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Radii problems for certain analytic functions

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Abstract In this paper, we study some radii problems for certain classes of analytic functions. These results generalize some of the previously known radii problems such as the radius of convexity for starlikness and radius of quasi-convexity for close-to-convex functions. Also, it is shown that some of these radii are best possible.

1. Introduction

Let A_n be the class of analytic functions f,

$$f(z) = z + \sum_{m=n}^{\infty} a_{m+1} z^{m+1}, \quad n \geqslant 1, \tag{1.1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. We shall need the following known classes (Noor, 2008) in our discussion. Let, for $0 \leqslant \gamma, \beta < 1$,

$$p(z) = 1 + \sum_{m=n}^{\infty} a_m z^m, \quad z \in E, \tag{1.2}$$

we have,

$$P(\gamma, n) = \{p : p \text{ is analytic in } E, \text{ given by (1.2), } Rep(z) > \gamma\}$$

$$S^*(\gamma, n) = \left\{ f : f \in A_n \text{ and } \frac{zf'}{f} \in P(\gamma, n) \right\}$$

$$C(\gamma, n) = \left\{ f : f \in A_n \text{ and } \left(1 + \frac{zf''}{f'}\right) \in P(\gamma, n) \right\}.$$

It is clear that

$$f \in C(\gamma, n) \Longleftrightarrow zf' \in S^*(\gamma, n). \tag{1.3}$$

For $n = 1$, these classes have been introduced by Robertson (1963).

$$K(\beta, \gamma) = \left\{ f : f \in A_n \text{ and } \frac{zf'}{g} \in P(\beta, n) \text{ for some } g \in S^*(\gamma, n) \right\}$$

$$C^*(\beta, \gamma) = \left\{ f : f \in A_n \text{ and } \frac{(zf')'}{g'} \in P(\beta, n) \text{ for some } g \in C(\gamma, n) \right\}.$$

We note here that

$$f \in C^*(\beta, \gamma) \Longleftrightarrow zf' \in K(\beta, \gamma). \tag{1.4}$$

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These classes have been studied by Noor (1987) for $n = 1$.

Now, we have the definition of following classes for $k \geqslant 2$,

$$P_k(\gamma,n) = \left\{p : p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z), \quad p_1,\ p_2 \in P(\gamma,n)\right\}$$

$$P'_k(\gamma,n) = \{p : p' \in P_k(\gamma,n)\}$$

$$V_k(\gamma,n) = \left\{f : f \in A_n \text{ and } \left(1+\frac{zf''}{f'}\right) \in P_k(\gamma,n)\right\}$$

$$R_k(\gamma,n) = \left\{f : f \in A_n \text{ and } \frac{zf'}{f} \in P_k(\gamma,n)\right\}.$$

We note that $P_2(\gamma,n) = P(\gamma,n)$. The class $P_k(0,1) = P_k$ was introduced and discussed by Pinchuk (1971), where he defined it as follows:

$$P_k = \left\{p : p(z) = \frac{1}{2}\int_{-\pi}^{\pi} \frac{1+ze^{-it}}{1-ze^{-it}}\, d\mu(t), \times \int_{-\pi}^{\pi} d\mu(t) = 2, \int_{-\pi}^{\pi} |d\mu(t)| \leqslant k\right\}.$$

It is easy to see that

$$P_k(\gamma,n) = (1-\gamma)P_k(0,n) + \gamma$$

and $P_2(0,1) = P$ is the class of functions with positive real part. It is clear that

$$f \in V_k(\gamma,n) \Longleftrightarrow zf' \in R_k(\gamma,n). \tag{1.5}$$

The classes $V_k(0,1) = V_k$ and $R_k(0,1) = R_k$ are the well-known classes of functions with bounded boundary rotation and bounded radius rotation, respectively.

Let $f_j(z),\ j = 1,2$ in A_n be given by

$$f_j(z) = z + \sum_{m=n}^{\infty} a_{m+1,j} z^{m+1}, \quad n \geqslant 1, \quad z \in E.$$

Then the Hadamard product or convolution $(f_1 * f_2)(z)$ of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = z + \sum_{m=n}^{\infty} a_{m+1,1}a_{m+1,2} z^{m+1}, \quad n \geqslant 1, \quad z \in E. \tag{1.6}$$

By using the Hadamard product, we define the well-known Ruscheweyh derivative (see Rucheweyh, 1975) as following.

Denote by $D^\alpha : A_n \to A_n$ the operator defined by

$$D^\alpha f(z) = \frac{z^n}{(1-z)^{1+\alpha}} * f(z), \quad \alpha > -1.$$

For $\alpha = m \in N_0 = \{0,1,2\ldots\}$, we can write

$$D^m f(z) = \frac{z}{(1-z)^{1+m}} * f(z) = \frac{z(z^{m-1}f(z))^{(m)}}{m!}.$$

Also, for Ruscheweyh derivative D^α, the following identity is known (see Fukui and Sakaguchi, 1980).

For a real number $\alpha(\alpha > -1)$, we have

$$z(D^\alpha f(z))' = (\alpha+1)D^{\alpha+1}f(z) - \alpha D^\alpha f(z). \tag{1.7}$$

Recently, the Ruscheweyh derivative has been studied in Noor and Hussain (2008).

We now have the following classes which have been introduced and studied in Noor (1991), for the case $n = 1$,

$$S^*_\alpha(\gamma,n) = \{f \in A_n : D^\alpha f \in S^*(\gamma,n),\ \alpha > -1,\ z \in E\}$$
$$C_\alpha(\gamma,n) = \{f \in A_n : D^\alpha f \in C(\gamma,n),\ \alpha > -1,\ z \in E\}$$
$$K_\alpha(\beta,\gamma) = \{f \in A_n : D^\alpha f \in K(\beta,\gamma),\ \alpha > -1,\ z \in E\}$$

and

$$C^*_\alpha(\beta,\gamma) = \{f \in A_n : D^\alpha f \in C^*(\beta,\gamma),\ \alpha > -1,\ z \in E\}.$$

2. Preliminary results

The following lemmas will be used.

Lemma 2.1. *Let $h \in P(0,n) = P_n$ for $z \in E$. Then*

(i) $\left|\frac{h'(z)}{h(z)}\right| \leqslant \frac{2n|z|^{n-1}}{1-|z|^{2n}}$

(ii) $|zh'(z)| \leqslant \frac{2n|z|^n Reh(z)}{1-|z|^{2n}}$

(iii) $\frac{1-|z|^n}{1+|z|^n} \leqslant Reh(z) \leqslant |h(z)| \leqslant \frac{1+|z|^n}{1-|z|^n}$.

For (i) we refer to MacGraw (1963), *(ii) will be found in* Bernardi (1974) *and for (iii) (see* Shah, 1972*).*

The following Lemma can easily be shown by using Lemma 2.1.

Lemma 2.2. *Let $p \in P(\gamma,n)$ for $z = re^{i\theta} \in E$. Then*

$$\frac{1+(2\gamma-1)r^n}{1+r^n} \leqslant Rep(z) \leqslant |p(z)| \leqslant \frac{1-(2\gamma-1)r^n}{1-r^n}$$

and

$$|zp'(z)| \leqslant \frac{2n(1-\gamma)r^n Rep(z)}{(1-r^n)[1+(1-2\gamma)r^n]}.$$

3. Main results

Theorem 3.1. *Let $f,\ g \in A_n$ and let $\frac{f'}{g'} \in P_n$, where $g \in V_k(\gamma,n)$. Then $f \in V_k(\gamma,n)$ for $|z| < r_0$, where*

$$r_0^n = r_0^n(\gamma,n) = \frac{1-\gamma}{(n-\gamma+1)+\sqrt{n^2-2\gamma n+2n}}. \tag{3.1}$$

This result is sharp.

Proof. We can write

$$f'(z) = g'(z)h(z), \text{ where } g \in V_k(\gamma,n),\ h \in P_n,\ z \in E.$$

Logarithmic differentiation yields,

$$\frac{(zf'(z))'}{f'(z)} = \frac{(zg'(z))'}{g'(z)} + \frac{zh'(z)}{h(z)} = H(z) + \frac{zh'(z)}{h(z)}, \quad h \in P_k(\gamma,n)$$
$$= \left(\frac{k}{4}+\frac{1}{2}\right)\{(1-\gamma)H_1(z)+\gamma\} - \left(\frac{k}{4}-\frac{1}{2}\right)\{(1-\gamma)H_2(z)+\gamma\} + \frac{zh'(z)}{h(z)},$$

where $H_i \in P_n,\ i = 1,2$.

This gives us

$$\frac{1}{1-\gamma}\left\{\frac{(zf'(z))'}{f'(z)}-\gamma\right\}=\left(\frac{k}{4}+\frac{1}{2}\right)\left\{H_1(z)+\frac{1}{1-\gamma}\frac{zh'(z)}{h(z)}\right\}-\left(\frac{k}{4}-\frac{1}{2}\right)\left\{H_2(z)+\frac{1}{1-\gamma}\frac{zh'(z)}{h(z)}\right\}.$$

Now, for $i=1,2$, we use Lemma 2.1, with $|z|=r$, to have

$$\begin{aligned}Re\left\{H_i(z)+\frac{1}{1-\gamma}\frac{zh'(z)}{h(z)}\right\} &\geqslant ReH_i(z)-\frac{1}{1-\gamma}\left|\frac{zh'(z)}{h(z)}\right| \\ &\geqslant \frac{1-r^n}{1+r^n}-\frac{1}{(1-\gamma)}\frac{2nr^n}{(1-r^{2n})} \\ &=\frac{(1-\gamma)(1-r^n)^2-2\gamma r^n}{(1-\gamma)(1-r^{2n})} \\ &=\frac{(1-\gamma)-2(n-\gamma+1)r^n+(1-\gamma)r^{2n}}{(1-\gamma)(1-r^{2n})}.\end{aligned}$$

Therefore, for $|z|<r_0$ where r_0 is as stated in (3.1),

$$Re\left\{H_i(z)+\frac{1}{1-\gamma}\frac{zh'(z)}{h(z)}\right\}>0.$$

This implies that

$$\frac{1}{1-\gamma}\left\{\frac{(zf'(z))'}{f'(z)}-\gamma\right\}\in P_k(0,n)\quad\text{for}\quad|z|<r_0,$$

which leads us to the required result that $f\in V_k(\gamma,n)$ for $|z|<r_0$, where r_0 is as given in (3.1).

Sharpness can be seen by taking

$$H_i(z)=h(z)=\frac{1-z^n}{1+z^n}\in P_n.\qquad\square$$

Theorem 3.2. *Let $f,\ g\in A_n$ and $g\in P_2'(\gamma,n)=P'(\gamma,n)$ in E. If $\frac{f'}{g'}\in P(\gamma,n)$, then $f\in V_2(0,n)$ for $|z|<r_1$ where r_1 is given by*

$$r_1^n=\frac{1}{2n(1-\gamma)+\sqrt{4n^2(1-\gamma)^2+1}}.\tag{3.2}$$

This result is also sharp.

Proof

Let $\frac{f'(z)}{g'(z)}=h(z),$ where $h\in P(\gamma,n)$.

Then

$$\frac{f''(z)}{f'(z)}=\frac{h'(z)}{h(z)}+\frac{g''(z)}{g'(z)}.$$

That is

$$\frac{(zf'(z))'}{f'(z)}=\frac{zh'(z)}{h(z)}+\frac{zp'(z)}{p(z)}+1,\ \text{where } g'=p\in P(\gamma,n)\text{ in }E.$$

Now, using Lemma 2.2, we have

$$\begin{aligned}Re\frac{(zf'(z))'}{f'(z)} &\geqslant 1-\frac{4n(1-\gamma)r^n}{(1-r^n)(1+r^n)} \\ &=\frac{1-4n(1-\gamma)r^n-r^{2n}}{1-r^{2n}}.\end{aligned}\tag{3.3}$$

The right hand side of the above inequality (3.3) is positive for $|z|<r_1$, where r_1 is given by (3.2).

The sharpness can seen by considering

$$g(z)=\int_0^z\left(\frac{1-(2\gamma-1)t^n}{1-t^n}\right)dt,$$

$$h(z)=\frac{1-(2\gamma-1)z^n}{1-z^n}$$

and

$$f(z)=\int_0^z\left(\frac{1-(2\gamma-1)t^n}{1-t^n}\right)^2dt.$$

The inclusion results for the classes $S_\alpha^*(\gamma,n),\ C_\alpha(\gamma,n),\ K_\alpha(\beta,\gamma)$ and $C_\alpha^*(\beta,\gamma)$, with $n=1$, have been studied by Noor (1991). We here deal with the converse case in general. $\square$

Theorem 3.3. *Let $f\in S_\alpha^*(\gamma,n),\ \alpha\geqslant 0$. Then $f\in S_{\alpha+1}^*(\gamma,n)$ for $|z|<r_0(\alpha,\gamma)$ is given as*

$$\begin{aligned}r_0^n &= r_0^n(\alpha,\gamma) \\ &=\frac{1+\alpha}{(1-\gamma+n)+\sqrt{(1-\gamma+n)^2-(1+\alpha)(1-2\gamma-\alpha)}}.\end{aligned}\tag{3.4}$$

Proof. Since $f\in S_\alpha^*(\gamma,n)$, so we can write it as

$$\frac{(zD^\alpha f(z))'}{D^\alpha f(z)}=H(z)=(1-\gamma)h(z)+\gamma,$$

where $H\in P(\gamma,n)$ and so $h\in P(0,n)=P_n$. Now using (1.7), we have

$$\begin{aligned}\frac{D^{\alpha+1}f(z)}{D^\alpha f(z)} &=\frac{1}{1+\alpha}\left\{\frac{(zD^\alpha f(z))'}{D^\alpha f(z)}+\alpha\right\} \\ &=\frac{1}{1+\alpha}\{(1-\gamma)h(z)+\gamma+\alpha\}.\end{aligned}$$

Differentiating both sides logarithmically, we have

$$\begin{aligned}\frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}f(z)} &=\frac{(zD^\alpha f(z))'}{D^\alpha f(z)}+\frac{(1-\gamma)zh'(z)}{(1-\gamma)h(z)+\gamma+\alpha} \\ &=(1-\gamma)h(z)+\gamma+\frac{(1-\gamma)zh'(z)}{(1-\gamma)h(z)+\gamma+\alpha},\end{aligned}$$

or

$$\frac{1}{1-\gamma}\left\{\frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}f(z)}-\gamma\right\}=h(z)+\frac{zh'(z)}{(1-\gamma)h(z)+\gamma+\alpha}.$$

Therefore,

$$Re\left\{\frac{1}{1-\gamma}\left(\frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}f(z)}-\gamma\right)\right\}=Re\left\{h(z)+\frac{zh'(z)}{(1-\gamma)h(z)+\gamma+\alpha}\right\}.$$

By using Lemma 2.1 for $|z|=r<1$, we have

$$\begin{aligned}Re\left\{\frac{1}{1-\gamma}\left(\frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}f(z)}-\gamma\right)\right\} &\geqslant Reh(z)\left\{1-\frac{2nr^n}{\frac{1-r^{2n}}{(1-\gamma)\left(\frac{1-r^n}{1+r^n}\right)+\gamma+\alpha}}\right\} \\ &=Reh(z)\left\{\frac{(1+\alpha)-2(1-\gamma+n)r^n+(1-2\gamma-\alpha)r^{2n}}{(1+\alpha)-2(1-\gamma)r^n+(1-2\gamma-\alpha)r^{2n}}\right\}.\end{aligned}$$

The right hand side of above inequality is positive if $r^n < r_0^n$ and so is given by (3.4).

As a special case, when $\alpha = 0,\ n = 1$ and $\gamma = 0$, we obtain the radius of convexity $2-\sqrt{3}$ for starlike functions. □

Theorem 3.4. *Let for $\alpha \geqslant 0$, $f \in C_\alpha(\gamma, n)$ in E. Then $f \in C_{\alpha+1}(\gamma, n)$ for $|z| < r_0 = r_0(\alpha, \gamma)$, where r_0 is given by (3.4).*

Proof. By using the definition of $C_\alpha(\gamma, n)$, we have

$$\begin{aligned} f \in C_\alpha(\gamma,n) &\Longleftrightarrow D^\alpha f \in C(\gamma,n) \ \text{ in } \ E. \\ &\Longleftrightarrow z(D^\alpha f)' \in S^*(\gamma,n) \ \text{ in } \ E. \\ &\Longleftrightarrow D^\alpha(zf') \in S^*(\gamma,n) \ \text{ in } \ E. \\ &\Longleftrightarrow zf' \in S^*_{\alpha+1}(\gamma,n) \ \text{ in } \ |z| < r_0. \\ &\Longleftrightarrow D^{\alpha+1}(zf') \in S^*(\gamma,n) \ \text{ in } \ |z| < r_0. \\ &\Longleftrightarrow z(D^{\alpha+1}f)' \in S^*(\gamma,n) \ \text{ in } \ |z| < r_0. \\ &\Longleftrightarrow D^{\alpha+1}f \in C(\gamma,n) \ \text{ in } \ |z| < r_0. \\ &\Longleftrightarrow f \in C_{\alpha+1}(\gamma,n) \ \text{ in } \ |z| < r_0, \end{aligned}$$

which is the required result. □

Theorem 3.5. *Let for $\alpha \geqslant 0$, $f \in K_\alpha(\beta, \gamma)$ in E. Then $f \in K_{\alpha+1}(\beta, \gamma)$ for $|z| < r_0 = r_0(\alpha, \gamma)$, where r_0 is given by (3.4).*

Proof. Since $f \in K_\alpha(\beta, \gamma)$, there exists $g \in S^*(\gamma, n)$ such that

$$\frac{z(D^\alpha f(z))'}{D^\alpha g(z)} = (1-\beta)h(z) + \beta, \quad h \in P(0,n). \tag{3.5}$$

Also, since $g \in S^*(\gamma, n)$, we can write

$$\frac{z(D^\alpha g(z))'}{D^\alpha g(z)} = (1-\gamma)H(z) + \gamma, \quad H \in P(0,n). \tag{3.6}$$

Using (1.7), we have

$$\frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}g(z)} = \frac{D^{\alpha+1}(zf'(z))}{D^{\alpha+1}g(z)} = \frac{\frac{1}{1+\alpha}z(D^\alpha(zf'(z)))' + \frac{\alpha}{1+\alpha}D^\alpha(zf'(z))}{\frac{1}{1+\alpha}z(D^\alpha g(z))' + \frac{\alpha}{1+\alpha}D^\alpha g(z)}$$
$$= \frac{\frac{z(D^\alpha(zf'(z)))'}{D^\alpha g(z)} + \alpha\frac{D^\alpha(zf'(z))}{D^\alpha g(z)}}{\frac{z(D^\alpha g(z))'}{D^\alpha g(z)} + \alpha}.$$

By using (3.5) and (3.6), we have

$$\frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}g(z)} = \frac{\frac{z(D^\alpha(zf'(z)))'}{D^\alpha g(z)} + \alpha\{(1-\beta)h(z)+\beta\}}{(1-\gamma)H(z)+\gamma+\alpha}. \tag{3.7}$$

From (3.5), we have

$$z(D^\alpha f(z))' = D^\alpha g(z)\{(1-\beta)h(z)+\beta\}.$$

Differentiating both sides, we have

$$z(z(D^\alpha f(z))')' = (1-\beta)zh'(z)(D^\alpha g(z)) + (D^\alpha g(z))'\{(1-\beta)h(z)+\beta\}.$$

That is,

$$\frac{z(D^\alpha(zf(z))')'}{D^\alpha g(z)} = (1-\beta)zh'(z) + \{(1-\gamma)H(z)+\gamma\}\{(1-\beta)h(z)+\beta\}. \tag{3.8}$$

Using (3.8) in (3.7), we obtain

$$\left\{\frac{1}{1-\beta}\left(\frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}g(z)} - \beta\right)\right\} = \left\{h(z) + \frac{zh'(z)}{(1-\gamma)H(z)+\gamma+\alpha}\right\}.$$

Now

$$Re\left\{\frac{1}{1-\beta}\left(\frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}f(z)} - \beta\right)\right\} = Reh(z)\left\{1 - \frac{\frac{2nr^n}{1-r^n}}{(1-\gamma)\frac{1-r^n}{1+r^n}+\gamma+\alpha}\right\}.$$

We note that right hand side is positive for $|z| < r_0 = r_0(\alpha, \gamma)$ given by (3.4) and also $g \in S^*_\alpha(\gamma, n)$ for $|z| < r_0$. Hence $f \in K_{\alpha+1}(\beta, \gamma)$ for $|z| < r_0$. □

As a special case, when $\alpha = 0, \beta = 0, \gamma = 0$ and $n = 1$, we obtain the radius of quasi-convexity $2-\sqrt{3}$ for close-to-convex functions.

Using the same method as in Theorem 3.4 with the relation (1.4), we can easily prove the following result.

Theorem 3.6. *Let $f \in C^*_\alpha(\beta, \gamma)$ for $\alpha \geqslant 0$, $z \in E$. Then $f \in C^*_{\alpha+1}(\beta, \gamma)$ for $|z| < r_0 = r_0(\alpha, \gamma)$, where r_0 is given by (3.4).*

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