\frac{\partial}{\partial x}\right), \tag{12}$$

is a differential operator.

Replacing the approximate solution (9) in 2,3,4 and (11) and using the above lemmas and theorem gives

$$\underline{u}\widehat{\Pi}_D=\underline{f}, \tag{13}$$

$$\underline{u}\widehat{B}_{c_1}=\underline{\varphi}_1, \tag{14}$$

$$\underline{u}\widehat{B}_{c_2}=\underline{\varphi}_2, \tag{15}$$

$$\underline{u}\widehat{I}_c=\underline{\psi}, \tag{16}$$

where $\widehat{\Pi}_D$, $\widehat{B}_{c_1}$, $\widehat{B}_{c_2}$ and $\widehat{I}_c$ are the corresponding matrix representation of Eq. (1) and supplementary conditions (2)–(4), respectively.

Remark 2.10. The matrix $\widehat{\Pi}_D$ in Eq. (13) has the following structure

$$\widehat{\Pi}_D=\rho c_p\hat{\eta}_t - k_0\left[\hat{\eta}_x+\alpha_1(\Psi M_1\Phi^{-1}+\widehat{M}_2)+\alpha_2(2\Phi\overline{M}_1\Phi^{-1}+\widehat{U}\widehat{M}_2)\right],$$

where

$$\overline{M}_1=\begin{pmatrix}\underline{u}(\widetilde{M}_1)_0 & \underline{u}(\widetilde{M}_1)_1 & \cdots & \underline{u}(\widetilde{M}_1)_{(N+1)^2-1}\\ 0 & \underline{u}(\widetilde{M}_1)_0 & \cdots & \underline{u}(\widetilde{M}_1)_{(N+1)^2-2}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \underline{u}(\widetilde{M}_1)_0\end{pmatrix},$$

and $\widetilde{M}_1=\Psi M_1$.

Set $\widehat{G}=(\widehat{\Pi}_D,\widehat{B}_{c_1},\widehat{B}_{c_2},\widehat{I}_c)$ and $\underline{g}=(\underline{f},\underline{\varphi}_1,\underline{\varphi}_2,\underline{\psi})$. Then, the nonlinear systems of Eqs. (13)–(16) can be written as

$$\underline{u}\widehat{G}=\underline{g}, \tag{17}$$

which is constructed as follows:

(i) Choose the corresponding equations obtained from supplementary conditions (14)–(16) ($3(N+1)$ equations).
(ii) Choose $(N+1)^2-3(N+1)$ equations from nonlinear system of Eq. (13).

Solving the system (17) gives the unknown coefficients $\{u_k\}_{k=0}^{(N+1)^2-1}$ and so, the approximate solution $u_N(x,t)$ is obtained.

3. Error estimation

Whenever the solution of a problem is not known, specially in nonlinear phenomena, an error estimator is needed as a vital component of the algorithm. To this end, an error estimator for the proposed method is presented in this section.

Define the error function as

$$e_N(x,t)=u(x,t)-u_N(x,t). \tag{18}$$

Substituting $u_N(x,t)$ into (2)–(4), (11), yields

$$Du_N(x,t)=f(x,t)+H_N(x,t), \tag{19}$$

$$B_{c_1}u_N(x,t)=\varphi_1(t), \tag{20}$$

$$B_{c_2}u_N(x,t)=\varphi_2(t), \tag{21}$$

$$I_c u_N(x,t)=\psi(x), \tag{22}$$

where $H_N(x,t)$ is a perturbation term. Subtracting (19)–(22) from (11), (2)–(4) respectively, gives

$$De_N(x,t)=-H_N(x,t), \tag{23}$$

$$B_{c_1}e_N(x,t)=0, \tag{24}$$

$$B_{c_2}e_N(x,t)=0 \tag{25}$$

$$I_c e_N(x,t)=0. \tag{26}$$

Now, we can proceed by the same way as we did in Section 2 to get the estimation $e_{N,M}(x,t)$ to the error function $e_N(x,t)$.

4. Numerical results

In this section, we illustrate by numerical examples the efficiency and accuracy of the proposed method where the ''Absolute Errors'' and ''Estimate Errors'' are reported at some arbitrary selected points. Note that, non-polynomial terms of the problem or supplementary conditions must be approximated by polynomials of suitable degrees.

Example 4.1. Consider the following nonlinear transient heat conduction equation

$$\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right)+f(x,t)=\frac{\partial u}{\partial t},\quad x\in[0,1],\quad t>0,$$

where

$$k(u)=1+u^2,$$

and

$$f(x,t)=1-2x-2t,$$

with the supplementary conditions

$$u(0,t)=t,\quad u(1,t)=1+t,\quad u(x,0)=x.$$

The exact solution of this problem is $u(x,t)=x+t$. For $N=2$ and using the standard basis, the proposed method gives the nonlinear system

$$\begin{cases} u_0=u_2=u_6=0,\\ u_1=u_3=1,\\ u_4+u_7=0,\\ u_5+u_8=0,\\ u_0(2u_0u_7+2u_1u_6+4u_3u_4)+u_1(2u_0u_6+2u_3^2)-2u_2+2u_7=2,\\ u_0(2u_0u_8+2u_1u_7+2u_2u_6+4u_3u_5+2u_4^2)\\ \quad+u_1(2u_0u_7+2u_1u_6+4u_3u_4)\\ +u_2(2u_0u_6+2u_3^2)+2u_8=0,\end{cases}$$

with the solution $\{u_1 = u_3 = 1, u_i = 0 \text{ for } i \neq 1,3\}$ which leads to the exact solution. Indeed, as mentioned in Ortiz and Samara (1981), the operational Tau method for equations with polynomial solution is exact, whenever the degree of the Tau approximation is at least equal to the degree of solution.

Example 4.2. Consider a plane wall having variable thermo-physical properties (thermal conductivity, specific heat and density) in which its surface temperatures are related to the time. We assume that a heat sink or heat source is presented in the wall in which the magnitude of released or dissipated energy is a nonlinear function of space and temperature (for examples chemical reaction or electrical resistance). Also, the ambient temperature is zero. The following nonlinear heat conduction equation can be obtained using the first law of thermodynamics

$$\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) + f(x,t) = u^2\frac{\partial u}{\partial t}, \quad x \in [0,1], \quad t > 0.$$

In a special case we assume that thermo-physical properties and sum of heat flux are

$$k(u) = 1 + u + \frac{1}{2}u^2,$$

and

$$f(x,t) = -\sin^2 t(1 + x\sin t - x^3\cos t).$$

The supplementary conditions are assumed to be of the Dirichlet kind, namely

$$u(0,t) = 0, \quad u(1,t) = \sin t, \quad u(x,0) = 0.$$

The exact solution of this problem is $u(x,t) = x\sin t$.

Table 1 shows the absolute errors with respect to shifted Chebyshev basis functions at the selected points for various choices of N.

Example 4.3. Consider the nonlinear transient heat conduction equation of the form

$$\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) + f(x,t) = \frac{\partial u}{\partial t}, \quad x \in [0,1], \quad t > 0.$$

with supplementary conditions

$$u(0,t) = 0, \quad u(1,t) + \frac{\partial u(1,t)}{\partial x} = 2e^t, \quad u(x,0) = x,$$

where thermo-physical properties and sum of heat flux are

$$k(u) = 1 + u^2,$$

Table 1 Absolute errors of Example 4.2.

(x,t)	$N = 4$	$N = 8$	$N = 12$
(0.1,0.1)	8.3313e−09	2.7555e−16	2.0000e−22
(0.2,0.3)	4.0413e−06	1.0839e−11	5.1200e−18
(0.4,0.3)	8.0827e−06	2.1679e−11	1.0230e−17
(0.5,0.5)	1.2944e−04	2.6850e−09	9.7900e−15
(0.6,0.8)	1.6136e−03	2.2064e−07	5.2810e−12
(0.7,0.8)	1.8826e−03	2.5741e−07	6.1612e−12
(0.8,0.9)	3.8616e−03	8.4784e−07	3.2530e−11
(0.9,0.8)	2.4205e−03	3.3096e−07	7.9215e−12
(1,1)	8.1376e−03	2.7308e−06	1.5983e−10
Cpu time (s)	3.80	7.10	19.80

Table 2 Absolute errors of Example 4.3.

(x,t)	$N = 4$	$N = 8$	$N = 10$
(0.1,0.1)	8.4742e−09	2.7835e−16	3.1880e−17
(0.2,0.2)	5.5163e−07	2.8794e−13	4.1700e−17
(0.3,0.3)	6.3923e−06	1.6774e−11	1.3453e−14
(0.4,0.4)	3.6546e−05	3.0095e−10	4.3508e−13
(0.5,0.5)	1.4189e−04	2.8321e−09	6.3847e−12
(0.6,0.6)	4.3128e−04	1.7720e−08	5.7408e−11
(0.7,0.7)	1.1073e−03	8.3660e−08	3.6819e−10
(0.8,0.8)	2.5127e−03	3.2141e−07	1.8440e−09
(0.9,0.9)	5.1891e−03	1.0550e−06	7.6460e−09
(1,1)	9.9485e−03	3.0586e−06	2.7314e−08
Cpu time (s)	2.90	5.75	15.50

and

Table 3 Estimate errors of Example 4.3.

(x,t)	$N = 4$	$N = 8$
(0.1,0.1)	7.5201e−09	2.2836e−16
(0.2,0.2)	5.4793e−07	2.3491e−13
(0.3,0.3)	6.3840e−06	1.4720e−11
(0.4,0.4)	3.6528e−05	2.8526e−10
(0.5,0.5)	1.4175e−04	2.6355e−09
(0.6,0.6)	4.3122e−04	1.7418e−08
(0.7,0.7)	1.1068e−03	8.3544e−08
(0.8,0.8)	2.5112e−03	3.2082e−07
(0.9,0.9)	5.1825e−03	1.0455e−06
(1,1)	9.9436e−03	3.0545e−06
Cpu time (s)	3.15	6.00

Table 4 Maximum absolute errors of perturb problem of Example 4.3.

ε	$N = 4$	$N = 8$
10^{-3}	9.9845e−03	3.3240e−06
10^{-5}	9.5500e−03	3.2132e−06
10^{-7}	9.2408e−03	3.1425e−06

$$f(x,t) = xe^t(1 - 2e^{2t}).$$

The exact solution of the problem is $u(x,t) = xe^t$.

The absolute and estimate errors for this example with respect to shifted Chebyshev basis for various choices of N, reported in Tables 2 and 3, show the accuracy and efficiency of the proposed method. These results confirm that the absolute and estimate errors are in good agreement.

To check the stability of the proposed method, we perturb the coefficients of approximate solution by $\varepsilon = 10^{-3}$, 10^{-5} and 10^{-7}. Then, we solve the perturbed problem by the method and find out that there are no total changes in the final results. Table 4 shows the maximum absolute errors of the perturbed problem with respect to shifted Chebyshev basis for various choices of N.

5. Conclusion

In this work, a computational method based on the operational Tau method is present for solving 1-D nonlinear transient heat conduction equations by converting it and necessary supplementary conditions to a nonlinear system of equations. Our results indicate the proposed algorithm can be regarded as a structurally simple algorithm and high superior performance that is conventionally applicable to the numerical solution of these type of equations. The accuracy of the method is improved as the degree of approximation is increased.

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