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ORIGINAL ARTICLE

The block by block method for the numerical solution of the nonlinear two-dimensional Volterra integral equations

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Abstract In this study, an efficient method is presented for solving nonlinear two-dimensional Volterra integral equations of the second kind. Using block by block method, nonlinear two-dimensional Volterra integral equations reduce to a algebraic equations. Also a theorem is proved for convergence analysis. Numerical examples are presented and results are compared with the analytical solution to demonstrate the validity and applicability of the method.

1. Introduction

Many problems in applied mathematics and physics give rise to nonlinear two-dimensional Volterra integral equations the second kind (Hanson and Phillips, 1978; Mckee et al., 2000)

$$u(x,y) = f(x,y) + \int_0^x \int_0^y k(x,y,s,t,u(s,t))dtds; \quad (x,y) \in D, \tag{1}$$

where $f(x,y)$ and $k(x,y,s,t,u)$ are given continuous functions defined, respectively on $D = [0,b] \times [0,b]$, $E = D \times D \times (-\infty, +\infty)$ and $u(x,y)$ is unknown on D. While several numerical methods for approximating the solution of one-dimensional Volterra integral equations are known, for two-dimensional only a few are discussed in the literature. The numerical solution of equations of the type of (1) seems to have first been considered by Bel' tyukov and Kuznechikhina (1976) where they proposed an explicit Rung–Kutta type method of order 3 without any convergence analysis. A bivariate cubic spline functions method of full continuity was obtained by Singh (1979). Brunner and Kauthen (1989) introduced collocation and iterated collocation methods for two-dimensional linear Volterra integral equations. An asymptotic error expansion of the iterated collocation solution for two-dimensional linear and nonlinear Volterra integral equations was obtained by Han and Zhang (1994) and Guoqiang et al. (2000), respectively. More recently, Hadizadeh and Moatamedi (2007) have investigated a differential transformation approach for nonlinear two-dimensional Volterra integral equations.

In the present paper, we apply block by block method (Katani and Shahmorad, 2010; Saberi-Nadjafi and Heidari, 2007), to solve the nonlinear two-dimensional Volterra integral equations (1).

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2. Development block by block method for solving nonlinear two-dimensional Volterra integral equations

The basic region $D=[0,b]\times[0,b]$ is divided into steps of width and Length h, such as $x_i=ih$, $y_j=jh;\quad i,j=0,1,2,\ldots,n$ and $nh=b$. In what follows, we denote by $U_{i,j}$ approximation of $u(x,y)$ at the mesh point $(x,y)=(x_i,y_j)$ and $U_{0,0}=f(0,0)$. We let the of blocks to be 2. From Eq. (1) we have

$$\begin{aligned}
u(x_{2m+1},y_{2m+1}) &= f(x_{2m+1},y_{2m+1})+\int_0^{x_{2m+1}}\\
&\quad\times\int_0^{y_{2m+1}}k(x_{2m+1},y_{2m+1},s,t,u(s,t))dtds\\
&= f(x_{2m+1},y_{2m+1})+\int_0^{x_{2m}}\int_0^{y_{2m}}\\
&\quad\times k(x_{2m+1},y_{2m+1},s,t,u(s,t))dtds+\int_{x_{2m}}^{x_{2m+1}}\int_0^{y_{2m}}\\
&\quad\times k(x_{2m+1},y_{2m+1},s,t,u(s,t))dtds+\int_0^{x_{2m}}\int_{y_{2m}}^{y_{2m+1}}\\
&\quad\times k(x_{2m+1},y_{2m+1},s,t,u(s,t))dtds+\int_{x_{2m}}^{x_{2m+1}}\int_{y_{2m}}^{y_{2m+1}}\\
&\quad\times k(x_{2m+1},y_{2m+1},s,t,u(s,t))dtds. \qquad (2)
\end{aligned}$$

Now, integration over $[0,x_{2m}]$ and $[0,y_{2m}]$ can be accomplished by Simpson's rule and the integral over $[x_{2m},x_{2m+1}]$ and $[y_{2m},y_{2m+1}]$ are computed by using fourth degree two-dimensional Lagrange interpolation of the integrand at the points $x_{2m},x_{2m+\frac{1}{2}},x_{2m+1}$ and $y_{2m},y_{2m+\frac{1}{2}},y_{2m+1}$. Hence

$$\begin{aligned}
U_{2m+1,2m+1} &= f(x_{2m+1},y_{2m+1})+\frac{h^2}{9}[k(x_{2m+1},y_{2m+1},x_0,y_0,U_{0,0})\\
&\quad+4k(x_{2m+1},y_{2m+1},x_1,y_0,U_{1,0})+\ldots\\
&\quad+k(x_{2m+1},y_{2m+1},x_{2m},y_0,U_{2m,0})]\\
&\quad+\frac{4h^2}{9}[k(x_{2m+1},y_{2m+1},x_0,y_1,U_{0,1})\\
&\quad+4k(x_{2m+1},y_{2m+1},x_1,y_1,U_{1,1})+\ldots\\
&\quad+k(x_{2m+1},y_{2m+1},x_{2m},y_1,U_{2m,1})]+\ldots\\
&\quad+\frac{h^2}{9}[k(x_{2m+1},y_{2m+1},x_0,y_{2m},U_{0,2m})\\
&\quad+4k(x_{2m+1},y_{2m+1},x_1,y_{2m},U_{1,2m})+\ldots\\
&\quad+k(x_{2m+1},y_{2m+1},x_{2m},y_{2m},U_{2m,2m})]\\
&\quad+\frac{h^2}{18}[k(x_{2m+1},y_{2m+1},x_{2m},y_0,U_{2m,0})\\
&\quad+4k(x_{2m+1},y_{2m+1},x_{2m+\frac{1}{2}},y_0,U_{2m+\frac{1}{2},0})\\
&\quad+k(x_{2m+1},y_{2m+1},x_{2m+1},y_0,U_{2m+1,0})]\\
&\quad+\frac{4h^2}{18}[k(x_{2m+1},y_{2m+1},x_{2m},y_1,U_{2m,1})\\
&\quad+4k(x_{2m+1},y_{2m+1},x_{2m+\frac{1}{2}},y_1,U_{2m+\frac{1}{2},1})\\
&\quad+k(x_{2m+1},y_{2m+1},x_{2m+1},y_1,U_{2m+1,1})]+\ldots\\
&\quad+\frac{h^2}{18}[k(x_{2m+1},y_{2m+1},x_{2m},y_{2m},U_{2m,2m})\\
&\quad+4k(x_{2m+1},y_{2m+1},x_{2m+\frac{1}{2}},y_{2m},U_{2m+\frac{1}{2},2m})\\
&\quad+k(x_{2m+1},y_{2m+1},x_{2m+1},y_{2m},U_{2m+1,2m})]\\
&\quad+\frac{h^2}{18}[k(x_{2m+1},y_{2m+1},x_0,y_{2m},U_{0,2m})\\
&\quad+4k(x_{2m+1},y_{2m+1},x_1,y_{2m},U_{1,2m})+\ldots\\
&\quad+k(x_{2m+1},y_{2m+1},x_{2m},y_{2m},U_{2m,2m})]\\
&\quad+\frac{4h^2}{18}[k(x_{2m+1},y_{2m+1},x_0,y_{2m+\frac{1}{2}},U_{0,2m+\frac{1}{2}})\\
&\quad+4k(x_{2m+1},y_{2m+1},x_1,y_{2m+\frac{1}{2}},U_{1,2m+\frac{1}{2}})+\ldots\\
&\quad+k(x_{2m+1},y_{2m+1},x_{2m},y_{2m+\frac{1}{2}},U_{2m,2m+\frac{1}{2}})]\\
&\quad+\frac{h^2}{18}[k(x_{2m+1},y_{2m+1},x_0,y_{2m+1},U_{0,2m+1})\\
&\quad+4k(x_{2m+1},y_{2m+1},x_1,y_{2m+1},U_{1,2m+1})+\ldots\\
&\quad+k(x_{2m+1},y_{2m+1},x_{2m},y_{2m+1},U_{2m,2m+1})]\\
&\quad+\frac{h^2}{36}[k(x_{2m+1},y_{2m+1},x_{2m}\cdot y_{2m},U_{2m,2m})\\
&\quad+4k(x_{2m+1},y_{2m+1},x_{2m+\frac{1}{2}},y_{2m},U_{2m+\frac{1}{2},2m})\\
&\quad+k(x_{2m+1},y_{2m+1},x_{2m+1},y_{2m},U_{2m+1,2m})]\\
&\quad+\frac{4h^2}{36}[k(x_{2m+1},y_{2m+1},x_{2m},y_{2m+\frac{1}{2}},U_{2m,2m+\frac{1}{2}})\\
&\quad+4k(x_{2m+1},y_{2m+1},x_{2m+\frac{1}{2}},y_{2m+\frac{1}{2}},U_{2m+\frac{1}{2},2m+\frac{1}{2}})\\
&\quad+k(x_{2m+1},y_{2m+1},x_{2m+1},y_{2m+\frac{1}{2}},U_{2m+1,2m+\frac{1}{2}})]\\
&\quad+\frac{h^2}{36}[k(x_{2m+1},y_{2m+1},x_{2m},y_{2m+1},U_{2m,2m+1})\\
&\quad+4k(x_{2m+1},y_{2m+1},x_{2m+\frac{1}{2}},y_{2m+1},U_{2m+\frac{1}{2},2m+1})\\
&\quad+k(x_{2m+1},y_{2m+1},x_{2m+1},y_{2m+1},U_{2m+1,2m+1})], \qquad (3)
\end{aligned}$$

where

$$\begin{aligned}
U_{2m+\frac{1}{2},2m+\frac{1}{2}} &= \frac{9}{64}U_{2m,2m}+\frac{9}{32}U_{2m,2m+1}-\frac{3}{64}U_{2m,2m+2}\\
&\quad+\frac{9}{32}U_{2m+1,2m}+\frac{9}{16}U_{2m+1,2m+1}\\
&\quad-\frac{3}{32}U_{2m+1,2m+2}+\frac{3}{64}U_{2m+2,2m}\\
&\quad-\frac{3}{32}U_{2m+2,2m+1}+\frac{1}{64}U_{2m+2,2m+2}, \qquad (4)
\end{aligned}$$

$$\begin{aligned}
U_{2m+\frac{1}{2},2m+1} &= \frac{9}{8}U_{2m,2m}-\frac{9}{8}U_{2m,2m+1}+\frac{3}{8}U_{2m,2m+2}\\
&\quad+\frac{9}{4}U_{2m+1,2m}-\frac{9}{4}U_{2m+1,2m+1}+\frac{3}{4}U_{2m+1,2m+2}\\
&\quad-\frac{3}{8}U_{2m+2,2m}+\frac{3}{8}U_{2m+2,2m+1}-\frac{1}{8}U_{2m+2,2m+2}, \qquad (5)
\end{aligned}$$

$$U_{2m+1,2m+\frac{1}{2}}=\frac{3}{8}U_{2m+1,2m}+\frac{3}{4}U_{2m+1,2m+1}-\frac{1}{8}U_{2m+1,2m+2}, \qquad (6)$$

$$U_{2m+\frac{1}{2},2m+2}=\frac{3}{8}U_{2m,2m+2}+\frac{3}{4}U_{2m+1,2m+2}-\frac{1}{8}U_{2m+2,2m+2}, \qquad (7)$$

$$U_{2m+2,2m+\frac{1}{2}}=\frac{3}{8}U_{2m+2,2m}+\frac{3}{4}U_{2m+2,2m+1}-\frac{1}{8}U_{2m+2,2m+2}, \qquad (8)$$

$$U_{2m+\frac{1}{2},2m}=\frac{3}{8}U_{2m,2m}+\frac{3}{4}U_{2m+1,2m}-\frac{1}{8}U_{2m+2,2m}, \qquad (9)$$

$$U_{2m,2m+\frac{1}{2}}=\frac{3}{8}U_{2m,2m}+\frac{3}{4}U_{2m,2m+1}-\frac{1}{8}U_{2m,2m+2}. \qquad (10)$$

In a similar manner we obtain

$$\begin{aligned}
U_{2m+2,2m+2} &= f(x_{2m+2}, y_{2m+2}) \\
&\quad + \int_0^{x_{2m+2}} \int_0^{y_{2m+2}} k(x_{2m+2}, y_{2m+2}, s, t, u(s,t))dtds \\
&= f(x_{2m+2}, y_{2m+2}) + \frac{h^2}{9}[k(x_{2m+2}, y_{2m+2}, x_0, y_0, U_{0,0}) \\
&\quad + 4k(x_{2m+2}, y_{2m+2}, x_1, y_0, U_{1,0}) + \ldots \\
&\quad + k(x_{2m+2}, y_{2m+2}, x_{2m+2}, y_0, U_{2m+2,0})] \\
&\quad + \frac{4h^2}{9}[k(x_{2m+2}, y_{2m+2}, x_0, y_1, U_{0,1}) \\
&\quad + 4k(x_{2m+2}, y_{2m+2}, x_1, y_1, U_{1,1}) + \ldots \\
&\quad + k(x_{2m+2}, y_{2m+2}, x_{2m+2}, y_1, U_{2m+2,1})] + \ldots \\
&\quad + \frac{h^2}{9}[k(x_{2m+2}, y_{2m+2}, x_0, y_{2m+2}, U_{0,2m+2}) \\
&\quad + 4k(x_{2m+2}, y_{2m+2}, x_1, y_{2m+2}, U_{1,2m+2}) + \ldots \\
&\quad + k(x_{2m+2}, y_{2m+2}, x_{2m+2}, y_{2m+2}, U_{2m+2,2m+2})]. \qquad (11)
\end{aligned}$$

From Eqs. (3)–(11) we have a nonlinear equations system for $m = 1, 2, \ldots$. For sufficiently small h there exists unique solution which can be obtained by iteration such as modified Newton–Raphson method.

3. Convergence analysis

Theorem 3.1. *The approximate block by block method given by the system (3) and (11) is convergent and its order of convergence is at least four.*

Proof. Let

$$\begin{aligned}
|\varepsilon_{2m+1,2m+1}| &= |U_{2m+1,2m+1} - u(x_{2m+1}, y_{2m+1})| \\
&= \Bigg| \frac{h^2}{9}\sum_{i=0}^{2m}\sum_{j=0}^{2m} w_{i,j}k(x_{2m+1}, y_{2m+1}, x_i, y_j, U_{i,j}) \\
&\quad + \frac{h^2}{18}\sum_{j=0}^{2m} w_j k(x_{2m+1}, y_{2m+1}, x_{2m}, y_j, U_{2m,j}) \\
&\quad + \frac{2h^2}{9}\sum_{j=0}^{2m} w_j k(x_{2m+1}, y_{2m+1}, x_{2m+\frac{1}{2}}, y_j, U_{2m+\frac{1}{2},j}) \\
&\quad + \frac{h^2}{18}\sum_{j=0}^{2m} w_j k(x_{2m+1}, y_{2m+1}, x_{2m+1}, y_j, U_{2m+1,j}) \\
&\quad + \frac{h^2}{18}\sum_{i=0}^{2m} w_i k(x_{2m+1}, y_{2m+1}, x_i, y_{2m}, U_{i,2m}) \\
&\quad + \frac{h^2}{36}k(x_{2m+1}, y_{2m+1}, x_{2m}, y_{2m}, U_{2m,2m}) \\
&\quad + \frac{2h^2}{18}k(x_{2m+1}, y_{2m+1}, x_{2m+\frac{1}{2}}, y_{2m}, U_{2m+\frac{1}{2},2m}) \\
&\quad + \frac{h^2}{36}k(x_{2m+1}, y_{2m+1}, x_{2m+1}, y_{2m}, U_{2m+1,2m}) \\
&\quad + \frac{2h^2}{9}\sum_{i=0}^{2m} w_i k(x_{2m+1}, y_{2m+1}, x_i, y_{2m+\frac{1}{2}}, U_{i,2m+\frac{1}{2}}) \\
&\quad + \frac{2h^2}{18}k(x_{2m+1}, y_{2m+1}, x_{2m}, y_{2m+\frac{1}{2}}, U_{2m,2m+\frac{1}{2}}) \\
&\quad + \frac{4h^2}{9}k(x_{2m+1}, y_{2m+1}, x_{2m+\frac{1}{2}}, y_{2m+\frac{1}{2}}, U_{2m+\frac{1}{2},2m+\frac{1}{2}}) \\
&\quad + \frac{2h^2}{18}k(x_{2m+1}, y_{2m+1}, x_{2m+1}, y_{2m+\frac{1}{2}}, U_{2m+1,2m+\frac{1}{2}}) \\
&\quad + \frac{h^2}{18}\sum_{i=0}^{2m} k(x_{2m+1}, y_{2m+1}, xi, y_{2m+1}, U_{i,2m+1}) \\
&\quad + \frac{h^2}{36}k(x_{2m+1}, y_{2m+1}, x_{2m}, y_{2m+1}, U_{2m,2m+1}) \\
&\quad + \frac{2h^2}{18}k(x_{2m+1}, y_{2m+1}, x_{2m+\frac{1}{2}}, y_{2m+1}, U_{2m+\frac{1}{2},2m+1}) \\
&\quad + \frac{h^2}{36}k(x_{2m+1}, y_{2m+1}, x_{2m+1}, y_{2m+1}, U_{2m+1,2m+1}) \\
&\quad - \int_0^{2m+1}\int_0^{2m+1} k(x_{2m+1}, y_{2m+1}, s, t, u(s,t))dtds \Bigg|,
\end{aligned}$$

using the Lipschitz condition (Deimling, 1985) it can be written as

$$\begin{aligned}
|\varepsilon_{2m+1,2m+1}| &\leqslant h^2 c\sum_{i=0}^{2m}\sum_{j=0}^{2m}|\varepsilon_{i,j}| + h^2 c'\sum_{j=0}^{2m}|\varepsilon_{2m,j}| + h^2 c'\sum_{j=0}^{2m}|\varepsilon_{2m+1,j}| \\
&\quad + h^2 c'\sum_{j=0}^{2m}|\varepsilon_{2m+2,j}| + h^2 c''\sum_{i=0}^{2m}|\varepsilon_{i,2m}| + h^2 c'' \\
&\quad \times \sum_{i=0}^{2m}|\varepsilon_{i,2m+1}| + h^2 c''\sum_{i=0}^{2m}|\varepsilon_{i,2m+2}| + h^2 c'''|\varepsilon_{2m+1,2m}| \\
&\quad + h^2 c'''|\varepsilon_{2m,2m+1}| + h^2 c'''|\varepsilon_{2m+2,2m}| + h^2 c'''|\varepsilon_{2m,2m+2}| \\
&\quad + h^2 c'''|\varepsilon_{2m+1,2m+2}| + h^2 c'''|\varepsilon_{2m+2,2m+1}| \\
&\quad + h^2 c'''|\varepsilon_{2m+2,2m+2}| + h^2 c'''|\varepsilon_{2m+1,2m+1}| \\
&\quad + |R_{2m+1,2m+1}|,
\end{aligned}$$

where $R_{2m+1,2m+1}$ is the error of integration rule. Without diminish of universality, we assume that

$$\|\varepsilon_{i,j}\|_\infty = \max_{i,j=2m,2m+1,2m+2} \max |\varepsilon_{i,j}| = |\varepsilon_{2m+1,2m+1}|,$$

then let $R = max\{R_{2m+1,2m+1}\}$, hence

$$\begin{aligned}
\|\varepsilon_{i,j}\| &\leqslant h^2 c\sum_{i=0}^{2m}\sum_{j=0}^{2m}|\varepsilon_{i,j}| \\
&\quad + h^2 c'\sum_{j=0}^{2m}(|\varepsilon_{2m,j}| + |\varepsilon_{2m+1,j}| + |\varepsilon_{2m+2,j}|) \\
&\quad + h^2 c''\sum_{i=0}^{2m}(|\varepsilon_{i,2m}| + |\varepsilon_{i,2m+1}| + |\varepsilon_{i,2m+2}|) \\
&\quad + 8h^2 c'''|\varepsilon_{2m+1,2m+1}| + R,
\end{aligned}$$

and

$$\begin{aligned}
\|\varepsilon_{i,j}\| &\leqslant \frac{h^2 c}{1-8h^2c'''}\sum_{i=0}^{2m}\sum_{j=0}^{2m}|\varepsilon_{i,j}| + \frac{h^2 c'}{1-8h^2c'''}\sum_{j=0}^{2m}(|\varepsilon_{2m,j}| + |\varepsilon_{2m+1,j}| \\
&\quad + |\varepsilon_{2m+2,j}|) + \frac{h^2 c''}{1-8h^2c'''}\sum_{i=0}^{2m}(|\varepsilon_{i,2m}| + |\varepsilon_{i,2m+1}| + |\varepsilon_{i,2m+2}|) \\
&\quad + \frac{R}{1-8h^2c'''},
\end{aligned}$$

then from Gronwall inequality (Mckee et al., 2000), we have :

$$\|\varepsilon_{i,j}\| \leqslant \frac{R}{1-8h^2c'''}e^{\frac{\gamma}{1-8h^2c'''}(x_n+y_n)}.$$

Hence we deduce that $\|\varepsilon_{i,j}\| \to 0$ as $h \to 0$ and for function $k(x,y,s,t,u)$ and $u(x,y)$ with at least fourth order derivatives, we have $R = O(h^4)$ hence, $\|\varepsilon_{i,j}\| = O(h^4)$ and this completes the proof.

4. Numerical results

In this section, we applied the method presented in this paper for solving linear and nonlinear two-dimensional Volterra integral equations (1) and solved two examples.

Example 1. Consider the following linear two-dimensional Volterra integral equation (Bongsoo, 2009):

$$u(x,y) = f(x,y) + \int_0^x \int_0^y \frac{2\sin(t+s)}{e^{s-t}} u(s,t)dtds; \quad (x,y) \in [0,\frac{1}{2}] \times [0,\frac{1}{2}],$$

where

$$f(x,y) = \sin(x+y)(e^{x-y}+1) - \sin(y)e^{-y} - e^x\sin(x),$$

the exact solution is $u(x,y) = \sin(x+y)$.

The comparisons between the approximation $U(x,y)$ and the exact solution $u(x,y) = \sin(x+y)$ at the given test points (x,y) are presented in the Tables 1 and 2.

Example 2. Consider the following nonlinear two-dimensional Volterra integral equation (Tari et al., 2009):

$$u(x,y) = f(x,y) + \int_0^x \int_0^y (s^2+e^{-2t})u^2(s,t)dtds; \quad (x,y) \in [0,1] \times [0,1],$$

Table 1 Numerical results of example 1 with block by block method and $h = 0.1$.

Nodes (x,y)	Exact solution	Presented method	Error presented method
(0,0)	0	0	0
(0.1,0.1)	0.19866933079506	0.198668890723798	4.40071263596575 e-007
(0.2,0.2)	0.38941834230865	0.389408629516421	9.71279222977683 e-006
(0.3,0.3)	0.56464247339504	0.564600658978419	4.18144166166590 e-005
(0.4,0.4)	0.71735609089952	0.717250180986031	1.05909913491353 e-004
(0.5,0.5)	0.84147098480790	0.841262440010650	2.08544797246857 e-004

Table 2 Numerical results of example 1 with block by block method and $h = 0.05$.

Nodes (x,y)	Exact solution	Presented method	Error presented method
(0,0)	0	0	0
(0.1,0.1)	0.19866933079506	0.19866900757444	3.23220622433507 e-007
(0.2,0.2)	0.38941834230865	0.38941468262463	3.65968401649930 e-006
(0.3,0.3)	0.56464247339504	0.56462949769581	1.29756992223174 e-005
(0.4,0.4)	0.71735609089952	0.71732543492286	3.06559766636294 e-005
(0.5,0.5)	0.84147098480790	0.84141258725745	5.83975504512280 e-005

Table 3 Numerical results of example 2 with block by block method and $h = 0.1$.

Nodes (x,y)	Exact solution	Presented method	Error presented method
(0,0)	0	0	0
(0.1,0.1)	0.01105170918076	0.01105170915948	2.12735003224385$e-011$
(0.2,0.2)	0.04885611032641	0.04885611113270	8.06287279997431$e-010$
(0.3,0.3)	0.12588434261122	0.12588405050312	2.92108099361754$e-007$
(0.4,0.4)	0.23869195162260	0.23869101718841	9.34434194438394$e-007$
(0.5,0.5)	0.41218031767503	0.41218375273070	3.43505567068636$e-006$
(0.6,0.6)	0.65596276814058	0.65585320373805	9.56440252886104$e-006$
(0.7,0.7)	0.98673882666053	0.98673517185294	3.65480758712788$e-006$
(0.8,0.8)	1.42434619423518	1.42344690409411	9.00709858926874$e-004$
(0.9,0.9)	1.99227852003713	1.99233157037561	5.30503384754688$e-004$
(1,1)	2.71828182845905	2.71807931245121	2.02516007826947$e-004$

Table 4 Numerical results of example 2 with block by block method and $h = 0.05$.

Nodes (x,y)	Exact solution	Presented method	Error presented method
(0,0)	0	0	0
(0.1,0.1)	0.01105170918076	0.01105170918224	1.47911023706814$e-012$
(0.2,0.2)	0.04885611032641	0.04885611086382	2.68872021869093$e-010$
(0.3,0.3)	0.12588434261122	0.12588430490061	3.77105729143512$e-008$
(0.4,0.4)	0.23869195162260	0.23868966522708	2.86395521797989$e-007$
(0.5,0.5)	0.41218031767503	0.41218149485421	1.17717917974547$e-006$
(0.6,0.6)	0.65596276814058	0.65586574533429	2.97719370179195$e-006$
(0.7,0.7)	0.98673882666053	0.98630507018100	3.65480758712789$e-006$
(0.8,0.8)	1.42434619423518	1.42483300433444	4.86810099263435$e-004$
(0.9,0.9)	1.99227852003713	1.99226440132187	1.41187152615796$e-005$
(1,1)	2.71828182845905	2.70829208944238	5.10260983333843$e-004$

where

$$f(x,y) = x^2e^y + \frac{1}{14}x^7 - \frac{1}{14}x^7e^{2y} - \frac{1}{5}x^5y,$$

the exact solution is $u(x,y) = x^2e^y$.

The comparisons between the approximation $U(x,y)$ and the exact solution $u(x,y) = x^2e^y$ at the given test points (x,y) are presented in the Tables 3 and 4.

5. Conclusion

In this paper, we have investigated the application of block by block method for solving the nonlinear two-dimensional Volterra integral equations. This technique is very simple. A similar manner is used for 4, 6 block. By increasing number of blocks for 4 and 6, and quadrature rules (Newton–Cotes quadrature rule) the order of convergence increases such that is would be at least $O(h^6)$ and $O(h^8)$, respectively. Also we can expand this method to higher dimensional problems. Note that the find system extracted from the nonlinear equations will be nonlinear and a proper technique such Newton–Raphson method could be applied.

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