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The homotopy perturbation method for solving neutral functional–differential equations with proportional delays

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Abstract The aim of this paper is to apply homotopy perturbation method (HPM) to solve delay differential equations. Some examples are presented to show the ability of the method. The results reveal that the method is very effective and simple.

1. Introduction

Neutral functional–differential equations with proportional delays represent a particular class of delay differential equation. Such functional–differential equations play an important role in the mathematical modeling of real world phenomena (Bellen and Zennaro, 2003). These equations have been investigated by some authors and several efficient numerical and analytical methods have been designed to approximate their solutions Ishiwata and Muroya used the rational approximation method

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(Ishiwata and Muroya, 2007) and the collocation method (Ishiwata et al., 2008), Wang et al. obtained approximate solutions by continuous Runge–Kutta methods (Wang et al., 2009) and one-leg θ-methods (Wang and Li, 2007; Wang et al., 2009).

Very recently, Chen and his collaborator applied the variational iteration method for solving a neutral functional–differential equation with proportional delays (Chen and Wang, 2010).

In this paper, we apply the homotopy perturbation method (HPM in short) to solve neutral differential equation with proportional delays as considered in Chen and Wang (2010),

$$(u(t) + a(t)u(p_m t))^{(m)} = \beta u(t) + \sum_{k=0}^{m-1} b_k(t)u^{(k)}(p_k t) + f(t),$$
$$t \geqslant 0, \tag{1}$$

with the initial conditions

$$u^{(k)}(0) = \lambda_k, \quad k = 0, 1, \ldots, m-1.$$

where a and b_k $(k = 0, 1, \ldots, m-1)$ are known analytical functions, and $\beta, p_k, c_{ik}, \lambda_k$ are given constants with $0 < p_k < 1$ for $k = 0, 1, \ldots, m$.

This paper is organized as follows: In Section 2, basic idea of HPM is presented. Applying HPM to solving (1) is discussed in Section 3. Section 4 is devoted to numerical comparisons

between the results obtained by HPM in this work and some existing methods. Finally, conclusions are stated in the last section.

2. Basic idea of He's homotopy perturbation method

The topic of the He's homotopy perturbation method (He, 2004, 2005, 2006) has been rapidly growing in recent years. In this method the solution of functional equations is considered as the summation of an infinite series usually converging to the solution.

Considerable research works have been conducted recently in applying this method to a class of linear and nonlinear equations. For example, nonlinear Schrodinger equations (Biazar and Ghazvini, 2007), integral equations (Abbasbandy, 2006), nonlinear oscillators with discontinuities (He, 1999), nonlinear wave equations (He, 2000). To see more applications of this method we refer the interested readers to He (2004, 2005, 2006, 1999, 2000), Biazar and Ghazvini (2007), Abbasbandy (2006) and references therein.

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{2}$$

with the boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma,$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω.

The operator A can be divided into two parts, which are L and N, where L is a linear, but N is nonlinear. Eq. (2) can be, therefore, rewritten as follows:

$$L(u) + N(u) - f(r) = 0.$$

By the homotopy technique, we construct a homotopy $U(r,p):\Omega \times [0,1] \to \mathbb{R}$, which satisfies:

$$H(U,p) = (1-p)[L(U) - L(u_0)] + p[A(U) - f(r)],$$
$$p \in [0,1], \ r \in \Omega, \tag{3}$$

or

$$H(U,p) = L(U) - L(u_0) + pL(u_0) + p[A(U) - f(r)],$$
$$p \in [0,1], \ r \in \Omega, \tag{4}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of Eq. (1), which satisfies the boundary conditions. Obviously, from Eqs. (3) and (4) we will have

$$H(U,0) = A(U) - L(u_0) = 0,$$
$$H(U,1) = A(U) - f(r) = 0.$$

The changing process of p form zero to unity is just that of $U(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called homotopy. According to the (HPM), we can first use the embedding parameter p as a small parameter, and assume that the solution of Eqs. (3) and (4) can be written as a power series in p:

$$U = U_0 + pU_1 + p^2U_2 + p^3U_3 + \cdots$$

Setting $p = 1$, results in the approximate solution of Eq. (1)

$$u = \lim_{p \to 1} U = U_1 + U_2 + U_3 + \cdots$$

3. Method of solution

For solving Eq. (1), by homotopy perturbation method we construct a homotopy as follows:

$$\begin{aligned} H(v,p) = (1-p)[v^{(m)}(t) - u_0^{(m)}(t)] \\ + p\Bigg[v^{(m)}(t) + a(t)v^{(m)}(p_m t) - \beta v(t) \\ - \sum_{k=0}^{m-1} b_k(t)v^{(k)}(p_k t) - f(t)\Bigg], \quad t \geqslant 0. \end{aligned} \tag{5}$$

Suppose the solution of Eq. (2) has the form

$$v = v_0 + pv + p^2v^2 + \cdots, \tag{6}$$

where v_i's are functions yet to be determined.

Whereas series (6) be a convergent series at $p = 1$, the exact solution of (1), reads as:

$$v = v_0 + v_1 + v_2 + \cdots$$

Substituting (6) into (5) and arranging the coefficients powers of p following initial value problems

$$p^0 : v_0^{(m)}(t) - u_0^{(m)}(t) = 0, \quad v_0^{(k)}(0) = \lambda_k, \ k = 0, 1, \ldots, m-1,$$

$$p^1 : v_1^{(m)}(t) + u_0^{(m)}(t) + a(t)u_0^{(m)}(p_m t) - \beta u_0(t) - \sum_{k=0}^{m-1} b_k(t)u_0^{(k)}(p_k t),$$

$$v_1^{(k)}(0) = 0, \ k = 0, \ldots, m-1,$$

$$\vdots$$

$$p^n : v_n^{(m)}(t) + a(t)u_{n-1}^{(m)}(p_m t) - \beta u_{n-1}(t) - \sum_{k=0}^{m-1} b_k(t)u_{n-1}^{(k)}(p_k t),$$

$$v_n^{(k)}(0) = 0, \ k = 0, 1, \ldots, m-1. \tag{7}$$

Identifying the components v_k's, the nth approximation of the exact solution can be obtained, as:

$$u_n = v_0 + v_1 + v_2 + \cdots + v_n.$$

4. Illustrative examples

In this part, some examples are provided to illustrate performance of proposed method. For the sake of comparing purposes, we consider the same examples as used in Chen and Wang (2010).
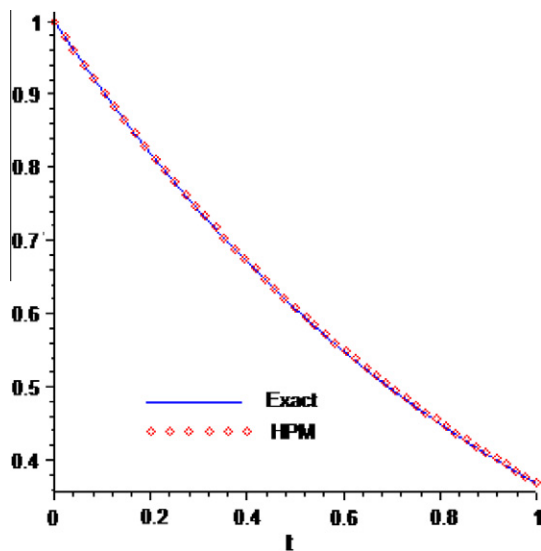
Example 1. Consider the following first-order neutral functional–differential equation with proportional delay:

$$\begin{cases} u'(t) = -u(t) + \frac{1}{2}u\left(\frac{t}{2}\right) + \frac{1}{2}u'\left(\frac{t}{2}\right), \quad t \in [0,1] \\ u(0) = 1, \end{cases}$$

Exact solution $u(t) = e^{-t}$.

In this example, starting with $u_0(t) = 1$, 7th order of HPM approximate solutions is obtained, as:

$$\begin{aligned} u_7(t) = 1 - \frac{127}{128}t + \frac{8001}{16384}t^2 - \frac{82677}{524288}t^3 + \frac{1240155}{33554432}t^4 \\ - \frac{1736217}{268435456}t^5 + \frac{1736217}{2147483648}t^6 - \frac{248031}{4294967296}t^7. \end{aligned} \tag{8}$$

Figure 1 Comparison of the approximate solutions with the exact solution for Example 1.

Figure 2 Comparison of the approximate solutions with the exact solution for Example 2.

The graphs of (8) and the exact solution are contained in Fig. 1. Also, we compare the absolute errors of homotopy perturbation method with the ones for the variational iteration method (Chen and Wang, 2010) and the two-stage order-one Runge–Kutta method of Wang et al. (2009) and the one-leg θ-method of Wang and Li (2007), Wang et al. (2009) with $\theta = 0.8$ using $h = 0.01$, in Table 1.
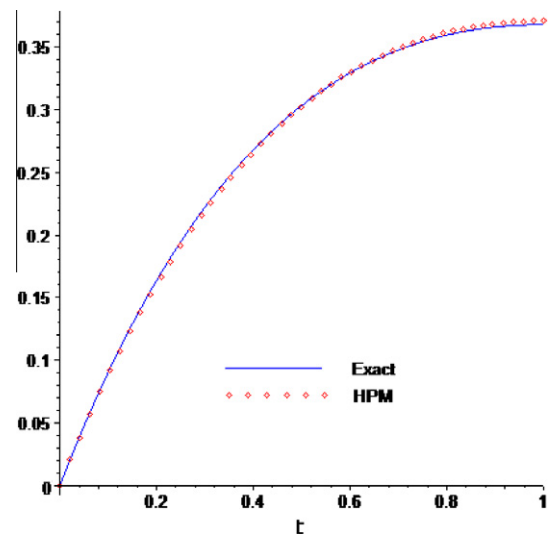
Example 2. Let us have the following first-order neutral functional–differential equation with proportional delay

$$\begin{cases} u'(t) = -u(t) + 0.1u(0.8t) + 0.5u'(0.8t) \\ \qquad + (0.32t - 0.5)\exp(-0.8t) + \exp(t), \quad t \geqslant 0 \\ u(0) = 0, \end{cases}$$

which has the exact solution $u(t) = te^{-t}$.

Considering the 5th order of HPM approximate solutions with $u_0(t) = 0$, we made comparison of such approximation with the exact solution in Fig. 2.

Comparing Table 2, it can be seen that the approximation solutions by HPM agree with the exact solution.

Example 3. As another example, let us consider the second-order neutral functional–differential equation with proportional delay

$$\begin{cases} u''(t) = u'\left(\frac{1}{2}t\right) - \frac{1}{2}tu''\left(\frac{1}{2}t\right) + 2, \quad t \in [0,1] \\ u(0) = 1, \quad u'(0) = 0. \end{cases}$$

Starting with $u_0(t) = 1$ in (7), we have

$$\begin{cases} u_1(t) = t^2, \\ u_n(0) = 0, \quad n \geqslant 2. \end{cases}$$

That is, we obtain $u(t) = u_0(t) + u_1(t) = 1 + t^2$, which coincides with the exact solution.

Example 4. In this example, we consider Eq. (1), as:

$$\begin{cases} u''(t) = \frac{3}{4}u(t) + u\left(\frac{1}{2}t\right) + u'\left(\frac{1}{2}t\right) \\ \qquad + \frac{1}{2}u''\left(\frac{1}{2}t\right) - t^2 - t + 1, \quad t \in [0,1] \\ u(0) = u'(0) = 0, \end{cases}$$

which enjoys exact solution $u(t) = t^2$.

Table 1 Comparison of the absolute errors for Example 1.

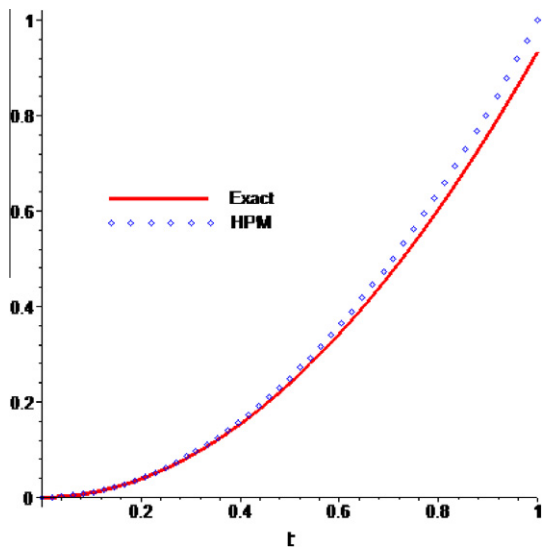| t	Two-stage order-one Runge–Kutta method	One-leg θ-method with $\theta = 0.8$	Variational iterative method		Homotopy perturbation method	
			$n = 7$	$n = 8$	$n = 7$	$n = 8$
0.1	4.55e−4	2.57e−3	7.43e−4	3.72e−4	6.73e−4	3.36e−4
0.2	8.24e−4	8.86e−3	1.42e−3	7.08e−4	1.16e−3	5.80e−4
0.3	1.12e−3	1.72e−2	2.02e−3	1.01e−3	1.50e−3	7.50e−4
0.4	1.35e−3	2.66e−2	2.58e−3	1.29e−3	1.73e−3	8.64e−4
0.5	1.52e−3	3.63e−2	3.07e−3	1.54e−3	1.86e−3	9.33e−4
0.6	1.66e−3	4.85e−2	3.52e−3	1.76e−3	1.94e−3	9.68e−4
0.7	1.75e−3	5.47e−2	3.93e−3	1.97e−3	1.95e−3	9.78e−4
0.8	1.81e−3	6.29e−2	4.30e−3	2.15e−3	1.93e−3	9.68e−4
0.9	1.84e−3	7.02e−2	4.64e−3	2.32e−3	1.89e−3	9.44e−4
1.0	1.85e−3	7.66e−2	4.94e−3	2.47e−3	1.82e−3	9.10e−4

Table 2 Comparison of the absolute errors for Example 2.

t	Two-stage order-one Runge–Kutta method	One-leg θ-method with $\theta = 0.8$	Variational iterative method		Homotopy perturbation method	
			$n = 5$	$n = 6$	$n = 5$	$n = 6$
0.1	8.68e−4	4.65e−3	2.62e−3	1.30e−3	2.17e−3	1.06e−3
0.2	1.49e−3	1.45e−2	4.36e−3	2.14e−3	2.87e−3	1.35e−3
0.3	1.90e−3	2.57e−2	5.40e−3	2.63e−3	2.63e−3	1.18e−3
0.4	2.16e−3	3.60e−2	5.89e−3	2.84e−3	1.83e−3	7.61e−4
0.5	2.28e−3	4.43e−2	5.96e−3	2.83e−3	7.76e−4	2.32e−4
0.6	2.31e−3	5.03e−2	5.71e−3	2.67e−3	3.33e−4	2.98e−4
0.7	2.27e−3	5.37e−2	5.23e−3	2.39e−3	1.35e−3	7.64e−4
0.8	2.17e−3	5.47e−2	4.59e−3	2.04e−3	2.20e−3	1.12e−3
0.9	2.03e−3	5.35e−2	3.84e−3	1.64e−3	2.82e−3	1.37e−3
1.0	1.86e−3	5.03e−2	3.04e−3	1.22e−3	3.21e−3	1.50e−3

In this example, starting with $u_0(t) = 0$, 5th order of HPM approximate solutions is obtained, as:

$$u_5(t) = \frac{31}{32}t^2 - \frac{31}{3072}t^3 - \frac{651}{262144}t^4 - \frac{23095}{37748736}t^5$$
$$- \frac{23095}{37748736}t^6 + \cdots \tag{9}$$

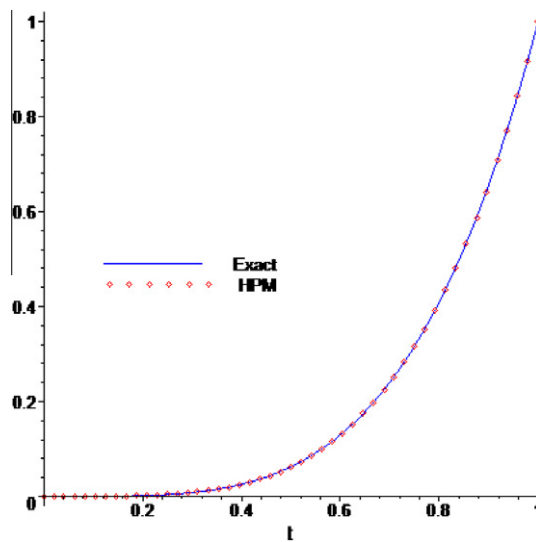The approximate solution (9) and exact solution are illustrated in Fig. 3.

Similar to above, we compute the absolute errors for different approaches, for example, 4 in Table 3.

Example 5. As last example, let's try the following third-order case of (1), as:

$$\begin{cases} u'''(t) = u(t) + u'\left(\frac{1}{2}t\right) + u''\left(\frac{1}{3}t\right) + \frac{1}{2}u'''\left(\frac{1}{4}t\right) - t^4 \\ \qquad\qquad - \frac{t^3}{2} - \frac{4}{3}t^2 + 21t, \quad t \in [0,1] \\ u(0) = u'(0) = u''(0) = 0, \end{cases}$$

Figure 3 Comparison of the approximate solutions with the exact solution for Example 4.

Figure 4 Comparison of the approximate solutions with the exact solution for Example 5.

Table 3 Comparison of the absolute errors for Example 4.

t	Two-stage order-one Runge–Kutta method	One-leg θ-method with $\theta = 0.8$	Variational iterative method		Homotopy perturbation method	
			$n = 5$	$n = 6$	$n = 5$	$n = 6$
0.1	8.68e−3	6.10e−3	3.34e−4	1.67e−4	3.33e−4	1.67e−4
0.2	1.49e−3	2.58e−2	1.43e−3	7.15e−4	1.42e−3	7.15e−4
0.3	1.90e−3	6.47e−2	2.45e−3	1.73e−3	3.44e−3	1.72e−3
0.4	2.16e−3	1.37e−1	6.58e−3	3.30e−3	6.57e−3	3.30e−3
0.5	2.28e−3	2.81e−1	1.11e−2	5.55e−3	1.10e−2	5.55e−3

Table 4 Comparison of the absolute errors for Example 5.

t	Two-stage order-one Runge–Kutta method	Variational iterative method			Homotopy perturbation method		
		$n = 4$	$n = 5$	$n = 6$	$n = 4$	$n = 5$	$n = 6$
0.1	4.97e−5	2.46e−8	3.07e−9	9.09e−12	2.50e−8	3.12e−9	3.90e−12
0.2	4.43e−4	4.03e−7	5.04e−8	2.98e−10	4.09e−7	5.12e−8	6.40e−10
0.3	1.57e−3	2.09e−6	2.62e−7	2.33e−9	2.12e−6	2.66e−7	2.32e−8
0.4	3.85e−3	6.80e−6	8.49e−7	1.01e−8	6.90e−6	8.63e−7	1.07e−7
0.5	7.78e−3	1.71e−5	2.13e−6	3.20e−8	1.73e−5	2.16e−6	2.70e−7
0.6	1.39e−2	3.64e−5	4.55e−6	8.24e−8	3.69e−5	4.62e−6	5.78e−7
0.7	2.28e−2	6.96e−5	8.69e−6	1.85e−7	7.06e−5	8.83e−6	1.10e−6
0.8	3.53e−2	1.23e−4	1.53e−5	3.76e−7	1.24e−4	1.55e−5	1.94e−6
0.9	5.19e−2	2.03e−4	2.54e−5	7.09e−7	2.06e−4	2.57e−5	3.22e−6
1.0	7.34e−2	3.21e−4	4.01e−5	1.26e−6	3.25e−4	4.07e−5	5.09e−6

In Fig. 4, we draw the diagrams of the 4th order of HPM approximate results obtained by HPM with $u_0(t) = 0$ and exact solution $u(t) = t^2$.

Furthermore, some result comparisons of this example are reported in Table 4.

5. Conclusion

In this paper, He's homotopy perturbation method has been successfully applied to find the solutions of neutral functional–differential equations. The efficiency and accuracy of the proposed decomposition method were demonstrated by some test problems. It is concluded from above tables and figures that the HPM is an accurate and efficient method to solve neutral functional–differential equations.

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