Journal of King Saud University – Science 31 (2019) 42–47

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Original article

Some new linear representations of matrix quaternions with some applications

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ARTICLE INFO

Article history:
Received 28 March 2017
Accepted 28 May 2017
Available online 5 June 2017

2010 MSC:
11R52
15A69
15A24

Keywords:
Quaternions
Kronecker product
Schur complement
Moore-Penrose inverse
Linear matrix quaternion equations

ABSTRACT

In this paper, we construct several new attractive and interested linear representations of matrix quaternions by using Kronecker structures in order to obtain the general partitioned linear representation form of matrix quaternions. In addition, we present the general solutions of three important partitioned quaternions systems by using our new representations and Kronecker structure. These systems are: the partitioned linear quaternion equations, general linear matrix quaternion system and coupled Sylvester matrix quaternion system.

1. Introduction

Throughout of the present paper, we denote that the set of all real numbers, the set of all complex numbers, the set of all quaternions and the set of all $n \times n$ matrices $\mathbb{R}$, $\mathbb{C}$, Q and $M_n(\mathbb{R})$, respectively. Also, the notations: $r(A)$, $R(A)$, A^T, A^{-1}, A^+ and $VecA$ stand for the rank, range, transpose, inverse, Moore-Penrose inverse and vector operator of matrix A, respectively.

The quaternions (Nie et al., 2017; Zeng, 2005; Tian and Styan, 2005; Farebrother et al., 2003; Zhang, 1997):

$$Q = \{a_1 + a_2 i + a_3 j + a_4 k : a_1, a_2, a_3, a_4 \in R\}, \tag{1-1}$$

where $\{1, i, j, k\}$ is set of basis of Q which satisfying the following "Hamilton conditions":

$$\begin{aligned} &i^2 = j^2 = k^2 = -1;\\ &ij = k;\ ki = j;\ jk = i;\\ &ji = -k;\ ik = -j;\ kj = -i, \end{aligned} \tag{1-2}$$

are a four dimensional generalization of two dimensional complex algebra:

$$\mathbb{C} = \{a_1 + a_2 i : a_1, a_2 \in R,\ i^2 = -1\}. \tag{1-3}$$

Similarly to how complex numbers can describe both points and linear operations in the plane, quaternions can describe both points and linear operations in three or four dimensions. Historically, the development of quaternions runs parallel to the development or real linear algebra and matrix theory. Thus they provided a framework for dealing with vector quantities before the wide spread popularization of matrices and vector calculus in mathematics and physics and have inspired the development of more general "hypercomplex" geometric algebras such as Clifford algebras (Farenick and Pidkowitch, 2003; Zhang, 1997, 2011; Sun et al., 2011; Lee and Song, 2010; Harauz, 1990; Behan and Mars, 2004; Kuipres, 2000; Farebrother et al., 2003; Alagoz et al., 2012; Song et al., 2014; Jafari et al., 2013; Li et al., 2014; Huang, 2000; Song and Wang, 2011; Wang, 2005; Bolat and Ipek, 2004; Wang and Song, 2007). In the other word, quaternions algebra have been

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Peer review under responsibility of King Saud University.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jksus.2017.05.017
1018-3647/

playing a central role in many fields of sciences such as differential geometry, human images, control theory, quantum physics, theory of relativity, simulation of particle motion, 3D geophones, multispectral images, signal processing include seismic velocity analysis, seismic waveform deconvolution, 3D anemometers, statistical signal processing and probability distributions (Farenick and Pidkowitch, 2003; Zhang, 1997; Sun et al., 2011; Lee and Song, 2010; Harauz, 1990; Behan and Mars, 2004; Kuipres, 2000; Took and Mandic, 2011; Ginzberg, 2013; Leo and Scolarici, 2000; Zhang and March, 2011).

Recently, matrix quaternion equations and systems play an important role in mathematics and other sciences such as engineering, statistics, control theory and quantum field theory in physics and chemistry (Behan and Mars, 2004; Took and Mandic, 2011; Sun et al., 2011; Lee and Song, 2010; Ginzberg, 2013; He and Wang, 2013; Nie et al., 2017; Wang et al., 2009, 2016; Rehman and Wang, 2015; Lin and Wang, 2013; Zhang, 2007, 2013; Lawrynowicz et al., 2010; Zhang and March, 2011). For example, Zhang (1997) studied and proved some properties on quaternions and quaternions on matrices; Lee and Song (2010) established matrix representations of Clifford algebra; Farebrother et al. (2003) established some matrix representations of quaternions; Tian and Styan (2005) established some matrix versions of the Cauchy-Schwarz and Frucht-Kantorovich inequalities over the quaternion algebra; Alagoz et al. (2012) studied the split quaternion matrices; Bolat and Ipek (2004) studied the singular value decomposition of quaternion matrices; Song and Wang (2011) solved some restricted some linear quaternion equations by using an alternative condensed Cramer rule method; He and Wang (2013), Nie et al. (2017), Wang et al. (2016) and Rehman and Wang (2015) established the necessary and sufficient conditions for the existence to the solutions of such matrix quaternion systems which include the coupled generalized Sylvester matrix equations and matrix quaternion equations with three and more variables. Moreover, the vector-sensor signal processing was studied by Behan and Mars (2004); a color human face image by quaternion matrix representations was recognized and reconstructed by Sun et al. (2011); the statistical properties of quaternion matrices was studied by Ginzberg (2013); quaternions random signals was studied by Took and Mandic (2011); the three-Dimensional (3D) Ising models was constructed and discussed by Zhang (2007, 2013); the order-disorder model and Ising lattice was studied by Lawrynowicz et al. (2010); and the temperature-time duality in the 3D Ising model was also presented by Zhang and March (2011).

The complex number as in (1-3) can be extended and defined as a real linear representation σ on matrix quaternions Q of order $2^n \times 2^n$ $(n \in \mathbb{N})$ as follows (Lee and Song, 2010; Farebrother et al., 2003; Song et al., 2014; Huang, 2000; Jafari et al., 2013; Zhang, 1997, 2011; Tian and Styan, 2005): $\sigma : Q \to M_{2^n}(\mathbb{R})$ by

$$Q_\sigma = \sigma(a_1 I + a_2 H) : H^2 = -I, \tag{1-4}$$

where $a_1, a_2 \in \mathbb{R}$, and H is not a unique matrix.

Also the Hamiltonian representation as in (1-1) and the quaternion representation as in (1-4) are extended as follow (Lee and Song, 2010; Farebrother et al., 2003; Song et al., 2014; Huang, 2000; Jafari et al., 2013; Zhang, 1997, 2011; Tian and Styan, 2005; Zeng, 2005): Let $\sigma : Q \to M_{2^n}(\mathbb{R})$ be a real linear representation on matrix quaternions Q defined by:

$$Q_\sigma = \sigma(a_1 I + a_2 H + a_3 J + a_4 K), \tag{1-5}$$

where $a_r \in \mathbb{R}$ $(r = 1, 2, 3, 4)$, I is an identity matrix of order $2^n \times 2^n$ and H, J, K are real matrices of order $2^n \times 2^n$ such that satisfying the following Hamilton conditions:

$$\begin{aligned} &H^2 = J^2 = K^2 = -I; \\ &HJ = K;\ JK = H;\ KH = J; \\ &JH = -K;\ KJ = -H;\ HK = -J. \end{aligned} \tag{1-6}$$

Note also that H, J and K are not unique real matrices and the Hamilton conditions as in (1-6) can be rearranged as in the following table:

Hamilton Conditions Table

$\times$	I	H	J	K
I	I	H	J	K
H	H	$-I$	K	$-J$
J	J	$-K$	$-I$	H
K	K	J	$-H$	$-I$

Finally, the Moore-Penrose inverse and Kronecker products of matrices as defined below, respectively, are playing a central role to obtain our results in the solutions of the linear quaternion systems.

(i) The Moore-Penrose inverse A^+ of a rectangular matrix A is defined to be satisfied the following equations (Wang, 1997; Kilicman and Al-Zhour, 2007, 2011)

$$AA^+A = A;\ A^+AA^+ = A^+;\ (AA^+)^T = AA^+;\ (A^+A)^T = A^+A. \tag{1-7}$$

Note that if A is a nonsingular square matrix, then $A^+ = A^{-1}$.

(i) The Kronecker product of $A = [a_{ij}]$ and $A = [b_{kl}]$ is defined by (Van Loan, 2000; Visick, 2000; Jódar and Abou-Kandil, 1989; Al-Zhour and Kilicman, 2007; Al-Zhour, 2012, 2014, 2015, 2016):

$$A \otimes B = [a_{ij}B]. \tag{1-8}$$

Note that the Kronecker product has the following nice properties (Kilicman and Al-Zhour, 2007; Al-Zhour and Kilicman, 2007; Al-Zhour, 2012, 2014, 2015):

$$\begin{aligned} &(A \otimes B)^+ = A^+ \otimes B^+; \\ &(A \otimes B)(C \otimes D) = AC \otimes BD; \\ &Vec(ABC^T) = (C \otimes A)VecB, \end{aligned} \tag{1-9}$$

where A B, C and D are matrices with compatible sizes.

One of the most important applications of quaternions, Kronecker products and the 2×2 Pauli spin matrices as in (3-1) later is the Hamiltonian of the 3D Ising model on a simple or thorhombic lattice which is written by (Zhang, 2013):

$$\begin{aligned} \hat{H} = &-J\sum_{\tau=1}^{n}\sum_{\rho=1}^{m}\sum_{\delta=1}^{l} s_\rho^{(\tau)}, \delta^s \rho^{(\tau+1)}, \delta \\ &-J'\sum_{\tau=1}^{n}\sum_{\rho=1}^{m}\sum_{\delta=1}^{l} s_\rho^{(\tau)}, \delta^s \rho^{(\tau)} + 1, \delta \\ &-J''\sum_{\tau=1}^{n}\sum_{\rho=1}^{m}\sum_{\delta=1}^{l} s_\rho^{(\tau)}, \delta^s \rho^{(\tau)}, \delta + 1. \end{aligned} \tag{1-10}$$

The partition function of (1-10) is given by Zhang (2007, 2013) as follows:

$$Z = Tr(T(V))^m = Tr(V_3V_2V_1)^m, \tag{1-11}$$

where the transfer matrices V_1, V_2, V_3 and all variables are given in equations (1)-(9) in Zhang (2013) works. Also another representation of a partition function of (1-10) is given by Zhang (2007, 2013) works as follows:

$$Z = (2\sin h2K)\frac{mnl}{2}Tr(V_3V_2V_1)^m = (2\sin h2K)\frac{mnl}{2}\sum_{i=1}^{2nl}\lambda_i^m, \quad (1\text{-}12)$$

where the transfer matrices V_1, V_2, V_3 and all variables are given in Eqs. (11)–(23) in Zhang (2013) works.

In the present paper, several new attractive and interested linear representations of matrix quaternions are constructed by using Kronecker structures which conclude to the general partitioned linear representation form of matrix quaternions. Furthermore, the general solutions of partitioned linear quaternion and linear matrix quaternion systems which includes the coupled Sylvester matrix quaternion equations are also presented by using our new effective approach.

2. Linear representations of matrix quaternions as in (1-4)

Since the matrix H (where $H^2 = -I$) as in (1-4) is not a unique, then in this Section we construct the 2-dimensional, 4-dimensional and 8 -dimensional matrix quaternions based on the Kronecker structure as in the following cases:

Case 1. Choose $H = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then it is easy to verify that $H^2 = -I_2$ and the real linear representation σ on matrix quaternions Q_2 is given by $\sigma : Q_2 \to M_2(\mathbb{R})$ and defined as follows:

$$Q_\sigma = \sigma(a_1I_2 + a_2H) = \begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix}. \quad (2\text{-}1)$$

Case 2. Consider the following matrices:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (2\text{-}2)$$

Here, $A^2 = -I_2$, $B^2 = I_2$ and the real linear representation σ on matrix quaternions Q_4 is given by $\sigma : Q_4 \to M_4(\mathbb{R})$ and defined as follows:

$$Q_\sigma = \sigma(a_1I_4 + a_2H) : H^2 = -I_4, \quad (2\text{-}3)$$

Now, we can generate the matrix quaternions of order 4×4 by using the Kronecker product as follow. Choose: $H = A \otimes B = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$, then $H^2 = A^2 \otimes B^2 = -I_4$. In this case,

$$Q_\sigma = \begin{bmatrix} a_1 & 0 & -a_2 & 0 \\ 0 & a_1 & 0 & a_2 \\ a_2 & 0 & a_1 & 0 \\ 0 & -a_2 & 0 & a_1 \end{bmatrix}. \quad (2\text{-}4)$$

Case 3. Consider the following matrices:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (2\text{-}5)$$

Here, $A^2 = -I_2$, $B^2 = C^2 = I_2$ and the real linear representation σ on matrix quaternions Q_8 is given by $\sigma : Q_8 \to M_8(\mathbb{R})$ which is defined as:

$$Q_\sigma = \sigma(a_1I_8 + a_2H) : H^2 = -I_8. \quad (2\text{-}6)$$

Now, we can generate the matrix quaternions of order 8×8 by using the Kronecker product in some different ways. For example,

(i) Choose:

$$H = A \otimes C \otimes B = \begin{bmatrix} 0&0&0&0&0&0&-1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&-1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&-1&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&-1&0&0&0&0&0&0 \end{bmatrix}, \quad \text{then}$$

$H^2 = A^2 \otimes C^2 \otimes B^2 = -I_8$. In this case,

$$Q_\sigma = \begin{bmatrix} a_1&0&0&0&0&0&-a_2&0 \\ 0&a_1&0&0&0&0&0&a_2 \\ 0&0&a_1&0&-a_2&0&0&0 \\ 0&0&0&a_1&0&a_2&0&0 \\ 0&0&a_2&0&a_1&0&0&0 \\ 0&0&0&-a_2&0&a_1&0&0 \\ a_2&0&0&0&0&0&a_1&0 \\ 0&-a_2&0&0&0&0&0&a_1 \end{bmatrix}. \quad (2\text{-}7)$$

(ii) Choose:

$$H = A \otimes B \otimes C = \begin{bmatrix} 0&0&0&0&0&-1&0&0 \\ 0&0&0&0&-1&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \\ 0&1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&-1&0&0&0&0 \\ 0&0&-1&0&0&0&0&0 \end{bmatrix}, \quad \text{then}$$

$H^2 = A^2 \otimes B^2 \otimes C^2 = -I_8$. In this case

$$Q_\sigma = \begin{bmatrix} a_1&0&0&0&0&-a_2&0&0 \\ 0&a_1&0&0&-a_2&0&0&0 \\ 0&0&a_1&0&0&0&0&a_2 \\ 0&0&0&a_1&0&0&a_2&0 \\ 0&a_2&0&0&a_1&0&0&0 \\ a_2&0&0&0&0&a_1&0&0 \\ 0&0&0&-a_2&0&0&a_1&0 \\ 0&0&-a_2&0&0&0&0&a_1 \end{bmatrix}. \quad (2\text{-}8)$$

(iii) Choose:

$$H = A \otimes A \otimes A = \begin{bmatrix} 0&0&0&0&0&0&0&-1 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&-1&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&-1&0&0&0&0&0 \\ 0&-1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \end{bmatrix}, \quad \text{then}$$

$H^2 = A^2 \otimes A^2 \otimes A^2 = -I_8$. In this case,

$$Q_\sigma = \begin{bmatrix} a_1&0&0&0&0&0&0&-a_2 \\ 0&a_1&0&0&0&0&a_2&0 \\ 0&0&a_1&0&0&a_2&0&0 \\ 0&0&0&a_1&-a_2&0&0&0 \\ 0&0&0&a_2&a_1&0&0&0 \\ 0&0&-a_2&0&0&a_1&0&0 \\ 0&-a_2&0&0&0&0&a_1&0 \\ a_2&0&0&0&0&0&0&a_1 \end{bmatrix}. \quad (2\text{-}9)$$

3. Linear representations of matrix quaternions as in (1-5)

Since H, J and K are not unique real matrices as in (1-5), then we discuss below some important cases for choosing these matrices such that the all Hamilton conditions as in (1-6) holds in order to get the 4-dimensional and 8-dimensional matrix quaternions from (1-5).Consider the following 2×2 Pauli spin matrices:

$$A=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},\quad B=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad C=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad D=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \tag{3-1}$$

Here, it is easy to verify that:

$$A^2=-I_2,\ B^2=C^2=D^2=I_2,$$
$$AD=DA=A,\ BD=DB=B,\ CD=DC=C,$$
$$AB=C,\ BA=-C,\ CA=B,\ AC=-B,\ CB=A,\ BC=-A.$$

Now by using the Kronecker products of A, B, C and D in some ways, then we can generate the imaginary parts H, J and $K \in M_4(\mathbb{R})$ in many different selections. For example, if we choose:

$$H=D\otimes A=\begin{bmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{bmatrix},\quad J=A\otimes B=\begin{bmatrix}0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{bmatrix},$$

$$K=HJ=A\otimes C=\begin{bmatrix}0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}.$$

It is easy to verify that the all Hamilton conditions are holds by:

$$H^2=D^2\otimes A^2=-I_4,$$
$$J^2=A^2\otimes B^2=-I_4,$$
$$K^2=A^2\otimes C^2=-I_4,$$
$$HJ=DA\otimes AB=A\otimes C=K,$$
$$JH=AD\otimes BA=A\otimes(-C)=-K,$$
$$JK=A^2\otimes BC=-(D)\otimes(-A)=H,$$
$$KJ=A^2\otimes CB=-(D)\otimes A=-H,$$
$$KH=AD\otimes CA=A\otimes B=J,$$
$$HK=DA\otimes AC=A\otimes(-B)=-J.$$

In this case,

$$Q_\sigma=\sigma(a_1I_4+a_2H+a_3J+a_4K)=\begin{bmatrix}a_1 & -a_2 & -a_3 & -a_4\\ a_2 & a_1 & -a_4 & a_3\\ a_3 & a_4 & a_1 & -a_2\\ a_4 & -a_3 & a_2 & a_1\end{bmatrix}. \tag{3-2}$$

Similarly, we can easily construct some possible cases for choosing the real matrices H,J and K which satisfying the all Hamilton conditions by rearranging the Kronecker products between 2×2 -matrices A, B, C and D as given in (3-1). Also based on the Kronecker products of A, B, C and D in (3-1), we can generate the imaginary parts H, J and $K \in M_8(\mathbb{R})$ in many different ways. For example,

(i) Choose: $H=A\otimes C\otimes D, J=D\otimes A\otimes C$ and $K=HJ=A\otimes B\otimes C$.
(ii) Choose: $H=A\otimes B\otimes D, J=B\otimes D\otimes A$ and $K=HJ=C\otimes B\otimes A$.

Note that H, J and K are matrices of order 8×8 with one nonzero element (1 or -1) in each row and one nonzero element (1 or -1) in each column and it is easy to check that the all Hamilton conditions as in (1-6) are holds and easy also to find the quaternion matrix $Q_\sigma=\sigma(a_1I_4+a_2H+a_3J+a_4K)\in M_8(\mathbb{R})$.

Note that we can obtain the higher dimensional quaternion matrices of order $2^n\times 2^n$ $(n=3,4,\ldots)$ by using m-fold Kronecker products of 2×2 matrices as same as chosen in (3-1).

4. General partitioned representations form of matrix quaternions

We note from the all above cases as in Sections 2 and 3 that any quaternion matrix $Q_\sigma \in M_{2^n}(\mathbb{R})$ can be rewritten as a partitioned quaternion matrix as follow:

$$Q_\sigma=\begin{bmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{bmatrix}: S_{11}=S_{22}, S_{12}=-S_{21}. \tag{4-1}$$

Now, we can construct the 8-dimensional matrix quaternions based on (4-1) and by letting T be a partitioned matrix of order 8×8 as follows:

$$T=\begin{bmatrix}T_{11} & T_{12} & T_{13} & T_{14}\\ T_{21} & T_{22} & T_{23} & T_{24}\\ T_{31} & T_{32} & T_{33} & T_{34}\\ T_{41} & T_{42} & T_{43} & T_{44}\end{bmatrix}=\begin{bmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{bmatrix}, \tag{4-2}$$

where each block of matrices T_{ij} is of order 2×2 and each block of matrices S_{kl} is of order 4×4.In fact,

$$S_{11}=\begin{bmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix},\quad S_{12}=\begin{bmatrix}T_{13} & T_{14}\\ T_{23} & T_{24}\end{bmatrix},\quad S_{21}=\begin{bmatrix}T_{31} & T_{32}\\ T_{41} & T_{42}\end{bmatrix},\quad S_{22}=\begin{bmatrix}T_{33} & T_{34}\\ T_{43} & T_{44}\end{bmatrix},$$

and $T_{11}=T_{22}, T_{14}=T_{23}, T_{41}=T_{32}, T_{33}=T_{44}, T_{21}=-T_{12}, T_{13}=-T_{24}, T_{31}=-T_{42}, T_{34}=-T_{43}$.

By this way, we can obtain the quaternion matrix $Q_\sigma \in M_8(\mathbb{R})$ as in the following form:

$$Q_\sigma=\begin{bmatrix}a_1 & -a_2 & -a_3 & -a_4 & -a_5 & a_6 & -a_7 & -a_8\\ a_2 & a_1 & -a_4 & a_3 & -a_6 & -a_5 & -a_8 & a_7\\ a_3 & a_4 & a_1 & -a_2 & -a_7 & -a_8 & a_5 & -a_6\\ a_4 & -a_3 & a_2 & a_1 & -a_8 & a_7 & a_6 & a_5\\ a_5 & -a_6 & a_7 & a_8 & a_1 & -a_2 & -a_3 & -a_4\\ a_6 & a_5 & a_8 & -a_7 & a_2 & a_1 & -a_4 & a_3\\ a_7 & a_8 & -a_5 & a_6 & a_3 & a_4 & a_1 & -a_2\\ a_8 & -a_7 & -a_6 & -a_5 & a_4 & -a_3 & a_2 & a_1\end{bmatrix}. \tag{4-3}$$

Similarly, we can extend this method as in (4-1) and (4-2) and depends on the quaternion matrix of order 8×8 as in (4-3) to obtain an quaternion matrix of order 16×16, 32×32, and so on to get the quaternion matrix of order $2^n\times 2^n$ $(n=4,5,\ldots)$.

5. Some applications of linear matrix quaternions

In this Section, we present the general vector solutions of the partitioned linear quaternion equations, general linear matrix quaternion system and coupled Sylvester matrix quaternion system by using Kronecker structure.

Lemma 5.1. *Let* $Q_\sigma = \begin{bmatrix} N & -M \\ M & N \end{bmatrix} \in M_{2^n}(\mathbb{R})$ *be a partitioned quaternion matrix and* $N, M \in M_{2^{n-1}}(\mathbb{R})$ *such that the all relevant inverses exist. Then* $Q_\sigma^{-1} = \begin{bmatrix} S_N^{-1} & N^{-1}MS_N^{-1} \\ -S_N^{-1}MN^{-1} & S_N^{-1} \end{bmatrix}$, where $S_N = N + MN^{-1}M$ is the Schur complement of N in Q_σ.

Lemma 5.2. *Let* $Q_\sigma = \begin{bmatrix} N & -M \\ M & N \end{bmatrix} \in M_{2^n}(\mathbb{R})$ *be a partitioned quaternion matrix such that* $r(Q_\sigma) = 2r(N)$, $R(N) = R(M)$ *and* $R(N^T) = R(M^T)$. *Then*

$$Q_\sigma^+ = \begin{bmatrix} S_N^+ & -S_M^+ \\ S_M^+ & S_N^+ \end{bmatrix}, \tag{5-1}$$

where

$$S_N^+ = (N + MN^+M^+)^+, \ S_M^+ = -(M + NM^+N^+)^+. \tag{5-2}$$

Theorem 5.1. *Let* $M, N \in M_{2^{n-1}}(\mathbb{R})$ *be given real full-rank matrices such that* $R(N) = R(M)$ *and* $R(N^T) = R(M^T)$, *and* $c, k \in M_{2^{n-1}\times 1}(\mathbb{R})$ *be given constant vectors, and* $x, y \in M_{2^{n-1}\times 1}(\mathbb{R})$ *be unknown vectors. Then the general solutions of the following linear quaternion system*:

$$Nx - My = c, \ Mx + Ny = k, \tag{5-3}$$

are given by:

$$x = c(N + MN^+M^+)^+ + k(M + NM^+N^+)^+,$$
$$y = k(N + MN^+M^+)^+ - c(M + NM^+N^+)^+. \tag{5-4}$$

Proof. The system as in (5-3) can be rewritten in matrix form as follows:

$$Q_\sigma z = r, \tag{5-5}$$

where $Q_\sigma = \begin{bmatrix} N & -M \\ M & N \end{bmatrix}$ is a quaternion matrix of order $2^n \times 2^n$, $r = \begin{bmatrix} c \\ k \end{bmatrix}$ is a constant vector of order $2^n \times 1$ and $z = \begin{bmatrix} x \\ y \end{bmatrix}$ is a vector of order $2^n \times 1$ to be solved. Now the system as in (5-5) is a partitioned linear matrix quaternion and satisfies the all assumptions of Lemma 5.2, then the unique solutions of quaternion system (5-3) is given by:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} N & -M \\ M & N \end{bmatrix}^+ \begin{bmatrix} c \\ k \end{bmatrix} = \begin{bmatrix} S_N^+ & -S_M^+ \\ S_M^+ & S_N^+ \end{bmatrix} \begin{bmatrix} c \\ k \end{bmatrix}. \tag{5-6}$$

Hence, it is easy from (5-6) to obtain the general solution x and y as in (5-4). □

Theorem 5.2. *Let* $A, B, C, D \in M_n$ *be given real full-rank matrices such that* $R(B \otimes A) = R(D \otimes C)$ *and* $R(B^T \otimes A^T) = R(D^T \otimes C^T)$, *and* $G, H \in M_n$ *be given constant matrices, and* $X, Y \in M_n$ *be unknown matrices. Then the general vector solutions of the following general linear matrix quaternion system*:

$$AXB^T - CYD^T = G, \ CXD^T + AYB^T = H, \tag{5-7}$$

are given by:

$$VecX = (B \otimes A + DB^+D^+ \otimes CA^+C^+)^+ VecG$$
$$+ (D \otimes C + AD^+A^+ \otimes BC^+B^+)^+ VecH,$$
$$VecY = -(D \otimes C + AD^+A^+ \otimes BC^+B^+)^+ VecG$$
$$+ (B \otimes A + DB^+D^+ \otimes CA^+C^+)^+ VecH. \tag{5-8}$$

Proof. By taking the vector operator for both sides of the system as in (5-7) and based on the properties of Kronecker product as in (1-9), then (5-7) can be rewritten in matrix form as follows:

$$Q_\sigma w = d, \tag{5-9}$$

where $Q_\sigma = \begin{bmatrix} N & -M \\ M & N \end{bmatrix} = \begin{bmatrix} B \otimes A & -D \otimes C \\ D \otimes C & B \otimes A \end{bmatrix}$ is a quaternion partitioned matrix, $d = \begin{bmatrix} VecG \\ VecH \end{bmatrix}$ is a constant vector and $w = \begin{bmatrix} VecX \\ VecY \end{bmatrix}$ is a vector to be solved.

Now the system as in (5-7) is a partitioned linear quaternion and satisfies the all assumptions of Lemma 5.2, then the unique solutions of quaternion system as in (5-7) is given by:

$$\begin{bmatrix} VecX \\ VecY \end{bmatrix} = \begin{bmatrix} B \otimes A & -D \otimes C \\ D \otimes C & B \otimes A \end{bmatrix}^+ \begin{bmatrix} VecG \\ VecH \end{bmatrix} = \begin{bmatrix} S_{B\otimes A}^+ & -S_{D\otimes C}^+ \\ S_{D\otimes C}^+ & S_{B\otimes A}^+ \end{bmatrix} \begin{bmatrix} VecG \\ VecH \end{bmatrix}$$
$$= \begin{bmatrix} (B \otimes A + DB^+D^+ \otimes CA^+C^+)^+ & (D \otimes C + AD^+A^+ \otimes BC^+B^+)^+ \\ -(D \otimes C + AD^+A^+ \otimes BC^+B^+)^+ & (B \otimes A + DB^+D^+ \otimes CA^+C^+)^+ \end{bmatrix}$$
$$\times \begin{bmatrix} VecG \\ VecH \end{bmatrix}. \tag{5-10}$$

Hence, it is easy by simple computations of (5-10) to obtain the general vector solutions of X and Y as in (5-8). □

One of the most important cases can be obtained from (5-7) is the following coupled Sylvester matrix quaternion equations:

$$AX - YB^T = G, \ XB^T + AY = H, \tag{5-11}$$

where $A, B \in M_n$ are given real full-rank matrices, $G, H \in M_n$ are given constant matrices, and $X, Y \in M_n$ are unknown matrices to be solved as in next result.

Corollary 5.1. *Let* $A, B \in M_n$ *be given real full-rank matrices such that* $R(I_n \otimes A) = R(B \otimes I_N)$ *and* $R(I_n \otimes A^T) = R(B^T \otimes I_n)$ *and let* $G, H \in M_n$ *be given constant matrices and* $X, Y \in M_n$ *be unknown matrices. The general vector solutions of the coupled Sylvester matrix quaternion equations as in* (5-11) *are given by*:

$$VecX = (I_n \otimes A + BB^+ \otimes A^+)^+ VecG$$
$$+ (B \otimes I_n + B^+ \otimes AA^+)^+ VecH,$$
$$VecY = -(B \otimes I_n + B^+ \otimes AA^+)^+ VecG$$
$$+ (I_n \otimes A + BB^+ \otimes A^+)^+ VecH. \tag{5-12}$$

Proof. Similarly by the same technique as in the proof of Theorem 5.2, then (5-11) can be rewritten in matrix form as follows:

$$Q_\sigma t = e, \tag{5-13}$$

where $Q_\sigma = \begin{bmatrix} N & -M \\ M & N \end{bmatrix} = \begin{bmatrix} I_n \otimes A & -B \otimes I_n \\ B \otimes I_n & I_n \otimes A \end{bmatrix}$ is a quaternion partitioned matrix, $e = \begin{bmatrix} VecG \\ VecH \end{bmatrix}$ is a constant vector and $t = \begin{bmatrix} VecX \\ VecY \end{bmatrix}$ is a vector to be solved. Applying Lemma 5.2, then the unique solutions of quaternion system as in (5-11) is given by:

$$\begin{bmatrix} VecX \\ VecY \end{bmatrix} = \begin{bmatrix} I_n \otimes A & -B \otimes I_n \\ B \otimes I_n & I_n \otimes A \end{bmatrix}^+ \begin{bmatrix} VecG \\ VecH \end{bmatrix} = \begin{bmatrix} S^+_{I_n \otimes A} & -S^+_{B \otimes I_n} \\ S^+_{B \otimes I_n} & S^+_{I_n \otimes A} \end{bmatrix} \begin{bmatrix} VecG \\ VecH \end{bmatrix}$$
$$= \begin{bmatrix} (I_n \otimes A + BB^+ \otimes A^+)^+ & (B \otimes I_n + B^+ \otimes AA^+)^+ \\ -(B \otimes I_n + B^+ \otimes AA^+)^+ & (I_n \otimes A + BB^+ \otimes A^+)^+ \end{bmatrix} \begin{bmatrix} VecG \\ VecH \end{bmatrix}. \tag{5-14}$$

Hence it is easy to obtain the general vector solutions of X and Y as in (5-12) by using simple computations of (5-14). □

Remark 5.1. If the all relevant inverse of submatrices N and M exists in the partitioned linear quaternion matrix $Q_\sigma = \begin{bmatrix} N & -M \\ M & N \end{bmatrix}$ that obtained in Theorems 5.1 and 5.2 and in Corollary 5.1. Then we can easily get the general (vector) solutions of linear (matrix) quaternion systems as in (5-3), (5-7) and (5-11) by using the same procedures as in the proofs of Theorems 5.1 and 5.2 and Corollary 5.1 together with using Lemma 5.1.

6. Conclusion

Several new attractive and interested linear representations of matrix quaternions are constructed and obtained as in Sections 2,3 and 4 by using Kronecker structures which conclude to the general linear representation form of matrix quaternions as in Section 5. Moreover, the general solutions of partitioned linear quaternion and linear matrix quaternion systems which includes the coupled Sylvester matrix quaternion system are also presented by using a new approach. How to extend the use of our new method to find the general solutions of such linear matrix quaternion systems of several variables as in (Wang et al., 2016; Nie et al., 2017; Rehman and Wang, 2015); and also how to apply our new method for dealing the partition functions for the Hamiltonian of the 3D Ising model as in (1-11) and (1-12) which is given in details in (Zhang, 2013; Lawrynowicz et al., 2010) still require further researches.

Acknowledgements

The author expresses his sincere thanks to referees for careful reading of the manuscript and several helpful suggestions.

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