Journal of King Saud University – Science (2015) 27, 161–167

King Saud University
Journal of King Saud University – Science
www.ksu.edu.sa
www.sciencedirect.com

ORIGINAL ARTICLE

Solving Abel integral equations of first kind via fractional calculus

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Received 30 August 2014; accepted 29 September 2014
Available online 12 October 2014

KEYWORDS

Abel integral equation;
Singular integral equations;
Caputo fractional derivatives;
Fractional integrals

Abstract We give a new method for numerically solving Abel integral equations of first kind. An estimation for the error is obtained. The method is based on approximations of fractional integrals and Caputo derivatives. Using trapezoidal rule and Computer Algebra System Maple, the exact and approximation values of three Abel integral equations are found, illustrating the effectiveness of the proposed approach.

1. Introduction

Consider the following generalized Abel integral equation of first kind:

$$f(x) = \int_0^x \frac{k(x,s)g(s)}{(x-s)^\alpha} ds, \quad 0 < \alpha < 1, \quad 0 \leqslant x \leqslant b, \tag{1.1}$$

where g is the unknown function to be found, f is a well behaved function, and k is the kernel. This equation is one of the most famous equations that frequently appears in many physical and engineering problems, like semi-conductors, heat conduction, metallurgy and chemical reactions (Gorenflo, 1996; Gorenflo and Vessella, 1991). In experimental physics, Abel's integral equation of first kind (1.1) finds applications in plasma diagnostics, physical electronics, nuclear physics, optics and astrophysics (Knill et al., 1993; Kosarev, 1980). To determine the radial distribution of the radiation intensity of a cylinder discharge in plasma physics, for example, one needs to solve an integral Eq. (1.1) with $\alpha = \frac{1}{2}$. Another example of application appears when one describes velocity laws of stellar winds (Knill et al., 1993). If $k(x,s) = \frac{1}{\Gamma(1-\alpha)}$, then (1.1) is a fractional integral equation of order $1-\alpha$ (Podlubny, 1999). This problem is a generalization of the tautochrone problem of the calculus of variations, and is related with the born of fractional mechanics (Riewe, 1997). The literature on integrals and

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Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jksus.2014.09.004
1018-3647

derivatives of fractional order is now vast and evolving (see, e.g., Diethelm et al., 2005; Diethelm and Freed, 2002; Tarasov, 2013; Tenreiro Machado et al., 2011; Wang et al., 2011). The reader interested in the early literature, showing that Abel's integral equations may be solved with fractional calculus, is referred to Gel'fand and Shilov (1964). For a concise and recent discussion on the solutions of Abel's integral equations using fractional calculus see Li and Zhao (2013).

Many numerical methods for solving (1.1) have been developed over the past few years, such as product integration methods (Baker, 1977; Baratella and Orsi, 2004), collocation methods (Brunner, 2004), fractional multi step methods (Lubich, 1985, 1986; Plato, 2005), backward Euler methods (Baker, 1977), and methods based on wavelets (Lepik, 2009; Saeedi et al., 2011a,b). Some semi analytic methods, like the Adomian decomposition method, are also available, which produce a series solution (Bougoffa et al., 2013). Unfortunately, the Abel integral Eq. (1.1) is an ill-posed problem. For $k(x,s)=\frac{1}{\Gamma(1-\alpha)}$, Gorenflo (1996) presented some numerical methods based on fractional calculus, e.g., using the Grunwald–Letnikov difference approximation

$$D^{\alpha}f \simeq h^{-\alpha}\sum_{r=0}^{n}(-1)^{r}\binom{\alpha}{r}f(x-rh). \tag{1.2}$$

If f is sufficiently smooth and vanishes at $x \leqslant 0$, then formula (1.2) has an accuracy of order $O(h^2)$, otherwise it has an accuracy of order $O(h)$. On the other hand, Lubich (1985, 1986) introduced a fractional multi-step method for the Abel integral equation of first kind, and Plato (2005) considered fractional multi-step methods for weakly singular integral equations of first kind with a perturbed right-hand side. Liu and Tao (2007) solved the fractional integral equation, transforming it into an Abel integral equation of second kind. A method based on Chebyshev polynomials is given in Avazzadeh et al. (2011). Here we propose a method to solve an Abel integral equation of first kind based on a numerical approximation of fractional integrals and Caputo derivatives of a given function f belonging to $C^n[a,b]$ (see Theorem 4.2).

The structure of the paper is as follows. In Section 2 we recall the necessary definitions of fractional integrals and derivatives and explain some useful relations between them. Section 3 reviews some numerical approximations for fractional integrals and derivatives. The original results are then given in Section 4, where we introduce our method to approximate the solution of the Abel equation at the given nodes and we obtain an upper bound for the error. In Section 5 some examples are solved to illustrate the accuracy of the proposed method.

2. Definitions, relations and properties of fractional operators

Fractional calculus is a classical area with many good books available. We refer the reader to Malinowska and Torres (2012) and Podlubny (1999).

Definition 2.1. Let $\alpha > 0$ with $n-1 < \alpha \leqslant n, n \in \mathbb{N}$, and $a < x < b$. The left and right Riemann–Liouville fractional integrals of order α of a given function f are defined by

$${}_aJ_x^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt$$

and

$${}_xJ_b^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,$$

respectively, where Γ is Euler's gamma function, that is,

$$\Gamma(x)=\int_0^{\infty} t^{x-1}e^{-t}dt.$$

Definition 2.2. The left and right Riemann–Liouville fractional derivatives of order $\alpha > 0$, $n-1 < \alpha \leqslant n, n \in \mathbb{N}$, are defined by

$${}_aD_x^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}f(t)dt$$

and

$${}_xD_b^{\alpha}f(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b (t-x)^{n-\alpha-1}f(t)dt,$$

respectively.

Definition 2.3. The left and right Caputo fractional derivatives of order $\alpha > 0$, $n-1 < \alpha \leqslant n, n \in \mathbb{N}$, are defined by

$${}_a^CD_x^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}f^{(n)}(t)dt$$

and

$${}_x^CD_b^{\alpha}f(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b (t-x)^{n-\alpha-1}f^{(n)}(t)dt,$$

respectively.

Definition 2.4. Let $\alpha > 0$. The Grunwald–Letnikov fractional derivatives are defined by

$$D^{\alpha}f(x)=\lim_{h\to 0}h^{-\alpha}\sum_{r=0}^{\infty}(-1)^{r}\binom{\alpha}{r}f(x-rh)$$

and

$$D^{-\alpha}f(x)=\lim_{h\to 0}h^{\alpha}\sum_{r=0}^{\infty}\begin{bmatrix}\alpha\\ r\end{bmatrix}f(x-rh),$$

where

$$\begin{bmatrix}\alpha\\ r\end{bmatrix}=\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+r-1)}{r!}.$$

Remark 2.5. The Caputo derivatives (Definition 2.3) have some advantages over the Riemann–Liouville derivatives (Definition 2.2). The most well known is related with the Laplace transform method for solving fractional differential equations. The Laplace transform of a Riemann–Liouville derivative leads to boundary conditions containing the limit values of the Riemann–Liouville fractional derivative at the lower terminal $x = a$. In spite of the fact that such problems can be solved analytically, there is no physical interpretation for such a type of boundary conditions. In contrast, the Laplace transform of a Caputo derivative imposes boundary conditions involving integer-order derivatives at $x = a$, which usually are acceptable physical conditions. Another advantage is that the Caputo derivative of a constant function is zero, whereas for the Riemann–Liouville it is not. For details see Sousa (2012).

Remark 2.6. The Grunwald–Letnikov definition gives a generalization of the ordinary discretization formulas for derivatives with integer order. The series in Definition 2.4 converge absolutely and uniformly for each $\alpha > 0$ and for every bounded function f. The discrete approximations derived from the Grunwald–Letnikov fractional derivatives, e.g., (1.2), present some limitations. First, they frequently originate unstable numerical methods and henceforth usually a shifted Grunwald–Letnikov formula is used instead. Another disadvantage is that the order of accuracy of such approximations is never higher than one. For details see Baker (1977).

The following relations between Caputo and Riemann–Liouville fractional derivatives hold Podlubny (1999):

$$ {}_{a}^{C}D_{x}^{\alpha}f(x) = {}_{a}D_{x}^{\alpha}f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha} $$

and

$$ {}_{x}^{C}D_{b}^{\alpha}f(x) = {}_{x}D_{b}^{\alpha}f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-x)^{k-\alpha}. $$

Therefore, if $f \in C^n[a,b]$ and $f^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$, then

$$ {}_{a}^{C}D_{x}^{\alpha}f = {}_{a}D_{x}^{\alpha}f, $$

if $f^{(k)}(b) = 0, \quad k = 0, 1, \ldots, n-1$, then

$$ {}_{x}^{C}D_{b}^{\alpha}f = {}_{x}D_{b}^{\alpha}f. $$

Other useful properties of fractional integrals and derivatives are: all fractional operators are linear, that is, if L is an arbitrary fractional operator, then

$$ L(tf + sg) = tL(f) + sL(g) $$

for all functions $f, g \in C^n[a,b]$ or $f, g \in L^p(a,b)$ and $t, s \in \mathbb{R}$; if $\alpha, \beta > 0$, then

$$ J^{\alpha}J^{\beta} = J^{\alpha+\beta}, \quad D^{\alpha}D^{\beta} = D^{\alpha+\beta}; $$

if $f \in L^{\infty}(a,b)$ or $f \in C^n[a,b]$ and $\alpha > 0$, then

$$ {}_{a}^{C}D_{x}^{\alpha}\, {}_{a}J_{x}^{\alpha}f = f \quad \text{and} \quad {}_{x}^{C}D_{b}^{\alpha}\, {}_{x}J_{b}^{\alpha}f = f. \tag{2.1} $$

On the other hand, if $f \in C^n[a,b]$ and $\alpha > 0$, then

$$ {}_{a}J_{x}^{\alpha}\, {}_{a}^{C}D_{x}^{\alpha}f(x) = f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^{k} $$

and

$$ {}_{x}J_{b}^{\alpha}\, {}_{x}^{C}D_{b}^{\alpha}f(x) = f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(b-x)^{k}. $$

From (2.1) it is seen that the Caputo fractional derivatives are the inverse operators for the Riemann–Liouville fractional integrals. This is the reason why we choose here to use the Caputo fractional derivatives for solving Abel integral equations.

3. Known numerical approximations to fractional operators

Diethelm (1997) (see also Diethelm and Freed, 2002, pp. 62–63) uses the product trapezoidal rule with respect to the weight function $(t_k - \cdot)^{\alpha-1}$ to approximate the Riemann–Liouville fractional integrals. More precisely, the approximation

$$ \int_{t_0}^{t_k}(t_k-u)^{\alpha-1}f(u)du \simeq \int_{t_0}^{t_k}(t_k-u)^{\alpha-1}\widetilde{f}_k(u)du, $$

where $\widetilde{f}_k$ is the piecewise linear interpolator for f whose nodes are chosen at $t_j = jh$, $j = 0, 1, \ldots, n$ and $h = \frac{b-a}{n}$, is considered. Odibat (2006, 2009) uses a modified trapezoidal rule to approximate the fractional integral ${}_{0}J_{x}^{\alpha}f(x)$ (Theorem 3.1) and the Caputo fractional derivative ${}_{a}^{C}D_{x}^{\alpha}f(x)$ (Theorem 3.2) of order $\alpha > 0$.

Theorem 3.1. [See Odibat, 2006, 2009] *Let $b > 0, \alpha > 0$, and suppose that the interval $[0,b]$ is subdivided into k subintervals $[x_j, x_{j+1}]$, $j = 0, \ldots, k-1$, of equal distances $h = \frac{b}{k}$ by using the nodes $x_j = jh$, $j = 0, 1, \ldots, k$. Then the modified trapezoidal rule*

$$ T(f,h,\alpha) = \frac{h^{\alpha}}{\Gamma(\alpha+2)}((k-1)^{\alpha+1} - (k-\alpha-1)k^{\alpha})f(0) + f(b) + \frac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=1}^{k-1}\left((k-j+1)^{\alpha+1} - 2(k-j)^{\alpha+1} + (k-j-1)^{\alpha+1}\right)f(x_j) $$

is an approximation to the fractional integral ${}_{0}J_{x}^{\alpha}f|_{x=b}$:

$$ {}_{0}J_{x}^{\alpha}f|_{x=b} = T(f,h,\alpha) - E_T(f,h,\alpha). $$

Furthermore, if $f \in C^2[0,b]$, then

$$ |E_T(f,h,\alpha)| \leqslant c'_{\alpha}\|f''\|_{\infty}b^{\alpha}h^2 = O(h^2), $$

where c'_{α} is a constant depending only on α.

The following theorem gives an algorithm to approximate the Caputo fractional derivative of an arbitrary order $\alpha > 0$.

Theorem 3.2. [See Odibat, 2006, 2009] *Let $b > 0, \alpha > 0$ with $n-1 < \alpha \leqslant n, n \in \mathbb{N}$, and suppose that the interval $[0,b]$ is subdivided into k subintervals $[x_j, x_{j+1}]$, $j = 0, \ldots, k-1$, of equal distances $h = \frac{b}{k}$ by using the nodes $x_j = jh$, $j = 0, 1, \ldots, k$. Then the modified trapezoidal rule*

$$ C(f,h,\alpha) = \frac{h^{n-\alpha}}{\Gamma(n+2-\alpha)}\Bigg[\left((k-1)^{n-\alpha+1} - (k-n+\alpha-1)k^{n-\alpha}\right)f^{(n)}(0) + f^{(n)}(b) + \sum_{j=1}^{k-1}\left((k-j+1)^{n-\alpha+1} - 2(k-j)^{n-\alpha+1} + (k-j-1)^{n-\alpha+1}\right)f^{(n)}(x_j)\Bigg] $$

is an approximation to the Caputo fractional derivative ${}_{0}^{C}D_{x}^{\alpha}f|_{x=b}$:

$$ {}_{0}^{C}D_{x}^{\alpha}f|_{x=b} = C(f,h,\alpha) - E_C(f,h,\alpha). $$

Furthermore, if $f \in C^{n+2}[0,b]$, then

$$ |E_C(f,h,\alpha)| \leqslant c'_{n-\alpha}\|f^{(n+2)}\|_{\infty}b^{n-\alpha}h^2 = O(h^2), $$

where $c'_{n-\alpha}$ is a constant depending only on α.

In the next section we use Theorem 3.2 to find an approximation solution to a generalized Abel integral equation. The reader interested in other useful approximations for fractional operators is referred to Pooseh et al., 2012; Pooseh et al., 2013; Pooseh et al., 2014 and references therein.

4. Main results

Consider the following Abel integral equation of first kind:

$$ f(x) = \int_0^x \frac{g(t)}{(x-t)^{\alpha}}dt, \quad 0 < \alpha < 1, \quad 0 \leqslant x \leqslant b, \tag{4.1} $$

where $f \in C^1[a,b]$ is a given function satisfying $f(0) = 0$ and g is the unknown function we are looking for.

Theorem 4.1. *The solution to problem (4.1) is*

$$g(x) = \frac{\sin(\alpha\pi)}{\pi}\int_0^x \frac{f'(t)}{(x-t)^{1-\alpha}}dt. \tag{4.2}$$

Proof. According to Definition 2.1, we can write (4.1) in the equivalent form

$$f(x) = \Gamma(1-\alpha)\, {}_0J_x^{1-\alpha} g(x).$$

Then, using (2.1), it follows that

$${}_0^C D_x^{1-\alpha} f(x) = \Gamma(1-\alpha) g(x). \tag{4.3}$$

Therefore,

$$\begin{aligned} g(x) &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^x \frac{f'(t)}{(x-t)^{1-\alpha}}dt \\ &= \frac{\sin(\alpha\pi)}{\pi}\int_0^x \frac{f'(t)}{(x-t)^{1-\alpha}}dt, \end{aligned}$$

where we have used the identity $\pi = \sin(\alpha\pi)\Gamma(\alpha)\Gamma(1-\alpha)$.

Our next theorem gives an algorithm to approximate the solution (4.2) to problem (4.1).

Theorem 4.2. *Let $0 < x < b$ and suppose that the interval $[0, x]$ is subdivided into k subintervals $[t_j, t_{j+1}], j = 0, \ldots, k-1$, of equal length $h = \frac{x}{k}$ by using the nodes $t_j = jh, j = 0, \ldots, k$. An approximate solution $\tilde{g}$ to the solution g of the Abel integral Eq. (4.1) is given by*

$$\begin{aligned} \tilde{g}(x) = \frac{h^\alpha}{\Gamma(1-\alpha)\Gamma(2+\alpha)}\Bigg[&\left((k-1)^{1+\alpha} - (k-1-\alpha)k^\alpha\right)f'(0) \\ &+ f'(x) + \sum_{j=1}^{k-1}\left((k-j+1)^{1+\alpha} - 2(k-j)^{1+\alpha} + (k-j-1)^{1+\alpha}\right)f'(t_j)\Bigg]. \end{aligned} \tag{4.4}$$

Moreover, if $f \in C^3[0, x]$, then $g(x) = \tilde{g}(x) - \frac{1}{\Gamma(1-\alpha)}E(x)$ with

$$|E(x)| \leqslant c'_\alpha \|f'''\|_\infty x^\alpha h^2 = O(h^2),$$

where c'_α is a constant depending only on α and $\|f'''\|_\infty = \max_{t\in[0,x]}|f'''(t)|$.

Proof. We want to approximate the Caputo derivative in (4.3), i.e., to approximate

$$g(x) = \frac{1}{\Gamma(1-\alpha)}\, {}_0^C D_x^{1-\alpha} f(x) = \frac{\sin(\alpha\pi)\Gamma(\alpha)}{\pi}\, {}_0^C D_x^{1-\alpha} f(x).$$

If the Caputo fractional derivative of order $1-\alpha$ for f, $0 < \alpha < 1$, is calculated at collocation nodes t_j, $j = 0, \ldots, k$, then the result is a direct consequence of Theorem 3.2. □

5. Illustrative examples

We exemplify the approximation method given by Theorem 4.2 with three Abel integral equations whose exact solutions are found from Theorem 4.1. The computations were done with the Computer Algebra System Maple. The complete code is provided in Appendix.

Example 5.1. Consider the Abel integral equation

$$e^x - 1 = \int_0^x \frac{g(t)}{(x-t)^{1/2}}dt. \tag{5.1}$$

The exact solution to (5.1) is given by Theorem 4.1:

$$g(x) = \frac{e^x}{\sqrt{\pi}} erf(\sqrt{x}), \tag{5.2}$$

where $erf(x)$ is the error function, that is,

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

In Table 1 we present the approximate values for $g(x_i)$, $x_i = 0.1, 0.2, 0.3$, obtained from Theorem 4.2 with $k = 1, 10, 100$.

Example 5.2. Consider the following Abel integral equation of first kind:

$$x = \int_0^x \frac{g(t)}{(x-t)^{4/5}}dt. \tag{5.3}$$

The exact solution (4.2) of (5.3) takes the form

$$g(x) = \frac{5}{4}\frac{\sin\left(\frac{\pi}{5}\right)}{\pi} x^{4/5}. \tag{5.4}$$

We can see the numerical approximations of $g(x_i)$, $x_i = 0.4, 0.5, 0.6$, obtained from Theorem 4.2 with $k = 1, 10$ in Table 2.

Example 5.3. Consider now the Abel integral equation

$$x^{7/6} = \int_0^x \frac{g(t)}{(x-t)^{1/3}}dt. \tag{5.5}$$

Theorem 4.2 asserts that

$$g(x) = \frac{7\sqrt{3}}{12\pi}\int_0^x \frac{t^{1/6}}{(x-t)^{2/3}}dt. \tag{5.6}$$

The numerical approximations of $g(x_i), x_i = 0.6, 0.7, 0.8$, obtained from Theorem 4.2 with $k = 10, 100, 1000$, are given in Table 3.

Table 1 Approximated values $\tilde{g}(x_i)$ (4.4) to the solution (5.2) of (5.1) obtained from Theorem 4.2 with $k = 1, 10, 100$.

x_i	$k = 1$	$k = 10$	$k = 100$	Exact solution (5.2)	Error $\Delta_{k=100}$
0.1	0.2154319668	0.2152921762	0.2152904646	0.2152905021	3.75×10^{-8}
0.2	0.3267280013	0.3258941876	0.3258841023	0.3258840762	2.61×10^{-8}
0.3	0.4300194238	0.4275954299	0.4275658716	0.4275656575	2.14×10^{-7}

Table 2 Approximated values $\widetilde{g}(x_i)$ (4.4) to the solution (5.4) of (5.3) obtained from Theorem 4.2 with $k = 1, 10$.

x_i	$k = 1$	$k = 10$	Exact solution (5.4)	Error $\Delta_{k=10}$
0.4	0.1123639036	0.1123639036	0.1123639037	1×10^{-10}
0.5	0.1343243751	0.1343243751	0.1343243752	1×10^{-10}
0.6	0.1554174667	0.1554174668	0.1554174668	$\leqslant 10^{-11}$

Table 3 Approximated values $\widetilde{g}(x_i)$ (4.4) to the solution (5.6) of (5.5) obtained from Theorem 4.2 with $k = 10, 100, 1000$.

x_i	$k = 10$	$k = 100$	$k = 1000$	Exact solution (5.6)	Error $\Delta_{k=1000}$
0.6	0.6921182258	0.6981839386	0.6985886509	0.6986144912	0.0000258403
0.7	0.7475731262	0.7541248475	0.7545620072	0.7545898940	0.0000278868
0.8	0.7991892838	0.8061933689	0.8066606797	0.8066905286	0.0000298489

Acknowledgements

This work is part of first author's PhD project and partially supported by the Islamic Azad University (Science and Research branch of Tehran), Iran; and CIDMA–FCT, Portugal, within project PEst-OE/MAT/UI4106/2014. Jahanshahi was supported by a scholarship from the Ministry of Science, Research and Technology of the Islamic Republic of Iran, to visit the University of Aveiro. The hospitality and the excellent working conditions at the University of Aveiro are here gratefully acknowledged. The authors would like also to thank three anonymous referees for valuable comments.

Appendix A. Maple code for examples of Section 5

We provide here all the definitions and computations done in Maple for the problems considered in Section 5. The definitions follow closely the notations introduced along the paper, and should be clear even for readers not familiar with the Computer Algebra System Maple.

```
> # Solution given by Theorem 4.1
> g := (f, alpha, x) -> sin(alpha*Pi)*(int((diff(f(t), t))/(x-t)^(1-alpha), t = 0..x))/Pi:
> # Approximation given by Theorem 4.2
> gtilde := (f, alpha, h, k, x) -> h^alpha
           *(((k-1)^(1+alpha)-(k-1-alpha)*k^alpha)
           *(D(f))(0)+(D(f))(x)+sum(((k-j+1)^(1+alpha)-2*(k-j)^(1+alpha)
           +(k-j-1)^(1+alpha))*(D(f))(j*h), j = 1 .. k-1))
           /(GAMMA(1-alpha)*GAMMA(2+alpha)):
> # Example 5.1
> f1 := x -> exp(x)-1:
> g(f1, 1/2, x);
```

$$\frac{\operatorname{erf}(\sqrt{x})e^x}{\sqrt{\pi}}$$

```
> ExactValues1 := evalf([g(f1,1/2,0.1), g(f1,1/2,0.2), g(f1,1/2,0.3)]);
```

$$[0.2152905021, 0.3258840762, 0.4275656575]$$

```
> ApproximateValues1 := k -> evalf([gtilde(f1, 1/2, 0.1/k, k, 0.1),
                               gtilde(f1, 1/2, 0.2/k, k, 0.2),
                               gtilde(f1, 1/2, 0.3/k, k, 0.3)]):
> ApproximateValues1(1);
```

$$[0.2154319668, 0.3267280013, 0.4300194238]$$

```
> ApproximateValues1(10);
```

$$[0.2152921762, 0.3258941876, 0.4275954299]$$

```
> ApproximateValues1(100);
```

$$[0.2152904646, 0.3258841023, 0.4275658716]$$

```
> # Example 5.2
> f2 := x -> x:
> g(f2, 4/5, x);
```

$$\frac{5}{4}\,\frac{\sin\left(\frac{1}{5}\,\pi\right)x^{4/5}}{\pi}$$

```
> ExactValues2 := evalf([g(f2,4/5,0.4), g(f2, 4/5, 0.5), g(f2, 4/5, 0.6)]);
```

$$[0.1123639037, 0.1343243752, 0.1554174668]$$

```
> ApproximateValues2 := k -> evalf([gtilde(f2, 4/5, 0.4/k, k, 0.4),
                                    gtilde(f2, 4/5, 0.5/k, k, 0.5),
                                    gtilde(f2, 4/5, 0.6/k, k, 0.6)]):
> ApproximateValues2(1);
```

$$[0.1123639036, 0.1343243751, 0.1554174667]$$

```
> ApproximateValues2(10);
```

$$[0.1123639036, 0.1343243751, 0.1554174668]$$

```
> # Example 5.3
> f3 := x -> x^(7/6):
> g(f3, 1/3, x);
```

$$\frac{1}{2}\,\frac{\sqrt{3}\int_0^x \frac{7}{6}\,\frac{t^{1/6}}{(x-t)^{2/3}}\,dt}{\pi}$$

```
> ExactValues3 := evalf([g(f3,1/3,0.6), g(f3,1/3,0.7), g(f3,1/3,0.8)]);
```

$$[0.6986144912, 0.7545898940, 0.8066905286]$$

```
> ApproximateValues3 := k -> evalf([gtilde(f3, 1/3, 0.6/k, k, 0.6),
                                    gtilde(f3, 1/3, 0.7/k, k, 0.7),
                                    gtilde(f3, 1/3, 0.8/k, k, 0.8)]):
> ApproximateValues3(1);
```

$$[0.5605130027, 0.6054232384, 0.6472246667]$$

```
> ApproximateValues3(10);
```

$$[0.6921182258, 0.7475731262, 0.7991892838]$$

```
> ApproximateValues3(100);
```

$$[0.6981839386, 0.7541248475, 0.8061933689]$$

```
> ApproximateValues3(1000);
```

$$[0.6985886509, 0.7545620072, 0.8066606797]$$

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