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On the soliton solutions to the modified Benjamin-Bona-Mahony and coupled Drinfel'd-Sokolov-Wilson models and its applications

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A B S T R A C T

In this article, the analytical solutions for modified Benjamin-Bona-Mahony and coupled Drinfel'd-Sokolov-Wilson equations have been extracted with the help of very simple transformation. These results hold numerous traveling wave solutions that are of key importance in elucidating some physical circumstance. The technique can also be functional to other sorts of nonlinear evolution equations in contemporary areas of research.

1. Introduction

Nonlinear evolution equations (NEEs) have been studied in last few decades. A verity of NEEs are integrated with the help of various interesting computational techniques. To understand the physical structure, described by nonlinear partial differential equations (PDEs), exact solutions to the nonlinear PDEs play a crucial role in the study of the nonlinear models appearing in diverse disciplines; for instance, electromagnetic theory, geochemistry, astrophysics, fluid dynamics, elastic media, nuclear physics, optical fibers, high-energy physics, gravitation and in statistical and condensed matter physics, biology, solid state physics, chemical kinematics, chemical physics, electrochemistry, fluid dynamics, acoustics, cosmology and plasma physics etc, see (Seadawy and El-Rashidy, 2013; Gardner et al., 1967; Su and Gardner, 1969; Ito, 1980; Zhibin and Mingliang, 1993; Liang, 2014; Seadawy, 2012a, b; Seadawy, 2016a,b).

In recent few decades, growing interest have been drawn to find the analytical solutions for nonlinear wave equations, such as the traveling wave solution (Xu and Li, 2005), Cole-Hopf transformation, Painlevé method, Bäcklund transformation, amplitude ansatz method (Seadawy and Lu, 2017), sine-cosine method, Darboux transformation, Hirota method, function transformation method, Lie group analysis, extended simple equation method (Lu et al., 2017), homogeneous balance method (Chen et al., 2003), similarity reduced method, tanh method, fractional direct algebraic function method (Seadawy, 2016), inverse scattering method (Ablowitz and Clarkson, 1991), Hirota's bilinear method (Hirota, 1971), the homogeneous balance method (Wang, 1995), variational method (Khater et al., 2003), algebraic method (Khater et al., 2000), sine-cosine method, Jacobi elliptic function method (Liu et al., 2001), the F-expansion method (Zhou et al., 2003), extended Fan Sub-Equation method (Kalim and Younis, 2017), the (G'/G) expansion method (Abazari, 2010; Kutluay et al., 2010), the tanh and extended tanh method, extended direct algebraic method (Seadawy et al., 2016), the auxiliary equation method (Kalim and Seadawy, 2017) and many more (Yan et al., 2012; Grey and Tom, 2014; Mohapatra et al., 2015; Kalim and Seadawy, 2019; Abu Arqub et al., 2015; El-Ajou et al., 2015a,b; Abu Arqub, 2017a,b).

In this paper, the auxiliary equation method (AEM) is applied to construct the traveling wave solutions to the modified Benjamin-Bona-Mahony (m-BBM) and coupled Drinfel'd-Sokolov-Wilson (c-DSW) equations. The aim of the study is to deal with the explicit solutions of NPDEs and to explore the configuration of the physical

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phenomena depending upon various parameters. As a result, some new and more general exact traveling wave solutions are obtained.

The Benjamin-Bona-Mahony equation (BBM) describes the unidirectional propagation of long waves in certain nonlinear dispersive media, as discussed in Seadawy (2018, 2017). The BBM equation is known as the modified BBM equation (mBBM) for $n = 2$. The governing equation is as follows:

$$u_t + u_x + \vartheta_1 u^n u_x + \vartheta_2 u_{xxt} = 0 \tag{1}$$

where the coefficients ϑ_i for $i = 1, 2$; are real constants.

The coupled Drinfel'd-Sokolov-Wilson system (cDSW) reads

$$\left.\begin{aligned} &u_t + \varsigma_1 v v_x = 0, \\ &v_t + \varsigma_2 v_{xxx} + \varsigma_3 u v_x + \varsigma_4 u_x v = 0 \end{aligned}\right\} \tag{2}$$

where the coefficients ς_i for $i = 1, 2, 3, 4$; are real constants, for details see Seadawy (2017), Seadawy et al. (2017), Wena et al. (2009), Seadawy and Alamri (2018), Khater et al. (2006).

This article has been devised as follows: in Section 2, the auxiliary equation method is introduced, while in Section 3, the solutions of the NPDEs have been presented. In last Section 4, the conclusions have been drawn.

2. The description of the auxiliary equation method

We will briefly present the AEM in the following steps:

Step 1. Let us have a general form of nonlinear PDE

$$F(u, u_t, u_x, u_{xx}, u_{xxx}, \dots) = 0. \tag{3}$$

where F is a polynomial function with respect to the indicated variables.

Step 2. The following wave variable is presented to solve (3)

$$u(x, t) = F(\xi), \tag{4}$$

The transformations (4) convert the PDE (3) to an ODE

$$O_i(F, F_\xi, F_{\xi\xi}, F_{\xi\xi\xi}, \dots), \tag{5}$$

where $F = F(\xi)$ is unknown function.

Step 3. The main idea of the auxiliary equation method based on expanding the traveling wave solution $F(\xi)$ of Eqs. (5) as a finite series

$$F(\xi) = \sum_{j=0}^{n} a_j \psi^j(\xi), \tag{6}$$

ψ satisfies

$$\frac{d\psi}{d\xi} = C_0 + C_1\psi(\xi) + C_2\psi^2(\xi) + C_3\psi^3(\xi) + C_4\psi^4(\xi), \tag{7}$$

$$\xi = \alpha x - \omega t \tag{8}$$

where $C_i (i = 0, 1, 2, 3, 4)$ and α are constants.

Step 4. Applying the homogenous balance to (3), the parameters n in (6) can be obtained.

Step 5. Substituting (6), (7) and (8) in (3) and collecting the coefficients of $\psi^j\psi^{(k)}$, then solving the system for ω and C_i.

Step 6. Substituting ω, C_i and $\psi(\xi)$ obtained in step 5 into (6), to obtain the solutions for (1) and (2).

3. Soliton extraction

3.1. Modified Benjamin-Bona-Mahony equation

Consider the transformation

$$u(x, t) = u(\xi), \quad \xi = \alpha x - \omega t, \tag{9}$$

using (9) into (1),

$$(\alpha - \omega)u' + \alpha\vartheta_1 u^2 u' - \alpha^2\omega\vartheta_2 u''' = 0 \tag{10}$$

integrating

$$(\alpha - \omega)u + \frac{1}{3}\alpha\vartheta_1 u^3 - \alpha^2\omega\vartheta_2 u'' = 0 \tag{11}$$

Consider the homogeneous balance between u^3 and u'', gives $n = 3$. Suppose the solution of (11), is of the form

$$u = a_0 + a_1\psi(\xi) + a_2\psi(\xi)^2 + a_3\psi(\xi)^3 \tag{12}$$

Substituting (6), (7) and (12) in (11) and collecting the coefficients of $\psi^j\psi^{(k)}$

Case I. $C_4 = 0$

(a).

$$\psi_1(\xi) = \frac{\sqrt{3}\sqrt{\alpha - \omega}\tan(\theta_1\xi)}{\sqrt{\alpha}a_1\sqrt{\vartheta_1}} + \frac{\sqrt{6}\sqrt{\alpha}\lambda_1\sqrt{\vartheta_2}\sqrt{\omega}}{2a_1\sqrt{\vartheta_1}}, \tag{13}$$

where

$$\theta_1 = \frac{1}{2}\left(\frac{\sqrt{2}\sqrt{\omega}}{\alpha\sqrt{\vartheta_2}\sqrt{\alpha - \omega}} - \frac{\sqrt{2}}{\sqrt{\vartheta_2}\sqrt{\omega}\sqrt{\alpha - \omega}}\right).$$

The parameters C_i and a_j become

$C_0 = \frac{-2\sqrt{6}\alpha - \sqrt{6}\alpha^2 c_1^2\vartheta_2\omega + 2\sqrt{6}\omega}{4\alpha^{3/2}a_1\sqrt{\vartheta_1}\sqrt{\vartheta_2}\sqrt{\omega}}, \quad C_1 = \lambda_1, \quad C_2 = -\frac{a_1\sqrt{\vartheta_1}}{\sqrt{6}\sqrt{\alpha}\sqrt{\vartheta_2}\sqrt{\omega}}, \quad C_3 = 0,$

$a_0 = \frac{\sqrt{6}\sqrt{\alpha}c_1\sqrt{\vartheta_1}\sqrt{\vartheta_2}\omega^{3/2} - \sqrt{6}\alpha^{3/2}c_1\sqrt{\vartheta_1}\sqrt{\vartheta_2}\sqrt{\omega}}{2(\alpha\vartheta_1 - \vartheta_1\omega)}$ $a_1 = \lambda_2, \quad a_2 = a_3 = 0,$ where λ_1 and λ_2 are arbitrary constants, hence the solution of (1) will be

$$\begin{aligned} u_1(x, t) = {} & \eta_1\eta_2\operatorname{sech}^2(\eta_1\xi)\Big(-2\alpha^2\eta_1^2\vartheta_2\omega(\cosh(2\eta_1\xi) - 2)\operatorname{sech}^2(\eta_1\xi) \\ & + \alpha\eta_2^2\vartheta_1\tanh^2(\eta_1\xi) + \alpha - \omega\Big), \end{aligned} \tag{14}$$

where

$$\eta_1 = \frac{\sqrt{\alpha - \omega}}{\sqrt{2}\alpha\sqrt{\vartheta_2}\sqrt{\omega}},$$

$$\eta_2 = -\frac{\sqrt{3}\sqrt{\alpha - \omega}}{\sqrt{\alpha}\sqrt{\vartheta_1}}.$$

(b).

$$\psi_2(\xi) = \frac{\sqrt{3}\sqrt{\alpha - \omega}\tan(\theta_1\xi)}{\sqrt{\alpha}a_1\sqrt{\vartheta_1}} - \frac{\sqrt{6}\sqrt{\alpha}\lambda_1\sqrt{\vartheta_2}\sqrt{\omega}}{2a_1\sqrt{\vartheta_1}} \tag{15}$$

where

$$\theta_1 = \frac{1}{2}\left(\frac{\sqrt{2}}{\sqrt{\vartheta_2}\sqrt{\omega}\sqrt{\alpha - \omega}} - \frac{\sqrt{2}\sqrt{\omega}}{\alpha\sqrt{\vartheta_2}\sqrt{\alpha - \omega}}\right). \tag{16}$$

The parameters C_i and a_j become

$C_0 = \frac{2\sqrt{6}\alpha + \sqrt{6}\alpha^2 c_1^2\vartheta_2\omega - 2\sqrt{6}\omega}{4\alpha^{3/2}a_1\sqrt{\vartheta_1}\sqrt{\vartheta_2}\sqrt{\omega}}, \quad C_1 = \lambda_1, \quad C_2 = \frac{a_1\sqrt{\vartheta_1}}{\sqrt{6}\sqrt{\alpha}\sqrt{\vartheta_2}\sqrt{\omega}},$

$C_3 = 0, a_0 = \frac{\sqrt{6}\alpha^{3/2}c_1\sqrt{\vartheta_1}\sqrt{\vartheta_2}\sqrt{\omega} - \sqrt{6}\sqrt{\alpha}c_1\sqrt{\vartheta_1}\sqrt{\vartheta_2}\omega^{3/2}}{2(\alpha\vartheta_1 - \vartheta_1\omega)}, \quad a_1 = \lambda_2, \quad a_2 = a_3 = 0,$

where λ_1 and λ_2 are arbitrary constants, hence the solution of (1) will be

$$\begin{aligned} u_2(x, t) = {} & \eta_1\eta_2\operatorname{sech}^2(\eta_1\xi)\Big(-2\alpha^2\eta_1^2\vartheta_2\omega(\cosh(2\eta_1\xi) - 2)\operatorname{sech}^2(\eta_1\xi) \\ & + \alpha\eta_2^2\vartheta_1\tanh^2(\eta_1\xi) + \alpha - \omega\Big), \end{aligned} \tag{17}$$

where

$$\eta_1 = \frac{\sqrt{\alpha-\omega}}{\sqrt{2}\alpha\sqrt{\vartheta_2}\sqrt{\omega}},$$
$$\eta_2 = \frac{\sqrt{3}\sqrt{\alpha-\omega}}{\sqrt{\alpha}\sqrt{\vartheta_1}}.$$

Case II. $C_3 = 0, C_4 = 0$
(a).

$$\psi_3(\xi) = \frac{\sqrt{6}\sqrt{\alpha}\lambda_1\sqrt{\vartheta_2}\sqrt{\omega}}{2a_1\sqrt{\vartheta_1}} - \frac{\sqrt{3}\sqrt{\alpha-\omega}\tan(\theta_1\xi)}{\sqrt{\alpha}a_1\sqrt{\vartheta_1}}, \tag{18}$$

where

$$\theta_1 = \frac{1}{2}\left(\frac{\sqrt{2}\xi}{\sqrt{\vartheta_2}\sqrt{\omega}\sqrt{\alpha-\omega}} - \frac{\sqrt{2}\xi\sqrt{\omega}}{\alpha\sqrt{\vartheta_2}\sqrt{\alpha-\omega}}\right). \tag{19}$$

The parameters C_i and a_j become
$C_0 = -\frac{\sqrt{\frac{3}{2}}(2\alpha+\alpha^2c_1^2\vartheta_2\omega-2\omega)}{2\alpha^{3/2}a_1\sqrt{\vartheta_1}\sqrt{\vartheta_2\sqrt{\omega}}}$, $C_1 = \lambda_1$, $C_2 = -\frac{a_1\sqrt{\vartheta_1}}{\sqrt{6}\sqrt{\alpha}\sqrt{\vartheta_2\sqrt{\omega}}}$, $a_0 = -\frac{\sqrt{\frac{3}{2}}\sqrt{\alpha}c_1\sqrt{\vartheta_2\sqrt{\omega}}}{\sqrt{\vartheta_1}}$, $a_1 = \lambda_2$, $a_2 = 0$, $a_3 = 0$, where λ_1 and λ_2 are arbitrary constants, hence the solution of (1) will be

$$u_3(x,t) = \eta_1\eta_2\text{sech}^2(\eta_1\xi)\Big(-2\alpha^2\eta_1^2\vartheta_2\omega(\cosh(2\eta_1\xi)-2)\text{sech}^2(\eta_1\xi) + \alpha\eta_2^2\vartheta_1\tanh^2(\eta_1\xi) + \alpha - \omega\Big) \tag{20}$$

where

$$\eta_1 = \frac{\sqrt{\alpha-\omega}}{\sqrt{2}\alpha\sqrt{\vartheta_2}\sqrt{\omega}},$$
$$\eta_2 = -\frac{\sqrt{3}\sqrt{\alpha-\omega}}{\sqrt{\alpha}\sqrt{\vartheta_1}}.$$

(b).

$$\psi_4(\xi) = \frac{1}{2}\left(\frac{2\sqrt{3}\sqrt{\alpha-\omega}\tan(\theta_1)}{\sqrt{\alpha}a_1\sqrt{\vartheta_1}} - \frac{\sqrt{6}\sqrt{\alpha}\lambda_1\sqrt{\vartheta_2}\sqrt{\omega}}{a_1\sqrt{\vartheta_1}}\right). \tag{21}$$

where

$$\theta_1 = \frac{1}{2}\left(\frac{\sqrt{2}\xi}{\sqrt{\vartheta_2}\sqrt{\omega}\sqrt{\alpha-\omega}} - \frac{\sqrt{2}\sqrt{\omega}}{\alpha\sqrt{\vartheta_2}\sqrt{\alpha-\omega}}\right). \tag{22}$$

The parameters C_i and a_j become
$C_0 = \frac{\sqrt{\frac{3}{2}}(2\alpha+\alpha^2c_1^2\vartheta_2\omega-2\omega)}{2\alpha^{3/2}a_1\sqrt{\vartheta_1}\sqrt{\vartheta_2\sqrt{\omega}}}$, $C_1 = \lambda_1$, $C_2 = \frac{a_1\sqrt{\vartheta_1}}{\sqrt{6}\sqrt{\alpha}\sqrt{\vartheta_2\sqrt{\omega}}}$, $a_0 = \frac{\sqrt{\frac{3}{2}}\sqrt{\alpha}c_1\sqrt{\vartheta_2\sqrt{\omega}}}{\sqrt{\vartheta_1}}$, $a_1 = \lambda_2$, $a_2 = 0$, $a_3 = 0$ where λ_1 and λ_2 are arbitrary constants, hence the solution of (1) will be

$$u_4(x,t) = \eta_1\eta_2(-\sec^2(\eta_1\xi))(-2\alpha^2\eta_1^2\vartheta_2\omega(\cos(2\eta_1\xi)-2)\sec^2(\eta_1\xi) - \alpha\eta_2^2\vartheta_1\tan^2(\eta_1\xi) - \alpha + \omega) \tag{23}$$

where

$$\eta_1 = \frac{\sqrt{\alpha-\omega}}{\sqrt{2}\alpha\sqrt{\vartheta_2}\sqrt{\omega}},$$
$$\eta_2 = \frac{\sqrt{3}\sqrt{\alpha-\omega}}{\sqrt{\alpha}\sqrt{\vartheta_1}}.$$

(c).

$$\psi_5(\xi) = -\frac{\sqrt{3}\sqrt{\alpha-\omega}\tan(\theta_1\xi)}{\sqrt{\alpha}a_1\sqrt{\vartheta_1}}, \tag{24}$$

where

$$\theta_1 = \frac{1}{2}\left(\frac{\sqrt{2}\xi}{\sqrt{\vartheta_2}\sqrt{\omega}\sqrt{\alpha-\omega}} - \frac{\sqrt{2}\sqrt{\omega}}{\alpha\sqrt{\vartheta_2}\sqrt{\alpha-\omega}}\right). \tag{25}$$

The parameters C_i and a_j become
$C_0 = -\frac{\sqrt{\frac{3}{2}}\sqrt{\alpha^2-2\alpha\omega+\omega^2}}{\alpha^{3/2}a_1\sqrt{\vartheta_1}\sqrt{\vartheta_2\sqrt{\omega}}}$, $C_1 = 0$, $C_2 = \frac{a_1\sqrt{\vartheta_1}\sqrt{(\alpha-\omega)^2}}{\sqrt{6}\sqrt{\alpha}\sqrt{\vartheta_2\sqrt{\omega}}(\omega-\alpha)}$, $a_0 = 0$, $a_1 = \lambda_1$, $a_2 = 0$, $a_3 = 0$ where λ_1 is arbitrary constant, hence the solution of (1) will be

$$u_5(x,t) = \eta_1\eta_2(-\sec^2(\eta_1\xi))(-2\alpha^2\eta_1^2\vartheta_2\omega(\cos(2\eta_1\xi)-2)\sec^2(\eta_1\xi) - \alpha\eta_2^2\vartheta_1\tan^2(\eta_1\xi) - \alpha + \omega), \tag{26}$$

where

$$\eta_1 = \frac{\sqrt{\alpha-\omega}}{\sqrt{2}\alpha\sqrt{\vartheta_2}\sqrt{\omega}},$$
$$\eta_2 = -\frac{\sqrt{3}\sqrt{(\alpha-\omega)^2}}{\sqrt{\alpha}\sqrt{\vartheta_1}\sqrt{\alpha-\omega}}.$$

Case III. $C_0 = 0, C_4 = 0$
(a).

$$\psi_6(\xi) = \sqrt{\frac{-6\sqrt{3}\sqrt{\alpha}a_2\sqrt{\vartheta_1}\sqrt{\omega-\alpha}e^{2\theta_1\xi} - 6\sqrt{\omega-\alpha}e^{\theta_1\xi}}{1-3\alpha a_2^2\vartheta_1e^{2\theta_1\xi}}}, \tag{27}$$

where

$$\theta_1 = \frac{\sqrt{2}\sqrt{\omega-\alpha}}{\alpha\sqrt{\vartheta_2}\sqrt{\omega}}. \tag{28}$$

The parameters C_i and a_j become
$C_1 = \frac{\sqrt{\omega-\alpha}}{\sqrt{2}\alpha\sqrt{\vartheta_2\sqrt{\omega}}}$, $C_2 = 0$, $C_3 = -\frac{a_2\sqrt{\vartheta_1}}{2\sqrt{6}\sqrt{\alpha}\sqrt{\vartheta_2\sqrt{\omega}}}$, $a_0 = \frac{\sqrt{3}\sqrt{\omega-\alpha}}{\sqrt{\alpha}\sqrt{\vartheta_1}}$, $a_1 = 0$, $a_2 = \lambda_1$, $a_3 = 0$, where λ_1 is arbitrary constant, hence the solution of (1) will be

$$\begin{aligned} u_6(x,t) = &\frac{6\theta_1\lambda_1\sqrt{\omega-\alpha}e^{\theta_1\xi}}{(1-3\alpha\lambda_1^2\vartheta_1e^{2\theta_1\xi})^4} \\ &(9\alpha^2\lambda_1^4\vartheta_1^2e^{4\theta_1\xi}(23\alpha^2\theta_1^2\vartheta_2\omega + 94(\alpha-\omega)) \\ &+ 3\alpha\lambda_1^2\vartheta_1e^{2\theta_1\xi}(23\alpha^2\theta_1^2\vartheta_2\omega + 26(\omega-\alpha)) \\ &+ 8\sqrt{3}\sqrt{\alpha}\lambda_1\sqrt{\vartheta_1}e^{\theta_1\xi}(\alpha^2\theta_1^2\vartheta_2\omega - \alpha + \omega) + \alpha^2\theta_1^2\vartheta_2\omega \\ &+ 48\sqrt{3}\alpha^{3/2}\lambda_1^3\vartheta_1^{3/2}e^{3\theta_1\xi}(2\alpha^2\eta_1^2\vartheta_2\omega + \alpha - \omega) \\ &+ 72\sqrt{3}\alpha^{5/2}\lambda_1^5\vartheta_1^{5/2}e^{5\theta_1\xi}(\alpha^2\eta_1^2\vartheta_2\omega + 11(\alpha-\omega)) \\ &+ 27\alpha^3\lambda_1^6\vartheta_1^3e^{6\theta_1\xi}(\alpha^2\theta_1^2\vartheta_2\omega + 26(\alpha-\omega)) + 2\alpha - 2\omega). \end{aligned} \tag{29}$$

(b).

$$\psi_7(\xi) = \frac{6\sqrt{\omega-\alpha}e^{\theta_1\xi}}{\sqrt{3}\sqrt{\alpha}a_1\sqrt{\vartheta_1}e^{\theta_1\xi} - 1}, \tag{30}$$

where

$$\theta_1 = \frac{\sqrt{2}\sqrt{\omega-\alpha}}{\alpha\sqrt{\vartheta_2}\sqrt{\omega}}. \tag{31}$$

The parameters C_i and a_j become

$C_1 = \frac{\sqrt{2}\sqrt{\omega-\alpha}}{\alpha\sqrt{\vartheta_2\sqrt{\omega}}}$, $C_2 = -\frac{a_1\sqrt{\vartheta_1}}{\sqrt{6}\sqrt{\alpha}\sqrt{\vartheta_2\sqrt{\omega}}}$, $C_3 = 0$, $a_0 = -\frac{\sqrt{3}\sqrt{\omega-\alpha}}{\sqrt{\alpha}\sqrt{\vartheta_1}}$, $a_1 = \lambda_1$, $a_2 = a_3 = 0$, where λ_1 is arbitrary constant, hence the solution of (1) will be

$$u_7(x,t) = -\frac{1}{\left(\sqrt{3}\sqrt{\alpha}\lambda_1\sqrt{\vartheta_1}e^{\theta_1\xi}-1\right)^4}\left(6\theta_1\lambda_1\sqrt{\omega-\alpha}e^{\theta_1\xi}\right.$$
$$(3\alpha\lambda_1^2\vartheta_1 e^{2\theta_1\xi}(\alpha^2\theta_1\vartheta_2\omega-2\alpha+2\omega)+\alpha^2\theta_1\vartheta_2(-\omega)$$
$$\left.-8\sqrt{3}\sqrt{\alpha}\lambda_1\sqrt{\vartheta_1}(\alpha-\omega)e^{\theta_1\xi}-2\alpha+2\omega)\right). \tag{32}$$

3.2. Coupled Drinfel'd-Sokolov-Wilson equation

Consider the transformation

$$u(x,t) = u(\xi), \quad v(x,t) = v(\xi), \quad \xi = \alpha x - \omega t, \tag{33}$$

using (33) into (2),

$$-\omega u' + \alpha\varsigma_1 v v' = 0 \tag{34a}$$

$$\alpha\varsigma_4 u' v + \omega v' + \alpha\varsigma_3 u v' + \alpha^3\varsigma_2 v''' = 0 \tag{34b}$$

integrating (34a)

$$u = \frac{\alpha\varsigma_1 v^2}{\omega} \tag{35}$$

and substituting in (34b)

$$\alpha^2\varsigma_1(\varsigma_3+2\varsigma_4)v^2 v' + \alpha^3\varsigma_2 v^{(3)}\omega - \omega^2 v' = 0 \tag{36}$$

integrating (36)

$$\frac{1}{3}\alpha^2\varsigma_1(\varsigma_3+2\varsigma_4)v^3 + \alpha^3\varsigma_2\omega v'' - \omega^2 v = 0 \tag{37}$$

Consider the homogeneous balance between v^3 and v'', gives $n = 3$. Suppose the solution of (37), is of the form

$$u = a_0 + a_1\psi(\xi) + a_2\psi(\xi)^2 + a_3\psi(\xi)^3 \tag{38}$$

Substituting (6), (7) and (38) in (37) and collecting the coefficients of $\psi^j\psi^{(k)}$

Case I. $C_4 = 0$
(a).

$$\psi_1(\xi) = \frac{\sqrt{3}\omega\tanh\left(\frac{\xi\sqrt{\omega}}{\sqrt{2}\alpha^{3/2}\sqrt{\varsigma_2}}\right)}{\alpha a_1\sqrt{\varsigma_1}\sqrt{\varsigma_3+2\varsigma_4}}. \tag{39}$$

The parameters C_i and a_j become

$C_0 = \frac{\sqrt{\frac{3}{2}}\sqrt{\varsigma_1}\sqrt{\varsigma_3+2\varsigma_4}\omega^{3/2}}{\sqrt{\alpha}\sqrt{\varsigma_2}\left(a_1\alpha^2\varsigma_1\varsigma_3+2a_1\alpha^2\varsigma_1\varsigma_4\right)}$, $C_1 = 0$, $C_2 = -\frac{a_1\sqrt{\varsigma_1}\sqrt{\varsigma_3+2\varsigma_4}}{\sqrt{6}\sqrt{\alpha}\sqrt{\varsigma_2}\sqrt{\omega}}$, $C_3 = 0$, $a_0 = 0$, $a_1 = \lambda_1$, $a_2 = a_3 = 0$, where λ_1 is arbitrary constant, hence the solution of (2) will be

$$v_1(x,t) = \frac{\sqrt{6}\omega^{7/2}\left(\cosh\left(\frac{\sqrt{2}\xi\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}\right)-2\right)\operatorname{sech}^4\left(\frac{\xi\sqrt{\omega}}{\sqrt{2}\alpha^{3/2}\sqrt{\varsigma_2}}\right)}{\alpha^{5/2}\sqrt{\varsigma_1}\sqrt{\varsigma_2}\sqrt{\varsigma_3+2\varsigma_4}}. \tag{40}$$

(b).

$$\psi_2(\xi) = -\frac{\sqrt{3}\omega\tanh\left(\frac{\xi\sqrt{\omega}}{\sqrt{2}\alpha^{3/2}\sqrt{\varsigma_2}}\right)}{\alpha a_1\sqrt{\varsigma_1}\sqrt{\varsigma_3+2\varsigma_4}}. \tag{41}$$

The parameters C_i and a_j become

$C_0 = -\frac{\sqrt{\frac{3}{2}}\sqrt{\varsigma_1}\sqrt{\varsigma_3+2\varsigma_4}\omega^{3/2}}{\sqrt{\alpha}\sqrt{\varsigma_2}\left(a_1\alpha^2\varsigma_1\varsigma_3+2a_1\alpha^2\varsigma_1\varsigma_4\right)}$, $C_1 = 0$, $C_2 = \frac{a_1\sqrt{\varsigma_1}\sqrt{\varsigma_3+2\varsigma_4}}{\sqrt{6}\sqrt{\alpha}\sqrt{\varsigma_2}\sqrt{\omega}}$, $C_3 = 0$, $a_0 = 0$, $a_1 = \lambda_1$, $a_2 = a_3 = 0$, where λ_1 is arbitrary constant, hence the solution of (2) will be

$$v_2(x,t) = -\frac{\sqrt{6}\omega^{7/2}\left(\cosh\left(\frac{\sqrt{2}\xi\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}\right)-2\right)\operatorname{sech}^4\left(\frac{\xi\sqrt{\omega}}{\sqrt{2}\alpha^{3/2}\sqrt{\varsigma_2}}\right)}{\alpha^{5/2}\sqrt{\varsigma_1}\sqrt{\varsigma_2}\sqrt{\varsigma_3+2\varsigma_4}}. \tag{42}$$

Case II. $C_0 = 0, C_4 = 0$
(a).

$$\psi_3(\xi) = \frac{\sqrt{6\omega\sqrt{\alpha^2\varsigma_1(\varsigma_3+2\varsigma_4)}(-e^{\theta_1\xi})\left(\sqrt{3}a_2\alpha^2\varsigma_1(\varsigma_3+2\varsigma_4)e^{\theta_1\xi}+1\right)}}{\sqrt{3\alpha^4a_2^2\varsigma_1^2(\varsigma_3+2\varsigma_4)^2e^{2\theta_1\xi}-1}} \tag{43}$$

where

$$\theta_1 = \frac{\sqrt{2}\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}. \tag{44}$$

The parameters C_i and a_j become

$C_1 = \frac{\sqrt{\omega}}{\sqrt{2}\alpha^{3/2}\sqrt{\varsigma_2}}$, $C_2 = 0$, $C_3 = \frac{a_2\sqrt{\alpha^2\varsigma_1(\varsigma_3+2\varsigma_4)}}{2\sqrt{6}\alpha^{3/2}\sqrt{\varsigma_2}\sqrt{\omega}}$, $a_0 = \frac{\sqrt{3}\omega}{\sqrt{\alpha^2\varsigma_1\varsigma_3+2\alpha^2\varsigma_1\varsigma_4}}$, $a_1 = 0$, $a_2 = \lambda_1$, $a_3 = 0$, where λ_1 is arbitrary constant, hence the solution of (1) will be

$$v_3(x,t) = \frac{24\sqrt{2}\lambda_1\omega^{7/2}\sqrt{\alpha^2\varsigma_1(\varsigma_3+2\varsigma_4)}e^{\frac{\sqrt{2}\xi\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}}}{\alpha^{3/2}\sqrt{\varsigma_2}\left(\sqrt{3}\alpha^2\lambda_1\varsigma_1(\varsigma_3+2\varsigma_4)e^{\frac{\sqrt{2}\xi\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}}-1\right)^4}$$
$$\left(\alpha^2\lambda_1\varsigma_1(\varsigma_3+2\varsigma_4)e^{\frac{\sqrt{2}\xi\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}}\left(3\alpha^2\lambda_1\varsigma_1(\varsigma_3+2\varsigma_4)e^{\frac{\sqrt{2}\xi\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}}+4\sqrt{3}\right)+1\right) \tag{45}$$

(b).

$$\psi_4(\xi) = -\frac{6\omega\sqrt{\alpha^2\varsigma_1(\varsigma_3+2\varsigma_4)}e^{\theta_1\xi}}{\sqrt{3}a_1\alpha^2\varsigma_1\varsigma_3e^{\theta_1\xi}+2\sqrt{3}a_1\alpha^2\varsigma_1\varsigma_4e^{\theta_1\xi}-1}, \tag{46}$$

where

$$\theta_1 = \frac{\sqrt{2}\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}. \tag{47}$$

The parameters C_i and a_j become

$C_1 = \frac{\sqrt{2}\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}$, $C_2 = \frac{a_1\sqrt{\alpha^2\varsigma_1(\varsigma_3+2\varsigma_4)}}{\sqrt{6}\alpha^{3/2}\sqrt{\varsigma_2}\sqrt{\omega}}$, $C_3 = 0$, $a_0 = \frac{\sqrt{3}\omega}{\sqrt{\alpha^2\varsigma_1\varsigma_3+2\alpha^2\varsigma_1\varsigma_4}}$, $a_1 = -\lambda_1$, $a_2 = a_3 = 0$ where λ_1 is arbitrary constant, hence the solution of (1) will be

$$v_4(x,t) = -\frac{24\sqrt{2}\lambda_1\omega^{7/2}\sqrt{\alpha^2\varsigma_1(\varsigma_3+2\varsigma_4)}e^{\frac{\sqrt{2}\xi\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}}}{\alpha^{3/2}\sqrt{\varsigma_2}\left(\sqrt{3}\alpha^2\lambda_1\varsigma_1(\varsigma_3+2\varsigma_4)e^{\frac{\sqrt{2}\xi\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}}+1\right)^4}$$
$$\left(\alpha^2\lambda_1\varsigma_1(\varsigma_3+2\varsigma_4)e^{\frac{\sqrt{2}\xi\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}}\left(3\alpha^2\lambda_1\varsigma_1(\varsigma_3+2\varsigma_4)e^{\frac{\sqrt{2}\xi\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}}-4\sqrt{3}\right)+1\right) \tag{48}$$

Case III. $C_3 = 0, C_4 = 0$
(a).

$$\psi_5(\xi) = \frac{6\omega\sqrt{\alpha^2\varsigma_1(\varsigma_3+2\varsigma_4)}e^{\theta_1\xi}}{\sqrt{3}a_1\alpha^2\varsigma_1\varsigma_3e^{\theta_1\xi}+2\sqrt{3}a_1\alpha^2\varsigma_1\varsigma_4e^{\theta_1\xi}-1} \tag{49}$$

where

$$\eta_1 = \frac{\sqrt{2}\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}. \tag{50}$$

The parameters C_i and a_j become

$C_0 = 0,\ \ C_1 = \frac{\sqrt{2}\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}},\ \ C_2 = -\frac{a_1\sqrt{\alpha^2\varsigma_1(\varsigma_3+2\varsigma_4)}}{\sqrt{6}\alpha^{3/2}\sqrt{\varsigma_2}\sqrt{\omega}},\ \ a_0 = -\frac{\sqrt{3}\omega}{\sqrt{\alpha^2\varsigma_1\varsigma_3+2\alpha^2\varsigma_1\varsigma_4}},$ $a_1 = \lambda_1$, $a_2 = a_3 = 0$ where λ_1 is arbitrary constant, hence the solution of (2) will be

$$v_5(x,t) = \frac{24\sqrt{2}\lambda_1\omega^{7/2}\sqrt{\alpha^2\varsigma_1(\varsigma_3+2\varsigma_4)}e^{\frac{\sqrt{2}\xi\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}}}{\alpha^{3/2}\sqrt{\varsigma_2}\left(\sqrt{3}\alpha^2\lambda_1\varsigma_1(\varsigma_3+2\varsigma_4)e^{\frac{\sqrt{2}\xi\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}}-1\right)^4} \left(\alpha^2\lambda_1\varsigma_1(\varsigma_3+2\varsigma_4)e^{\frac{\sqrt{2}\xi\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}}(3\alpha^2\lambda_1\varsigma_1(\varsigma_3+2\varsigma_4)e^{\frac{\sqrt{2}\xi\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}} (3\sqrt{3}\alpha^2\lambda_1\varsigma_1(\varsigma_3+2\varsigma_4)e^{\frac{\sqrt{2}\xi\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}}+2)-4\sqrt{3})-1\right), \tag{51}$$

(b).

$$\psi_6(\xi) = \frac{\sqrt{3}\omega\tanh\left(\frac{\xi\sqrt{\omega}}{\sqrt{2}\alpha^{3/2}\sqrt{\varsigma_2}}\right)}{\alpha a_1\sqrt{\varsigma_1}\sqrt{\varsigma_3+2\varsigma_4}} - \frac{a_0}{a_1} \tag{52}$$

The parameters C_i and a_j become

$C_0 = \frac{3\omega^2-\alpha^2a_0^2\varsigma_1(\varsigma_3+2\varsigma_4)}{\sqrt{6}\alpha^{5/2}a_1\sqrt{\varsigma_1}\sqrt{\varsigma_2}\sqrt{\varsigma_3+2\varsigma_4}\sqrt{\omega}},\qquad C_1 = -\frac{\sqrt{\frac{2}{3}}a_0\sqrt{\varsigma_1}\sqrt{\varsigma_3+2\varsigma_4}}{\sqrt{\alpha}\sqrt{\varsigma_2}\sqrt{\omega}},$

$C_2 = -\frac{a_1\sqrt{\varsigma_1}\sqrt{\varsigma_3+2\varsigma_4}}{\sqrt{6}\sqrt{\alpha}\sqrt{\varsigma_2}\sqrt{\omega}}$, $a_0 = \lambda_1$, $a_1 = \lambda_2$, $a_2 = a_3 = 0$ where λ_1 and λ_2 are arbitrary constants, hence the solution of (2) will be

$$v_6(x,t) = \frac{\sqrt{6}\omega^{7/2}\left(\cosh\left(\frac{\sqrt{2}\xi\sqrt{\omega}}{\alpha^{3/2}\sqrt{\varsigma_2}}\right)-2\right)\mathrm{sech}^4\left(\frac{\xi\sqrt{\omega}}{\sqrt{2}\alpha^{3/2}\sqrt{\varsigma_2}}\right)}{\alpha^{5/2}\sqrt{\varsigma_1}\sqrt{\varsigma_2}\sqrt{\varsigma_3+2\varsigma_4}}, \tag{53}$$

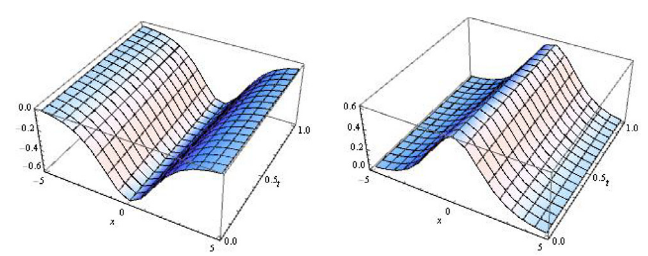
therefore, for $j = 1, 2, \ldots 6$

$$u_j(x,t) = \frac{\alpha\varsigma_1 v_j^2(x,t)}{\omega} \tag{54}$$

4. Discussions and results

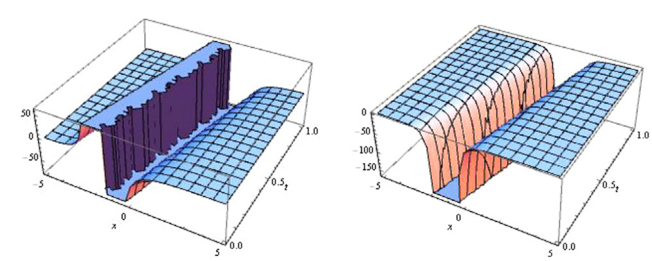
The graphical representation of solitons have been illustrated in the following figures, for various values of the parameters. Mathematica 10.4 is used to carried out simulations and to visualize the behavior of nonlinear waves. In Case I, the solution for the Eq. (1), is shown in Fig. 1 obtained from the Eq. (14) with $\vartheta_1 = 1$, $\vartheta_2 = 2$, $\alpha = 4$, $\omega = 2$, and Eq. (17) with $\vartheta_1 = 1$, $\vartheta_2 = 2$, $\alpha = 4$, $\omega = 2$, while in Case II, the solution for the Eq. (1), is shown in Fig. 2 obtained from the Eq. (20) with $\vartheta_1 = 3$, $\vartheta_2 = -1$, $\alpha = 4$, $\omega = -2$, Eq. (23) with $\vartheta_1 = 1$, $\vartheta_2 = -1$, $\alpha = 2$, $\omega = -3$ and Eq. (26) with $\vartheta_1 = 1$, $\vartheta_2 = -1$, $\alpha = 1$, $\omega = -1$. Moreover, in Case III, the solution for the Eq. (1), is shown in Fig. 3 obtained from the Eq. (29) with $\vartheta_1 = 3$, $\vartheta_2 = 2$, $\lambda_1 = 1$, $\alpha = 1$, $\omega = 2$ and Eq. (32) with $\vartheta_1 = 1$, $\vartheta_2 = 1$, $\lambda_1 = 2$, $\alpha = 1$, $\omega = 2$.

Similarly, the solution to the Eq. (2) for Case I, is shown in Fig. 4 obtained from the Eq. (40) with $\varsigma_1 = 1$, $\varsigma_2 = 1$, $\varsigma_3 = 1$, $\varsigma_4 = 1$, $\alpha = 2$, $\omega = 3$ and Eq. (42) with $\varsigma_1 = 1$, $\varsigma_2 = 2$, $\varsigma_3 = 3$, $\varsigma_4 = 4$, $\alpha = 2$, $\omega = 3$, while In Case II, the solution for the Eq. (2), is shown in Fig. 5 obtained from the Eqs. (45) with $\varsigma_1 = 1$, $\varsigma_2 = 1$, $\varsigma_3 = 1$, $\varsigma_4 = 1$, $\lambda_1 = 1$, $\alpha = 1$, $\omega = 1$ and Eq. (48) with $\varsigma_1 = 1$, $\varsigma_2 = -1$, $\varsigma_3 = 2$, $\varsigma_4 = 3$, $\lambda_1 = 1$, $\alpha = -1$, $\omega = 2$. Moreover, in Case III, the solution for the Eq. (2), is shown in Fig. 6 obtained from the Eq. (51) with $\varsigma_1 = 1$, $\varsigma_2 = 1$, $\varsigma_3 = 1$, $\varsigma_4 = 1$, $\lambda_1 = 1$, $\alpha = 1$, $\omega = 1$ and Eq. (53) with $\varsigma_1 = 1$, $\varsigma_2 = -5$, $\varsigma_3 = 4$, $\varsigma_4 = 3$, $\alpha = 5$, $\omega = -2$.



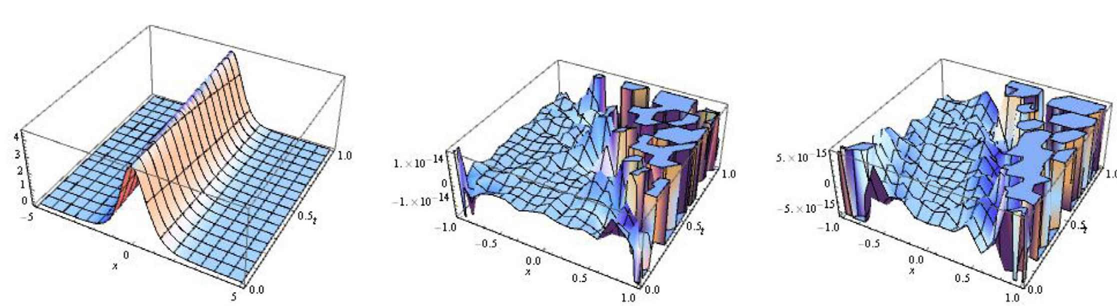
(a) $u_1(x,t)$: $\vartheta_1=1$, $\vartheta_2=2$, $\alpha=4$, $\omega=2$ (b) $u_2(x,t)$: $\vartheta_1=1$, $\vartheta_2=2$, $\alpha=4$, $\omega=2$

Fig. 1. mBBM equation (Case I).



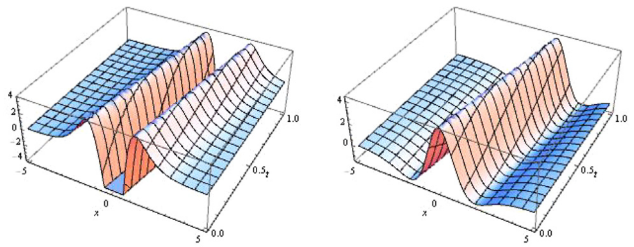
(a) $u_6(x,t)$: $\vartheta_1=3$, $\vartheta_2=2$, $\lambda_1=1$, $\alpha=1$, $\omega=2$ (b) $u_7(x,t)$: $\vartheta_1=1$, $\vartheta_2=1$, $\lambda_1=2$, $\alpha=1$, $\omega=2$

Fig. 3. mBBM equation (Case III).



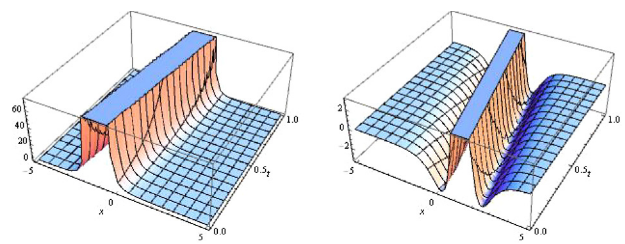
(a) $u_3(x,t)$: $\vartheta_1=3$, $\vartheta_2=-1$, $\alpha=4$, $\omega=-2$ (b) $u_4(x,t)$: $\vartheta_1=1$, $\vartheta_2=-1$, $\alpha=2$, $\omega=-3$ (c) $u_5(x,t)$: $\vartheta_1=1$, $\vartheta_2=-1$, $\alpha=1$, $\omega=-1$

Fig. 2. mBBM equation (Case II).



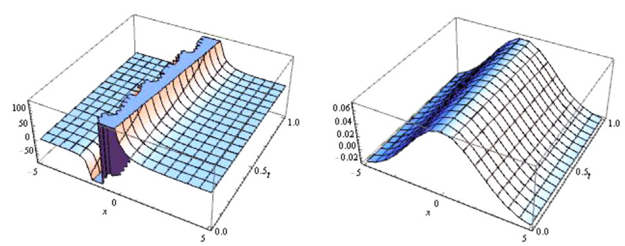
(a) $v_1(x,t)$: $\varsigma_1=1$, $\varsigma_2=1$, $\varsigma_3=1$, $\varsigma_4=1$, $\alpha=2$, $\omega=3$ (b) $v_2(x,t)$: $\varsigma_1=1$, $\varsigma_2=2$, $\varsigma_3=3$, $\varsigma_4=4$, $\alpha=2$, $\omega=3$

Fig. 4. cDSW system (Case I).



(a) $v_3(x,t)$: $\varsigma_1=1$, $\varsigma_2=1$, $\varsigma_3=1$, $\varsigma_4=1$, $\lambda_1=1$, $\alpha=1$, $\omega=1$ (b) $v_4(x,t)$: $\varsigma_1=1$, $\varsigma_2=-1$, $\varsigma_3=2$, $\varsigma_4=3$, $\lambda_1=1$, $\alpha=-1$, $\omega=2$

Fig. 5. cDSW system (Case II).



(a) $v_5(x,t)$: $\varsigma_1=1$, $\varsigma_2=1$, $\varsigma_3=1$, $\varsigma_4=1$, $\lambda_1=1$, $\alpha=1$, $\omega=1$ (b) $v_6(x,t)$: $\varsigma_1=1$, $\varsigma_2=-5$, $\varsigma_3=4$, $\varsigma_4=3$, $\alpha=5$, $\omega=-2$

Fig. 6. cDSW system (Case III).

5. Conclusion

The aim of the study is to find some new traveling-wave solutions for modified Benjamin-Bona-Mahony and Drinfel'd-Sokolov-Wilson equations. It is observed that the auxiliary equation method is one of the most powerful tools to find a variety of analytical solutions for more complex problems. Depending on the real parameters, a collection of new exact solutions are obtained, for details see Figs. 1–6 These results are very auspicious for further investigation and stances on a strong basis for the solution of NPDEs.

References

Abazari, R., 2010. Application of (G'/G)-expansion method to travelling wave solutions of three nonlinear evolution equation. Comput. Fluids 39, 1957–1963.

Ablowitz, M.J., Clarkson, P.A., 1991. Soliton, Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press, USA.

Abu Arqub, O., 2017a. Numerical solutions for the Robin time-fractional partial differential equations of heat and fluid flows based on the reproducing kernel algorithm. Int. J. Numer. Meth. Heat Fluid Flow. https://doi.org/10.1108/HFF-07-2016-0278.

Abu Arqub, O., 2017b. Fitted reproducing kernel Hilbert space method for the solutions of some certain classes of time-fractional partial differential equations subject to initial and Neumann boundary conditions. Comput. Math. Appl. 73, 1243–1261.

Abu Arqub, O., El-Ajou, A., Momani, S., 2015. Constructing and predicting solitary pattern solutions for nonlinear time-fractional dispersive partial differential equations. J. Comput. Phys. 293, 385–399.

Chen, Yong, Yan, Zhenya, Zhan, Honging, 2003. New explicit solitary wave solutions for (2 + 1)-dimensional Boussinesq equation and (3 + 1)-dimensional KP equation. Phys. Lett. A 307, 107–113.

El-Ajou, A., Abu Arqub, O., Momani, S., Baleanu, D., Alsaedi, A., 2015a. A novel expansion iterative method for solving linear partial differential equations of fractional order. Appl. Math. Comput. 257, 119–133.

El-Ajou, A., Abu Arqub, O., Momani, S., 2015b. Approximate analytical solution of the nonlinear fractional KdV-Burgers equation: a new iterative algorithm. J. Comput. Phys. 293, 81–95.

Gardner, C., Greene, J., Kruskal, M., Miura, R., 1967. Method for solving the Korteweg-de Vries equation. Phys. Rev. Lett. 19, 1095–1097.

Grey Jacob, Michael M. Tom, 2014. On the Solutions of the BBM-KP and BBM model Equations, oai:arXiv.org: 1410.3158.

Hirota, R., 1971. Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons. Phys. Rev. Lett. 27, 1192–1194.

Ito, M., 1980. An extension of nonlinear evolution equation of the K-dV(mKdV) type to higher order. J. Phys. Soc. Jpn. 49, 771–778.

Kalim, U.T., Seadawy, A.R., 2017. Bistable Bright-Dark Solitary Wave Solutions of the (3 + 1)-Dimensional Breaking Soliton, Boussinesq Equation with Dual Dispersion and Modified KdV-KP equations and their applications. Results Phys. 7, 1143–1149.

Kalim, U.T., Seadawy, A.R., 2019. Soliton Solution of (3+1)-Dimensional KdV-BBM, KP-BBM and mKdV-ZK equations and their applications in water waves. J. King Saud Univ.- Sci. 31, 8–13.

Kalim, U.T., Younis, M., 2017. Bright, dark and other optical solitons with second order spaotetimporal dispersion. Optik 142, 446–450.

Khater, A.H., Callebaut, D.K., Seadawy, A.R., 2000. General soliton solutions of an n-dimensional Complex Ginzburg-Landau equation. Phys. Scr. 62, 353–357.

Khater, A.H., Callebaut D.K., Seadawy A.R., 2003. Nonlinear Dispersive Kelvin-Helmholtz Instabilities in Magnetohydrodynamic Flows Presented in International Conference on Computational and Applied Mathematics (Belgium) ICCAM 2000 and published in Physica Scripta, 67, 340–349.

Khater, A.H., Callebaut, D.K., Helal, M.A., Seadawy, A.R., 2006. Variational method for the nonlinear dynamics of an elliptic magnetic stagnation line. Eur. Phys. J. D 39, 237–245.

Kutluay, S., Esen, A., Tasbozan, O., 2010. The (G'/G)-expansion method for some nonlinear evolution equations. Appl. Math. Comput. 217, 384–391.

Liang, Yuan, 2014. Exact solutions of the (3 + 1)-dimensional modified KdV-Zakharov-Kuznetsev equation and Fisher equations using the modified simple equation method. J. Interdisciplinary Math. 17, 565–578.

Liu, S.K., Fu, Z.T., Liu, S.D., Zhao, Q., 2001. Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. Phys. Lett. A 289, 69–74.

Lu, Dianchen, Seadawy, A.R., Arshad, M., 2017. Applications of extended simple equation method on unstable nonlinear Schrödinger equations. Optik 140, 136–144.

Mohapatra, S.C., Guedes Soares, C., 2015. Comparing solutions of the coupled Boussinesq equations in shallow water. In: Guedes Soares, Santos (Eds.), Maritime Technology and Engineering. Taylor & Francis Group, London, ISBN 978-1-138-02727-5.

Seadawy, A.R., 2012a. Exact solutions of a two-dimensional nonlinear Schrödinger equation. Appl. Math. Lett. 25, 687–691.

Seadawy, A.R., 2012b. Traveling wave solutions of the Boussinesq and generalized fifth-order KdV equations by using the direct algebraic method. Appl. Math. Sci. 6 (82), 4081–4090.

Seadawy, A.R., 2016. Three-dimensional nonlinear modified Zakharov-Kuznetsov equation of ion-acoustic waves in a magnetized plasma. Comput. Math. Appl. 71, 201–212.

Seadawy, A.R., 2016a. Stability analysis solutions for nonlinear three-dimensional modified Korteweg-de Vries-Zakharov-Kuznetsov equation in a magnetized electron-positron plasma. Physica A 455, 44–51.

Seadawy, A.R., 2016b. Stability analysis of traveling wave solutions for generalized coupled nonlinear KdV equations. Appl. Math. Inf. Sci. 10, 209–214.

Seadawy, Aly R., 2017. Solitary wave solutions of tow-dimensional nonlinear Kadomtsev-Petviashvili dynamic equation in a dust acoustic plasmas. The Pramana – J. Phys. 89 (3). 49:1–11.

Seadawy, Aly R., 2017. Two-dimensional interaction of a shear flow with a free surface in a stratified fluid and its a solitary wave solutions via mathematical methods. Eur. Phys. J. Plus 132. 518:12, 1–11.

Seadawy, Aly R., 2018. Three-dimensional weakly nonlinear shallow water waves regime and its travelling wave solutions. Int. J. Comput. Methods 15 (1).

Seadawy, Aly R., Alamri, Sultan Z., 2018. Mathematical methods via the nonlinear two-dimensional water waves of Olver dynamical equation and its exact solitary wave solutions. Results Phys. 8, 286–291.

Seadawy, A.R., El-Rashidy, K., 2013. Traveling wave solutions for some coupled nonlinear evolution equations. Math. Comput. Model. 57, 1371–1379.

Seadawy, A.R., Lu, Dianchen, 2017. Bright and dark solitary wave soliton solutions for the generalized higher order nonlinear Schrödinger equation and its stability. Results Phys. 7, 43–48.

Seadawy, A.R., El-Kalaawy, O.H., Aldenari, R.B., 2016. Water wave solutions of Zufiria?s higher-order Boussinesq type equations and its stability. Appl. Math. Comput. 280, 57–71.

Su, C.H., Gardner, C.S., 1969. Drivation of the Korteweg-de Vries and Burgers Equation. J. Math. Phys. 10, 536–539.

Wang, M.L., 1995. Solitary wave solutions for variant Boussinesq equations. Phys. Lett. A 199, 169–172.

Wena, Zhenshu, Liu, Zhengrong, Song, Ming, 2009. New exact solutions for the classical Drinfel'd-Sokolov-Wilson equation. Appl. Math. Comput. 215, 2349–2358.

Xu, G.Q., Li, Z.B., 2005. Exact travelling wave solutions of the Whitham-Broer-Kaup and Broer-Kaup-Kupershmidt equations. Chaos, Solitons Fractals 24, 549–556.

Yan, Fang, Hua, Cuncai, Liu, Haihong, Liu, Zengrong, 2012. The exact traveling wave solutions and their bifurcations in the Gardner and Gardner-KP Equations. Int. J. Bifurcation Chaos 22 (5), 125–126.

Zhibin, L., Mingliang, W., 1993. Travelling wave solutions to the two-dimensional Kdv-Burgers equation. J. Phys. A: Math. Gen. 26, 6027–6031.

Zhou, Y.B., Wang, M.L., Wang, Y.M., 2003. Periodic wave solutions to a coupled KdV equations with variable coefficients. Phys. Lett. A 308, 31–36.