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Soliton solutions of (3 + 1)-dimensional Korteweg-de Vries Benjamin–Bona–Mahony, Kadomtsev–Petviashvili Benjamin–Bona–Mahony and modified Korteweg de Vries–Zakharov–Kuznetsov equations and their applications in water waves

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ABSTRACT

In this article, the analytical solution of (3 + 1)-dimensional Korteweg-de Vries Benjamin–Bona–Mahony equation, Kadomtsev–Petviashvili Benjamin–Bona–Mahony equation and modified Korteweg-de Vries–Zakharov–Kuznetsov equation have been extracted. These results hold numerous traveling wave solutions that are of key importance in elucidating some physical circumstance. The technique can also be functional to other sorts of nonlinear evolution equations in contemporary areas of research.

1. Introduction

The propagation of nonlinear wave is one of the key phenomenon of nature and a growing interest has been drawn to the study of nonlinear waves in the dynamical system. The nonlinear equations have plenty of applications in sciences and engineering like electrochemistry, electromagnetic theory, fluid dynamics, acoustics, cosmology, astrophysics and plasma physics etc., see for references (Eslami, 2015, 2016a,b; Helal and Seadawy, 2009; Helal and Seadawy, 2011; Seadawy, 2015).

In the last few eras great improvement have been made in the progress of methods for finding the exact solutions of nonlinear equations but the advancement achieved is inadequate. Taking into account the merits and demerits of analytic methods, it is observed that there is no single outstanding preferable method which can be applied to any kind of nonlinear problems to obtain exact solutions. Consequently, it is apprehended that all of these methods are problem dependent, viz. some approaches work well with certain problems but not the others. Therefore, it is rather substantial to relate some established techniques in the literature to nonlinear partial differential equations, for details see also (Zabusky, 1967; Zhu, 1996; Seadawy, 2012; Seadawy and Sayed, 2013; Johnson, 1997).

The dynamics of shallow water waves is an important area of research in oceanography. There are several models that describe this kind of dynamics. A few of them are the Korteweg-de Vries (KdV) equation (Gardner et al., 1967), Korteweg-de Vries Burgers (KdV-B) equation (Zhibin and Mingliang, 1993), modified KdV (mKdV) equation (Ito, 1980), modified Korteweg-de Vries Zakharov-Kuznetsov (mKdv-ZK) equation, Boussinesq equation (Wang, 1995), Perergrine equation (Triki et al., 2010), Kawahara equation (Wazwaz, 2007a,b), Benjamin–Bona–Mahoney equation (Seadawy and Sayed, 2014), coupled Boussinesq equation (Mohapatra and Soares, 2015) and many others. Another model that is also considered and studied at times is the Gardner equation (GE) it is a combination of KdV and mKdV equation (Li and Wang, 2007). Therefore, occasionally, GE is referred to as the KdVmKdV equation, Kadomtsev–Petviashvili (KP) equation (Yong et al.,

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2003) and Gardner-Kadomtsev–Petviashvili (G-KP) equation (Yan et al., 2012).

Numerous dominant approaches have been offered, such as, Cole-Hopf transformation, Painleve method, Backlund transformation, sine–cosine method, Darboux transformation, Hirota method, Lie group analysis, homogeneous balance method (HBM), similarity reduced method, tanh method and so on, for details see also (Ma, 2011; Gai et al., 2012; Dutykha and Pelinovsky, 2014; Seadawy, 2014, 2016a,b, 2017; Seadawy and El-Rashidy, 2016).

The Korteweg-de Vries equation within the scope of the local fractional derivative formulation was investigated. The exact traveling wave solutions of non-differentiable type with the generalized functions defined on Cantor sets were analyzed (Yang et al., 2016). A family of local fractional two-dimensional Burgers-type equations was investigated. The local fractional Riccati differential equation method was proposed here for the first time. The travelling wave transformation of the non-differentiable type was presented. The non-differentiable exact travelling wave solutions for the problems were obtained (Yang et al., 2017).

Recently, there has been a growing interest in finding exact analytical solutions to nonlinear wave equations by using appropriate techniques. The investigation of exact traveling wave solutions for nonlinear partial differential equations (NPDEs) plays an important role in studying nonlinear physical phenomena (Xu and Li, 2005). These exact solutions can help better understand the mechanism of the complicated physical phenomena and dynamical processes modeled by nonlinear evolution equations (El-Wakil et al., 2006). Meanwhile, many powerful methods have been established and developed to construct exact solutions of NPDEs, leading to one of the most exciting advances in nonlinear science and theoretical physics. In fact, many kinds of exact soliton solutions have been obtained by using the inverse scattering method (Ablowitz and Clarkson, 1991), Hirota's bilinear method (Hirota, 1971), the homogeneous balance method (Hu, 2005), variational method (Helal and Seadawy, 2009; Seadawy, 2011), algebraic method (Hu, 2005), sine–cosine method (Wazwaz, 2007a,b), the Jacobi elliptic function method, the F-expansion method, the (G'/G) expansion method, the tanh and extended tanh method, conferred traveling wave solutions, including periodic traveling wave solutions, and rational solutions, the exact traveling wave solutions and their bifurcations obtained the solutions by using the new generalized transformation in HBM and so on (Kutluay et al., 2010; Abazari, 2010; Liu et al., 2001; Zhou et al., 2003).

The analytical solution of (3 + 1)-dimensional Korteweg-de Vries Benjamin–Bona–Mahony (KdV-BBM), Kadomtsev–Petviashvili Benjamin–Bona–Mahony (KP-BBM) and modified Kortewegde Vries–Zakharov–Kuznetsov (mKdv-ZK) equations are considered, the governing equations are as follows:

$$u_{tx} + \mu_1(uu_x)_x + \mu_2 u_{xxxx} - \mu_3 u_{xxtx} + \mu_4 u_{yy} - \mu_5 u_{zz} = 0, \tag{1.1}$$

$$u_{tx} + \mu_1 u_{xx} + \mu_2(uu_x)_x - \mu_3 u_{xxtx} + \mu_4 u_{yy} + \mu_5 u_{zz} = 0, \tag{1.2}$$

$$u_t + \mu_1 u^2 u_x + \mu_2 u_{xxx} + \mu_3(u_{yy} + u_{zz})_x = 0, \tag{1.3}$$

where the coefficients μ_i for $i = 1, 2, 3, 4, 5$; are real constants.

This article has been devised as follows: in Section 2, the auxiliary equation method is introduced, while in Section 3, the solutions of three nonlinear PDEs have been presented. In last Section 4, the conclusions have been drawn.

2. The description of the auxiliary equation method

We will briefly present the main steps of the AEM that will be applied to the non-linear equations (1.1)–(1.3), as in the following steps:

Step 1. Let us have a general form of nonlinear PDE

$$F(u, u_t, u_x, u_y, u_{xx}, u_{yy}, \ldots) = 0. \tag{2.4}$$

where F is a polynomial function with respect to the indicated variables.

Step 2. The following wave variable is presented to find the traveling wave solutions of the system (2.4)

$$u(x, y, t) = F(\xi), \tag{2.5}$$

The transformations (2.5) convert the PDE (2.4) to an ODE

$$O_i(F, F_\xi, F_{\xi\xi}, F_{\xi\xi\xi}, \ldots), \tag{2.6}$$

where $F = F(\xi)$ is unknown function.

Step 3. The main idea of the auxiliary equation method based on expanding the traveling wave solution $F(\xi)$ of Eqs. (2.6) as a finite series

$$F(\xi) = \sum_{i=0}^{n} e^{it}\psi^i(\xi), \tag{2.7}$$

ψ satisfies

$$\frac{d\psi}{d\xi} = c_0 + c_1\psi(\xi) + c_2\psi^2(\xi) + c_3\psi^3(\xi) + c_4\psi^4(\xi), \tag{2.8}$$

$$\xi = x + y + y - \omega t \tag{2.9}$$

where $c_i (i = 0, 1, 2, 3, 4)$ are constants.

Step 4. Balancing the highest order derivative term and the highest order nonlinear term of Eq. (2.4) with homogeneous balance method, the parameters n in (2.7) can be determined.

Step 5. Substituting (2.7)–(2.9) in (2.4) and collecting the coefficients of $\psi^j\psi^{(k)}$, then setting coefficients equal zero, we will obtain a set of over-determined equations for ω and c_i. By solving the system, we may determine these parameters.

Step 6. Substituting ωc_i and $\psi(\xi)$ obtained in step 5 into (2.7), to obtain the solution of Eq. 1.1,1.2.1.3.

3. Applications of the method

3.1. (3 + 1)-D KdV- BMM equation

Consider the transformation

$$u(x, y, z, t) = u(\xi), \xi = x + y + z - \omega t, \tag{3.10}$$

using (3.10) into (1.1),

$$(\mu_4 + \mu_5 - \omega)u'' + \mu_1(u'^2 + uu'') + (\mu_2 + \omega\mu_3)u'''' = 0 \tag{3.11}$$

integrating

$$(\mu_4 + \mu_5 - \omega)u' + \mu_1 uu' + (\mu_2 + \omega\mu_3)u''' = 0 \tag{3.12}$$

consider the homogeneous balance between uu' and u''', gives $n = 2$. Suppose the solution of (3.12), is of the form

$$u = 1 + e^t\psi(\xi) + e^{2t}\psi^2(\xi) \tag{3.13}$$

Substituting (2.7), (2.8) and (3.13) in (3.12) and collecting the coefficients of $\psi^j\psi^{(k)}$, Mathematica 10.4 is used to carry out symbolic computations.

Case I.

$$\psi_1(\xi) = -\frac{e^{-t}}{4}\left(2 + \sqrt{\frac{6(3\mu_1 + 4\mu_4 + 4\mu_5 - 4\omega)}{\mu_1}}\tan\xi\theta\right) \tag{3.14}$$

where

$$\theta=\sqrt{\frac{(\mu_2+\omega\mu_3)(\mu_1+\mu_4+\mu_5-\omega)(3\mu_1+4\mu_4+4\mu_5-4\omega)}{32(\mu_2+\omega\mu_3)^2(\omega-\mu_1-\mu_4-\mu_5)}} \tag{3.15}$$

The parameters C_i become

$$C_0=\frac{e^{-t}(11\mu_1+12(\mu_4+\mu_5-\omega))\sqrt{\omega-\mu_1-\mu_4-\mu_5}}{16\sqrt{3\mu_1(\mu_2+\omega\mu_3)(\mu_1+\mu_4+\mu_5-\omega)}} \tag{3.16}$$

$$C_1=\sqrt{\frac{\mu_1(\omega-\mu_1-\mu_4-\mu_5)}{12(\mu_2+\omega\mu_3)(\mu_1+\mu_4+\mu_5-\omega)}} \tag{3.17}$$

$$C_2=e^t\sqrt{\frac{\mu_1(\omega-\mu_1-\mu_4-\mu_5)}{12(\mu_2+\omega\mu_3)(\mu_1+\mu_4+\mu_5-\omega)}} \tag{3.18}$$

$$C_3=C_4=0 \tag{3.19}$$

so the solution of (1.1) will be

$$u_1=-\frac{147}{4096}\sec^4\frac{\sqrt{7}}{8}\times\xi\left(32-16\cos\frac{\sqrt{7}}{4}\xi+\sec^2\frac{\sqrt{7}}{8}\xi\left(111\cos\frac{\sqrt{7}}{4}\xi-\frac{27}{2}\times\cos\frac{\sqrt{7}}{2}\xi-\frac{171}{2}\right)+\frac{7}{2}\sec^2\frac{\sqrt{7}}{8}\xi\left(33-26\cos\frac{\sqrt{7}}{4}\xi+\cos\frac{\sqrt{7}}{2}\xi\right)\right) \tag{3.20}$$

Case II.

$$\psi_2(\xi)=\frac{e^{-t}}{4}\left(-2+\sqrt{\frac{6(3\mu_1+4\mu_4+4\mu_5-4\omega)}{\mu_1}}\tan\xi\theta\right) \tag{3.21}$$

where

$$\theta=\sqrt{\frac{(\mu_2+\omega\mu_3)(\mu_1+\mu_4+\mu_5-\omega)(3\mu_1+4\mu_4+4\mu_5-4\omega)}{32(\mu_2+\omega\mu_3)^2(\omega-\mu_1-\mu_4-\mu_5)}} \tag{3.22}$$

The parameters C_i become

$$C_0=-\frac{e^{-t}(11\mu_1+12(\mu_4+\mu_5-\omega))\sqrt{\omega-\mu_1-\mu_4-\mu_5}}{16\sqrt{3\mu_1(\mu_2+\omega\mu_3)(\mu_1+\mu_4+\mu_5-\omega)}} \tag{3.23}$$

$$C_1=-\sqrt{\frac{\mu_1(\omega-\mu_1-\mu_4-\mu_5)}{12(\mu_2+\omega\mu_3)(\mu_1+\mu_4+\mu_5-\omega)}} \tag{3.24}$$

$$C_2=-e^t\sqrt{\frac{\mu_1(\omega-\mu_1-\mu_4-\mu_5)}{12(\mu_2+\omega\mu_3)(\mu_1+\mu_4+\mu_5-\omega)}} \tag{3.25}$$

$$C_3=C_4=0 \tag{3.26}$$

so the solution of (1.1) will be

$$u_2=-\frac{147}{4096}\sec^4\frac{\sqrt{7}}{8}\times\xi\left(32-16\cos\frac{\sqrt{7}}{4}\xi+\sec^2\frac{\sqrt{7}}{8}\xi\left(111\cos\frac{\sqrt{7}}{4}\xi-\frac{27}{2}\cos\frac{\sqrt{7}}{2}\xi-\frac{171}{2}\right)+\frac{7}{2}\sec^2\frac{\sqrt{7}}{8}\xi\left(33-26\cos\frac{\sqrt{7}}{4}\xi+\cos\frac{\sqrt{7}}{2}\xi\right)\right) \tag{3.27}$$

3.2. (3 + 1)-D KP-BBM equation

Consider the transformation

$$u(x,y,z,t)=u(\xi),\ \xi=x+y+z-\omega t, \tag{3.28}$$

using (3.28) into (1.2),

$$(\mu_1+\mu_4+\mu_5-\omega^2)u''+\mu_2(u'^2+uu'')+\mu_3u''''=0 \tag{3.29}$$

integrating

$$(\mu_1+\mu_4+\mu_5-\omega^2)u'+\mu_2(uu')+\mu_3u'''=0 \tag{3.30}$$

consider the homogeneous balance between uu' and u''', gives $n=2$. Suppose the solution of (3.30), is of the form

$$u=1+e^t\psi(\xi)+e^{2t}\psi^2(\xi) \tag{3.31}$$

Substituting (2.7), (2.8) and (3.31) in (3.30) and collecting the coefficients of $\psi^j\psi^{(k)}$, Mathematica 10.4 is used to carry out symbolic computations.

Case I.

$$\psi_1(\xi)=\frac{e^{-t}}{4}\left(-2-\sqrt{\frac{6(4\mu_1+3\mu_2+4\mu_4+4\mu_5-4\omega)}{\mu_2}}\tan\xi\theta\right) \tag{3.32}$$

where

$$\theta=\sqrt{\frac{\omega(\mu_1+\mu_2+\mu_4+\mu_5-\omega)(4\mu_1+3\mu_2+4\mu_4+4\mu_5-4\omega)}{32\omega^2(\omega-\mu_1-\mu_2-\mu_4-\mu_5)}} \tag{3.33}$$

The parameters C_i become

$$C_0=\frac{e^{-t}(12\mu_1+11\mu_2+12(\mu_4+\mu_5-\omega))\sqrt{\omega-\mu_1-\mu_2-\mu_4-\mu_5}}{16\sqrt{3\mu_2\omega(\mu_1+\mu_2+\mu_4+\mu_5-\omega)}} \tag{3.34}$$

$$C_1=\sqrt{\frac{\mu_2(\omega-\mu_1-\mu_2-\mu_4-\mu_5)}{12\omega(\mu_1+\mu_2+\mu_4+\mu_5-\omega)}} \tag{3.35}$$

$$C_2=e^t\sqrt{\frac{\mu_2(\omega-\mu_1-\mu_2-\mu_4-\mu_5)}{12\omega(\mu_1+\mu_2+\mu_4+\mu_5-\omega)}} \tag{3.36}$$

$$C_3=C_4=0 \tag{3.37}$$

so the solution of (1.2) will be

$$u_1=-\frac{363}{2048}\sec^4\sqrt{\frac{11}{32}}\xi\left(64-32\cos\sqrt{\frac{11}{8}}\xi+\sec^2\sqrt{\frac{11}{32}}\xi\left(171\cos\sqrt{\frac{11}{8}}\xi-\frac{39}{2}\cos\sqrt{\frac{11}{2}}\xi-\frac{279}{2}\right)+\frac{11}{2}\sec^2\sqrt{\frac{11}{32}}\xi\left(33-26\cos\sqrt{\frac{11}{8}}\xi+\cos\sqrt{\frac{11}{2}}\xi\right)\right) \tag{3.38}$$

Case II.

$$\psi_2(\xi)=\frac{e^{-t}}{4}\left(-2+\sqrt{\frac{6(4\mu_1+3\mu_2+4\mu_4+4\mu_5-4\omega)}{\mu_2}}\tan\xi\theta\right) \tag{3.39}$$

where

$$\theta=\sqrt{\frac{\omega(\mu_1+\mu_2+\mu_4+\mu_5-\omega)(4\mu_1+3\mu_2+4\mu_4+4\mu_5-4\omega)}{32\omega^2(\omega-\mu_1-\mu_2-\mu_4-\mu_5)}} \tag{3.40}$$

The parameters C_i become

$$C_0 = -\frac{e^{-t}(12\mu_1 + 11\mu_2 + 12(\mu_4 + \mu_5 - \omega))\sqrt{\omega - \mu_1 - \mu_2 - \mu_4 - \mu_5}}{16\sqrt{3\mu_2\omega(\mu_1 + \mu_2 + \mu_4 + \mu_5 - \omega)}} \quad (3.41)$$

$$C_1 = -\sqrt{\frac{\mu_2(\omega - \mu_1 - \mu_2 - \mu_4 - \mu_5)}{12\omega(\mu_1 + \mu_2 + \mu_4 + \mu_5 - \omega)}} \quad (3.42)$$

$$C_2 = -e^{t}\sqrt{\frac{\mu_2(\omega - \mu_1 - \mu_2 - \mu_4 - \mu_5)}{12\omega(\mu_1 + \mu_2 + \mu_4 + \mu_5 - \omega)}} \quad (3.43)$$

$$C_3 = C_4 = 0 \quad (3.44)$$

so the solution of (1.2) will be

$$u_2 = -\frac{363}{2048}\sec^4\sqrt{\frac{11}{32}}\xi\left(64 - 32\cos\sqrt{\frac{11}{8}}\xi + \sec^2\sqrt{\frac{11}{32}}\xi\left(171\cos\sqrt{\frac{11}{8}}\xi - \frac{39}{2}\cos\sqrt{\frac{11}{2}}\xi - \frac{279}{2}\right) + \frac{11}{2}\sec^2\sqrt{\frac{11}{32}}\xi\left(33 - 26\cos\sqrt{\frac{11}{8}}\xi + \cos\sqrt{\frac{11}{2}}\xi\right)\right) \quad (3.45)$$

3.3. (3 + 1)-D mKdv-ZK equation

Consider the transformation

$$u(x,y,z,t) = u(\xi),\ \xi = x + y + z - \omega t, \quad (3.46)$$

using (3.46) into (1.3),

$$-\omega u' + \mu_1 u^2 u' + (\mu_2 + 2\mu_3)u''' = 0 \quad (3.47)$$

integrating

$$-\omega u + \mu_1\frac{u^3}{3} + (\mu_2 + 2\mu_3)u'' = 0 \quad (3.48)$$

consider the homogeneous balance between u^3 and u'', gives $n = 1$. Suppose the solution of (3.48), is of the form

$$u = 1 + e^t\psi(\xi) \quad (3.49)$$

Substituting (2.7), (2.8) and (3.49) in (3.48) and collecting the coefficients of $\psi^j\psi^{(k)}$, Mathematica 10.4 is used to carry out symbolic computations.

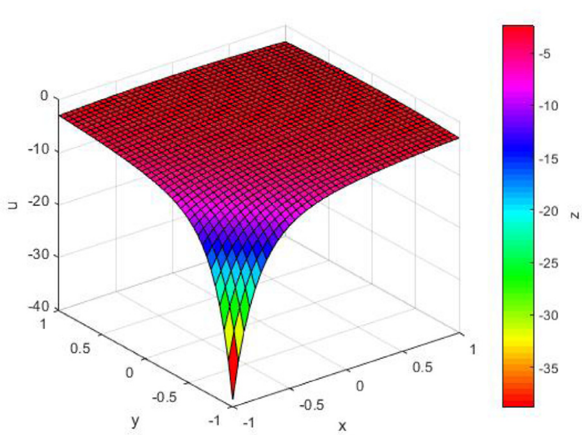
Case I.

$$\psi_1(\xi) = -e^{-t}\left(1 + \sqrt{\frac{3\omega}{\mu_1}}\tan\xi\theta\right) \quad (3.50)$$

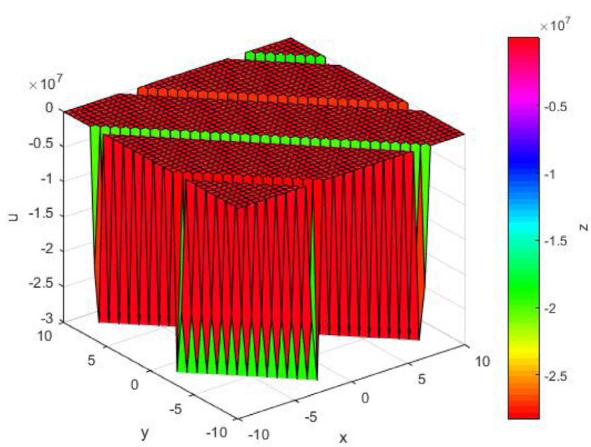
where

$$\theta = \frac{\sqrt{\omega(-\mu_2 - 2\mu_3)}}{\sqrt{2}(\mu_2 + 2\mu_3)} \quad (3.51)$$

The parameters C_i become

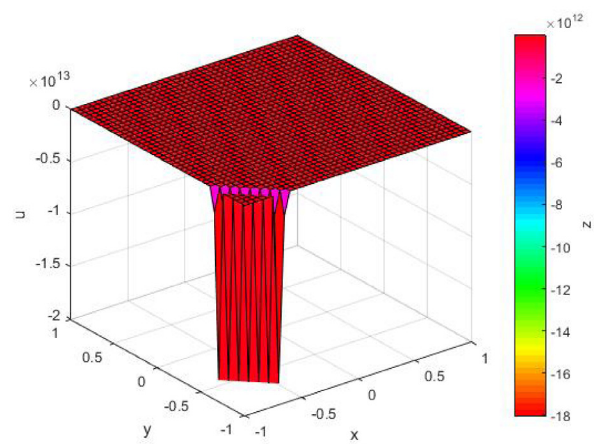


(a) Case I: $\mu_1=1,\ \mu_2=1,\ \mu_3=1,\ \mu_4=1,\ \mu_5=1,\ \omega=1,\ t=1$

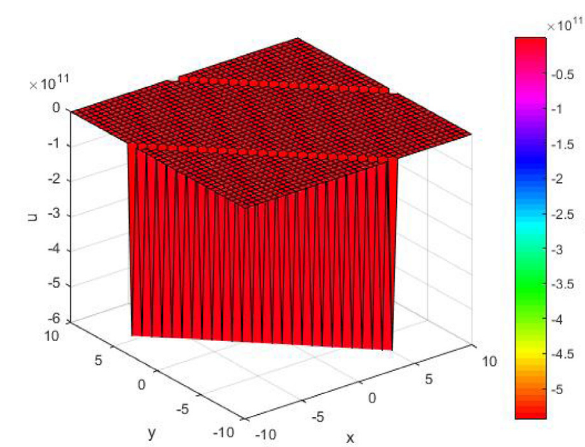


(b) Case II: $\mu_1=1,\ \mu_2=1,\ \mu_3=1,\ \mu_4=1,\ \mu_5=1,\ \omega=1,\ t=1$

Fig. 1. (3 + 1)-D KdV-BBM equation.

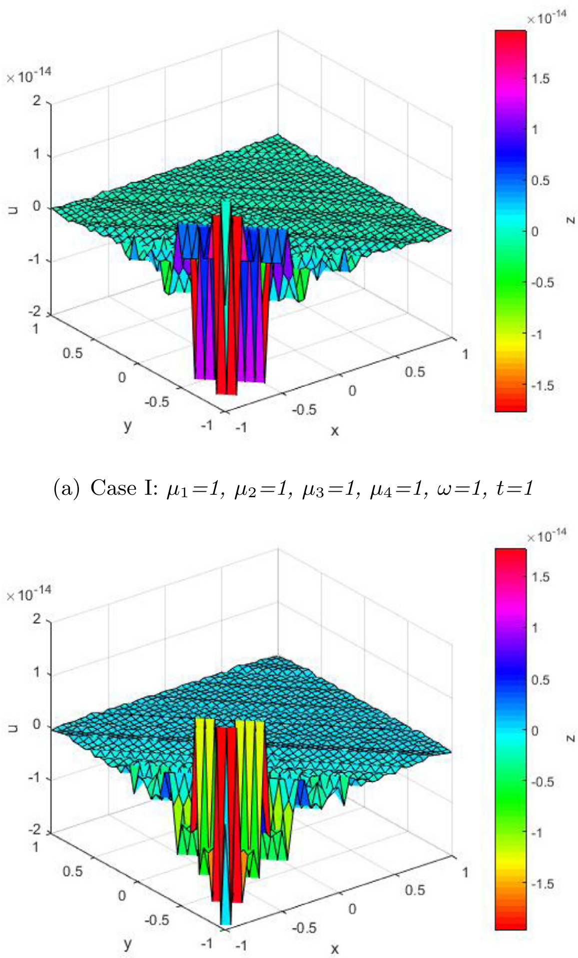


(a) Case I: $\mu_1=1,\ \mu_2=1,\ \mu_3=1,\ \mu_4=1,\ \mu_5=1,\ \omega=1,\ t=1$



(b) Case II: $\mu_1=1,\ \mu_2=1,\ \mu_3=1,\ \mu_4=1,\ \mu_5=1,\ \omega=1,\ t=1$

Fig. 2. (3 + 1)-D KP-BBM equation.



(a) Case I: $\mu_1=1,\ \mu_2=1,\ \mu_3=1,\ \mu_4=1,\ \omega=1,\ t=1$

(b) Case II: $\mu_1=1,\ \mu_2=1,\ \mu_3=1,\ \mu_4=1,\ \omega=1,\ t=1$

Fig. 3. (3 + 1)-D KdV-ZK equation.

$$C_0 = -\frac{e^{-t}(\mu_1 - 3\omega)}{\sqrt{6\mu_1(-\mu_2 - 2\mu_3)}} \tag{3.52}$$

$$C_1 = -\sqrt{\frac{2\mu_1}{3(-\mu_2 - 2\mu_3)}} \tag{3.53}$$

$$C_2 = -e^{-t}\sqrt{\frac{\mu_1}{6(-\mu_2 - 2\mu_3)}} \tag{3.54}$$

$$C_3 = C_4 = 0 \tag{3.55}$$

so the solution of (1.3) will be

$$u_1 = \frac{1}{\sqrt{2}}\sec^2\frac{\xi}{\sqrt{6}}\left(1 - \sec^2\frac{\xi}{\sqrt{6}}\left(2 - \cos\sqrt{\frac{2}{3}}\xi\right) + 3\tan^2\frac{\xi}{\sqrt{6}}\right) \tag{3.56}$$

Case II.

$$\psi_2(\xi) = e^{-t}\left(-1 + \sqrt{\frac{3\omega}{\mu_1}}\tan\xi\theta\right) \tag{3.57}$$

where

$$\theta = \frac{\sqrt{\omega(-\mu_2 - 2\mu_3)}}{\sqrt{2}(\mu_2 + 2\mu_3)} \tag{3.58}$$

The parameters C_i become

$$C_0 = \frac{e^{-t}(\mu_1 - 3\omega)}{\sqrt{6\mu_1(-\mu_2 - 2\mu_3)}} \tag{3.59}$$

$$C_1 = \sqrt{\frac{2\mu_1}{3(-\mu_2 - 2\mu_3)}} \tag{3.60}$$

$$C_2 = e^{-t}\sqrt{\frac{\mu_1}{6(-\mu_2 - 2\mu_3)}} \tag{3.61}$$

$$C_3 = C_4 = 0 \tag{3.62}$$

so the solution of (1.3) will be

$$u_2 = \frac{1}{\sqrt{2}}\sec^2 \times \frac{\xi}{\sqrt{6}}\left(-1 + \sec^2\frac{\xi}{\sqrt{6}}\left(2 - \cos\sqrt{\frac{2}{3}}\xi\right) - 3\tan^2\frac{\xi}{\sqrt{6}}\right) \tag{3.63}$$

4. Conclusion

In this article, the analytical solutions to (3 + 1)-dimensional Korteweg-de Vries Benjamin–Bona–Mahony equation, Kadomtsev–Petviashvili Benjamin–Bona–Mahony equation and modified Kortewegde Vries–Zakharov–Kuznetsov equations have been extracted with the help of the auxiliary equation method. These results are very auspicious for further investigation and stances on a strong basis for the solution of NPDEs (see Figs. 1–3).

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