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Strong GP-continuity and weakly GP-closed functions on GPT spaces

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ABSTRACT

In this research article, we present the procedure of generating GPT spaces in two different ways: using the generalized neighbourhood system and the monotonic operator. Then, we introduce several types of generalized primal continuous functions. Some characteristics have been dissected, and the relationships among them have been studied. We use the technique of Császár, which changes the "generalized topology" to other "generalized topologies" weaker than it, to show some important results. Furthermore, we show that the notion of "strong GP-continuity" coincides with the notion "GP-continuity" under some conditions. We present these results on a simple graph to make it easier for the reader. Finally, study the preservation of the notions of "GP-connected" and "GP-hyperconnected" by different types of generalized primal continuous functions.

1. Introduction

Continuity has been a core concept in the branch of pure mathematics in general and in the theory of topology. A function between two topological spaces is continuous if the pre-image of every open set is open. To judge whether a function is continuous, we need to examine the structure of the spaces. This nice kind of function is quite important for the role that it plays in preserving the topological structure of the domain space in the co-domain space.

In Császár (1997) the concept of generalization was established when Császár introduced the concept of generalized open set, which led to the definition of a new mathematical structure weaker than the topology, that is, having relaxing conditions. Following the scientific approach, any "generalization" to the basic concepts in the "theory of topology" must be studied. This happened when many articles drew attention to this generalized space and examined every topological characteristic. Moreover, the notion of "generalized continuous" functions between two generalized topological spaces is defined.

In Al-Saadi and Al-Malki (2023), the same methodology was followed. The author gave a new mathematical structure that connected the generalized topological properties with the primal set $\mathcal{P}$ given on the same set. The basic definitions were given, and some operators with nice behaviour are present in Al-Saadi and Al-Malki (2024).

In this paper, we continue our study of this new space. The study focuses on "generalized continuity" under the influence of the primal set. The topic of continuity is one of the major topics in topology theory, and for that, we investigate in depth many types of continuous functions. These types are already studied in the theory of "generalization". Here we will study them in the sense of primal collection. Also, we will study the relationship between them and state the necessary conditions to develop some weak kinds of them into strong types. The importance of continuous functions in topology comes from their effective role in preserving topological properties.

This article contains five sections. Section 1 is divided in two parts: the first one summarizes the previous studies in this research; the second part is recalling the basic definitions and the fundamental theorems. In Section 2, we present three types of continuous functions on GPT spaces. First, we generate GPT spaces in two different ways. One by using the generalized neighbourhood system, and the other by using the monotonic operator. Then, we give the definitions of GPN-continuous, GP-θ-continuous, and almost GP-continuous; and we discuss their characteristics. The relationship among them and counterexamples have been studied. In Section 3, we introduce the notions of "strong GP-continuous function", "strongly GP-θ-continuous", and "super GP-continuous". Then, study the relationship among them. Also, we present the concepts of "weakly GP-closed function" and "GP-regular space"; and we discuss their characteristics. In Section 4, we present the notions of "GP-connected" and "GP-hyperconnected". Then, we study the preservation of these notions by different types of generalized primal continuous functions. Section 5 is a discussion of all theimportant results that we present.

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1.1. Literature review

In Császár (2002) the notion of "generalized topology" was presented. The collection $\mathfrak{g}$ of the power set 2^T of a non-empty set T is named a generalized topology GT if the empty set belongs to $\mathfrak{g}$, and the countable union of every set that belongs to $\mathfrak{g}$ also belongs to $\mathfrak{g}$. Thus, the pair $(T, \mathfrak{g})$ is assumed to be a generalized topological space, denoted by GT space.

According to Császár (2008b), this space member is named $\mathfrak{g}$-open sets; however, $\mathfrak{g}$-closed means its complements. Also, $C_{\mathfrak{g}}(T)$ refers to the whole set of $\mathfrak{g}$-closed sets, whereas $c_{\mathfrak{g}}(M)$ and $i_{\mathfrak{g}}(M)$ mean the closure and interior of $M \subset T$, respectively, which are defined as in the general case. Studies continued in this generalized space. Császár (2004a) presented the concept of topological disconnection under the influence of generalization, while Császár (2005) presented the concept of generalized continuous functions. Császár (2008a) introduced the notion of "strong generalized topological space", which is given as follows: A generalized topology is named strong if $T \in \mathfrak{g}$.

Moreover, this field has received a lot of attention since the literature has studied and examined its topological characteristics in detail. For example, Császár (2004b) gave the separation axioms by replacing the notion of open sets with a more general expression, while a lot of research gave the definition of generalized continuity with different types of more general kinds of continuity, for example, Min (2009b) presented the notion of almost continuity, Min (2009a) introduced the concept of $\theta(\mathfrak{g}, \mathfrak{g}^\backprime)$-continuity, and Jayanthi (2012) gave the definition of contra continuous function.

In another direction, literature has spread that presents new mathematical structures as useful tools that lead to a deeper understanding and broader application of the concepts of topological spaces in various fields. For example, the mathematical structure "ideal" in Janković and Hamlett (1990) and its dual structure "filter" in Kuratowski (2014). These two structures have applications in science and society. The notion of ideal m-space is given in Al-Omari and Noiri (2012), including some type of operator.

On the same approach, the notion of "grill" appeared in Choquet (1947). The associated topology for the grill is given in Roy and Mukherjee (2007). New types of mapping were presented and compactness in the light of a grill in Roy et al. (2008). The generalized type of continuity and its decomposition via grill set are defined and studied in Hatir and Jafari (2010).

In 2022, the dual structure of the "grill" is defined and named "primal" in Acharjee et al. (2022). A family $\mathcal{P} \subseteq 2^T$ is called a primal over T, if the next are satisfied for $M, N \subseteq T$: (i) $T \notin \mathcal{P}$. (ii) Whenever $M \subseteq N$ and $N \in \mathcal{P}$, thus $M \in \mathcal{P}$. (iii) Whenever $M \cap N \in \mathcal{P}$, thus $M \in \mathcal{P}$ or $N \in \mathcal{P}$. Some operators are given via a "primal set" and presented in AL-Omari et al. (2022). Also, Modak (2013) follows the same approach and definition of new space via the notion of filter and grill. In Al-shami et al. (2023) a new space is defined by joining the nice properties of primal and soft sets.

1.2. GPT *spaces*

Through this part, we will review the basic definitions of GPT spaces, that are presented in Al-Saadi and Al-Malki (2023) and Al-Saadi and Al-Malki (2024)

Definition 1. The triple $(T, \mathfrak{g}, \mathcal{P})$ refers to the generalized primal topological space GPT space for short. Moreover, $(\mathfrak{g}, \mathcal{P})$-open sets is the symbol for the element of this space, and $(\mathfrak{g}, \mathcal{P})$-closed sets denote their complement. The entire set of $(\mathfrak{g}, \mathcal{P})$-closed set is symbolized by $C_{(\mathfrak{g},\mathcal{P})}(T)$ and $cl_{(\mathfrak{g},\mathcal{P})}(M)$ denotes the closure of $M \subseteq T$, that is qualified as in the general situation.

Definition 2. Let $(T, \mathfrak{g}, \mathcal{P})$ be a GPT space. Consider an operator $\psi : T \to 2^{2^T}$ given as $t \in O$, for all $O \in \psi(t)$. Hence, $O \in \psi(t)$ is a generalized primal neighbourhood or GPN for $t \in T$, and ψ is a generalized primal neighbourhood system over T or GPN system. The set of all GPN systems on T is denoted by $\Psi(T)$.

Definition 3. Let $(T, \mathfrak{g}, \mathcal{P})$ be a GPT space with $M \subseteq T$. The operator $(.)^\diamond : 2^T \to 2^T$ is defined by

$$M^\diamond(T, \mathfrak{g}, \mathcal{P}) = \{t \in T : M^c \cup O^c \in \mathcal{P}, \forall\ O \in \psi(t)\}.$$

Theorem 4. *Let $(T, \mathfrak{g}, \mathcal{P})$ be a* GPT *space. The next holds for $M, N \subseteq T$.*

(i) $\phi^\diamond = \phi$.
(ii) $M^\diamond$ *is $(\mathfrak{g}, \mathcal{P})$-closed i.e., $cl_{(\mathfrak{g},\mathcal{P})}(M^\diamond) = M^\diamond$.*
(iii) $(M^\diamond)^\diamond \subseteq M^\diamond$.
(iv) $M^\diamond \subseteq N^\diamond$, *whenever,* $M \subseteq N$.
(v) ${}^\diamond M \cup N^\diamond = (M \cup N)^\diamond$.
(vi) $(M \cap N)^\diamond \subseteq M^\diamond \cap N^\diamond$.

Definition 5. Let $(T, \mathfrak{g}, \mathcal{P})$ be a GPT space with $M \subseteq T$. We define the operator $cl^\diamond : 2^T \to 2^T$ by $cl^\diamond(M) = M \cup M^\diamond$.

Theorem 6. *Let $(T, \mathfrak{g}, \mathcal{P})$ be a* GPT *space. The next holds for $M, N \subseteq T$.*

(i) $cl^\diamond(\phi) = \phi$,
(ii) $M \subseteq cl^\diamond(M)$,
(iii) $cl^\diamond(cl^\diamond(M)) = cl^\diamond(M)$,
(iv) $cl^\diamond(M) \subseteq cl^\diamond(N)$, *whenever* $M \subseteq N$,
(v) $cl^\diamond(M) \cup cl^\diamond(N) = cl^\diamond(M \cup N)$.

Theorem 7. *Let $(T, \mathfrak{g}, \mathcal{P})$ be a* GPT *space with $M, N \subseteq T$. We have $M \cap N^\diamond \subseteq (M \cap N)^\diamond$, whenever M is $(\mathfrak{g}, \mathcal{P})$-open.*

Definition 8. Let $(T, \mathfrak{g}, \mathcal{P})$ be a GPT space and $M \subseteq T$.

(i) Whenever $M \subseteq cl^\diamond(i_{\mathfrak{g}} M)$, M is called $(\mathfrak{g}, \mathcal{P})$-semi-open set.
(ii) Whenever $M \subseteq i_{\mathfrak{g}}(cl^\diamond(M))$, M is called $(\mathfrak{g}, \mathcal{P})$-pre-open set.
(iii) Whenever $M = i_{\mathfrak{g}}(cl^\diamond(M))$, M is called $(\mathfrak{g}, \mathcal{P})$-regular open set.
(iv) Whenever $M \subseteq c_{\mathfrak{g}}(i_{\mathfrak{g}}(cl^\diamond(M)))$, M is called $(\mathfrak{g}, \mathcal{P})$-$\beta$-open set.
(v) Whenever $M \subseteq i_{\mathfrak{g}}(cl^\diamond(i_{\mathfrak{g}}(M)))$, M is called $(\mathfrak{g}, \mathcal{P})$-$\alpha$-openset.

The entire set of $(\mathfrak{g}, \mathcal{P})$-semi-opensets is denoted via $\mathcal{S}\mathfrak{g}_{\mathcal{P}}$, and the entire set of $(\mathfrak{g}, \mathcal{P})$-pre-open setsis denoted via $\mathcal{P}\mathfrak{g}_{\mathcal{P}}$. Moreover, the entire set of $(\mathfrak{g}, \mathcal{P})$-$\alpha$-open sets is denoted via $\alpha\mathfrak{g}_{\mathcal{P}}$, while $\beta\mathfrak{g}_{\mathcal{P}}$ represents the whole set of $(\mathfrak{g}, \mathcal{P})$-$\beta$-open sets. Finally, $\mathcal{R}\mathfrak{g}_{\mathcal{P}}$ represents the whole set of $(\mathfrak{g}, \mathcal{P})$-regularopen sets.

Definition 9. Let $(T, \mathfrak{g}, \mathcal{P})$ be a GPT space and $M \subseteq T$. If $(T \setminus M)$ is a $(\mathfrak{g}, \mathcal{P})$-semi-open (resp. $(\mathfrak{g}, \mathcal{P})$-pre-open, $(\mathfrak{g}, \mathcal{P})$-regular open, $(\mathfrak{g}, \mathcal{P})$-$\alpha$-open, $(\mathfrak{g}, \mathcal{P})$-$\beta$-open), hence M is called a $(\mathfrak{g}, \mathcal{P})$-semi-closed (resp. $(\mathfrak{g}, \mathcal{P})$-pre-closed, $(\mathfrak{g}, \mathcal{P})$-regular closed, $(\mathfrak{g}, \mathcal{P})$-$\alpha$-closed, $(\mathfrak{g}, \mathcal{P})$-$\beta$-closed).

Proposition 10. *Let $(T, \mathfrak{g}, \mathcal{P})$ be a* GPT *space. We have:*

(i) $\bigcup_{\delta\in\Delta} M_\delta$ *such that each* $M_\delta \in \mathcal{S}\mathfrak{g}_{\mathcal{P}}$ *also belongs to* $\mathcal{S}\mathfrak{g}_{\mathcal{P}}$.
(ii) $\bigcup_{\delta\in\Delta} M_\delta$ *such that each* $M_\delta \in \mathcal{P}\mathfrak{g}_{\mathcal{P}}$ *also belongs to* $\mathcal{P}\mathfrak{g}_{\mathcal{P}}$.
(iii) $\bigcup_{\delta\in\Delta} M_\delta$ *such that each* $M_\delta \in \alpha\mathfrak{g}_{\mathcal{P}}$ *also belongs to* $\alpha\mathfrak{g}_{\mathcal{P}}$.
(iv) $\bigcup_{\delta\in\Delta} M_\delta$ *such that each* $M_\delta \in \beta\mathfrak{g}_{\mathcal{P}}$ *also belongs to* $\beta\mathfrak{g}_{\mathcal{P}}$.
(v) $\bigcup_{\delta\in\Delta} M_\delta$ *such that each* $M_\delta \in \mathcal{R}\mathfrak{g}_{\mathcal{P}}$ *also belongs to* $\mathcal{R}\mathfrak{g}_{\mathcal{P}}$.

Corollary 11. *Let $(T, \mathfrak{g}, \mathcal{P})$ be a* GPT *space. All of the families $\mathcal{S}\mathfrak{g}_{\mathcal{P}}, \mathcal{P}\mathfrak{g}_{\mathcal{P}}, \alpha\mathfrak{g}_{\mathcal{P}}, \beta\mathfrak{g}_{\mathcal{P}}$ and $\mathcal{R}\mathfrak{g}_{\mathcal{P}}$ constitute a* GPT *space with a primal set $\mathcal{P}$ over T.*

Definition 12. Let $(T, \mathfrak{g}, \mathcal{P})$ be a GPT space and $(T^\backprime, \mathfrak{g}^\backprime)$ is GT space. The mapping $p : T \to T^\backprime$ is called $(\mathfrak{g}, \mathcal{P})$-semi-continuous (resp. $(\mathfrak{g}, \mathcal{P})$-pre-continuous, $(\mathfrak{g}, \mathcal{P})$-$\alpha$-continuous, $(\mathfrak{g}, \mathcal{P})$-$\beta$-continuous)if $p^{-1}(M)$, for all M is $\mathfrak{g}$-open is $(\mathfrak{g},\mathcal{P})$-semi-open (resp. $(\mathfrak{g},\mathcal{P})$-pre-open, $(\mathfrak{g},\mathcal{P})$-$\alpha$-open, $(\mathfrak{g},\mathcal{P})$-$\beta$-open).

Theorem 13. *Let $(T,\mathfrak{g},\mathcal{P})$ be a GPT space and $M \subseteq T$. We find:*

(i) $M \in \alpha\mathfrak{g}_{\mathcal{P}}$ if and only if $M \in \mathcal{S}\mathfrak{g}_{\mathcal{P}}$ and $M \in \mathcal{P}\mathfrak{g}_{\mathcal{P}}$.
(ii) If $M \in \mathcal{S}\mathfrak{g}_{\mathcal{P}}$, then $M \in \beta\mathfrak{g}_{\mathcal{P}}$.
(iii) If $M \in \mathcal{P}\mathfrak{g}_{\mathcal{P}}$, then $M \in \beta\mathfrak{g}_{\mathcal{P}}$.

Corollary 14. *Let $(T,\mathfrak{g},\mathcal{P})$ be a GPT space. We have:*

(i) $\mathcal{P}\mathfrak{g}_{\mathcal{P}} \cap \mathcal{S}\mathfrak{g}_{\mathcal{P}} = \alpha\mathfrak{g}_{\mathcal{P}}$.
(ii) $\mathfrak{g}-open \subset \alpha\mathfrak{g}_{\mathcal{P}} \subset \mathcal{S}\mathfrak{g}_{\mathcal{P}} \subset \beta\mathfrak{g}_{\mathcal{P}}$.
(iii) $\alpha\mathfrak{g}_{\mathcal{P}} \subset \mathcal{P}\mathfrak{g}_{\mathcal{P}} \subset \beta\mathfrak{g}_{\mathcal{P}}$.

Definition 15. Let $(T,\mathfrak{g},\mathcal{P})$ be a GPT space. Consider $M \subseteq T$. If $cl^{\diamond}(M) = T$, hence M is called $(\mathfrak{g},\mathcal{P})$-dense set.

2. Main results

This part is about demonstrating the possibility of determining new types of continuity on a GPT space. First, let us make a supplement to the results from Al-Saadi and Al-Malki (2023) about a GPT space characteristic.

Definition 16. Consider $\eta : 2^T \to 2^T$ as a map that has the monotonic property, that is, if $M \subset N$ of T, then $\eta(M) \subset \eta(N)$. The whole set of monotonic maps on a set T is denoted by $\Gamma(T)$.

Remark 17. Let $M \subset T$, then the next is true:

(i) M is η-open iff $M \subset \eta(M)$. Thus, ϕ is η-open.
(ii) The countableunion of η-open setsis η-open; see Császár (2002). Hence, the set of all η-open sets forms a GT and is denoted by $\mathfrak{g}_{\eta}$. Consequentially, $(T,\mathfrak{g}_{\eta},\mathcal{P})$ is a GPT space, where $\mathcal{P}$ is a primal set over T.

By the above consideration, it is obvious that we can produce all GPT space over T for some $\eta \in \Gamma$.

Corollary 18. *Let $(T,\mathfrak{g},\mathcal{P})$ be a GPT space. Then, there exists $\eta \in \Gamma$ satisfying the following properties:*

(i) $\eta(\phi) = \phi$ and $\eta(M) \subset M$,
(ii) $\eta(\eta(M)) = \eta(M)$.

Proof. Consider $\mathfrak{g}_{\eta} = \mathfrak{g}$ and $\eta(M) = \{\cup\, G : G \subset M, \forall G \in \mathfrak{g}\}$. Hence, $\eta(M) \in \mathfrak{g}$. So, $\eta(M) \subset M$ and $\eta(\phi) = \phi$, hence we prove (i).

Moreover, from the definition $M \subset \eta(M)$, but from (i) $\eta(M) \subset M$ implies $\eta(M) = M$. Therefore, $\eta(M) \in \mathfrak{g}$ and $\eta(\eta(M)) = \eta(M)$. □

Another method to obtain a GPT space is by using the map ψ, that is mentioned in Definition 2 (for deep details, see Császár, 2002).

Definition 19. If $\psi \in \Psi(T)$ and $G \in \mathfrak{g}$ satisfy the condition: if $t \in G$, there exists $O \in \psi(t)$ satisfying $O \subset G$. Then, $\mathfrak{g}$ is a GT denoted by $\mathfrak{g}_{\psi}$. Consequentially, $(T,\mathfrak{g}_{\psi},\mathcal{P})$ is a GPT space, where $\mathcal{P}$ is a primal set over T. Conversely, let $\mathfrak{g}$ be a GT, then $\exists\ \psi \in \Psi(T)$ such that $\mathfrak{g} = \mathfrak{g}_{\psi}$, which means ψ satisfies the condition: $\forall t \in T, \exists\ O \in \psi(t), O \in \mathfrak{g}$.

In the first case, consider $\psi = \psi_{\mathfrak{g}}$ and for the second case consider $\psi \in \Psi_{\mathfrak{g}}(T)$. For $\mathfrak{g} = \mathfrak{g}_{\eta}$, where $\eta \in \Gamma$, considering $i_{\mathfrak{g}_{\eta}}$ as i_{η} and $c_{\mathfrak{g}_{\eta}}$ as c_{η}. Also, for $\mathfrak{g} = \mathfrak{g}_{\psi}$, where $\psi \in \Psi(T)$, considering $i_{\mathfrak{g}_{\psi}}$ as i_{ψ} and $c_{\mathfrak{g}_{\psi}}$ as c_{ψ}.

Definition 20. Let $(T,\mathfrak{g},\mathcal{P})$ be a GPT space, $\psi \in \Psi$, and $M \subseteq T$. Then,

(i) $\iota_{\psi}(M) = \{t \in T : \exists\ O \in \psi(t)\ satisfies\ O \subset M\}$.
(ii) $\eta_{\psi}(M) = \{t \in T : O^c \cup M^c \in \mathcal{P},\ forall\ O \in \psi(t)\}$.

Note that $\iota_{\psi}, \eta_{\psi} \in \Gamma$. Moreover, $i_{\psi}(M) \subset \iota_{\psi}(M)$ as well as $cl^{\diamond}_{\psi}(M) \subset \eta_{\psi}(M)$. However, when $\mathfrak{g}_{\psi} = \mathfrak{g}$, we have $\iota_{\psi} = i_{\psi}$ and $\eta_{\psi} = cl^{\diamond}_{\psi}$.

2.1. GP-continuous function

This part of our article is a discussion of three different types of generalized primal continuity. We depend on two types of GPT spaces: those generated by a generalized neighbourhood system and monotonic maps.

Definition 21. Let $(T,\mathfrak{g},\mathcal{P})$ and $(T^{\backprime},\mathfrak{g}^{\backprime},\mathcal{P}^{\backprime})$ be two GPT spaces. Consider O as $(\mathfrak{g}^{\backprime},\mathcal{P}^{\backprime})$-open. Define a function $p : T \to T^{\backprime}$. Then, p is named GP-Continuous if and only if $p^{-1}(O)$ is $(\mathfrak{g},\mathcal{P})$-open.

The more general type of GP-continuity based on the concept of generalized neighbourhood systems as follows:

Definition 22. Consider $\psi \in \Psi(T)$ and $\psi^{\backprime} \in \Psi(T^{\backprime})$. Hence, p is named GPN-continuous iff for every $t \in T$ and $O^{\backprime} \in \psi^{\backprime}(p(t))$, there is $O \in \psi(t)$ satisfying $p(O) \subset O^{\backprime}$.

Since $cl^{\diamond}(M) \subset cl^{\diamond}(N)$ if $M \subseteq N$, then $cl^{\diamond} \in \Gamma$. The generalized primal neighbourhood system is defined as following:

Definition 23. Let $(T,\mathfrak{g},\mathcal{P})$ be GPT space, $M \subset T$, G be $(\mathfrak{g},\mathcal{P})$-open set, and $t \in G$. If $O = cl^{\diamond}(G)$, then $O \in \psi(cl^{\diamond},\mathfrak{g})(t)$. Therefore, $\psi(cl^{\diamond},\mathfrak{g}) \in \Psi(T)$.

The connection between GPN-continuity and GP-continuity is described in the following.

Theorem 24. *Every GPN-continuous function is GP-continuous, where $\mathfrak{g}$ produce by the neighbourhood system.*

Proof. Consider p as a function from $(T,\mathfrak{g}_{\psi},\mathcal{P})$ to $(T^{\backprime},\mathfrak{g}^{\backprime}_{\psi},\mathcal{P}^{\backprime})$. Let $G^{\backprime} \in \mathfrak{g}^{\backprime}_{\psi^{\backprime}}$, $t \in T$ and $p(t) \in G^{\backprime}$. Thus, there is $O^{\backprime} \in \psi^{\backprime}(p(t))$ satisfies $O^{\backprime} \subset G^{\backprime}$. Hence, $\exists\ O \in \psi(t)$ such that $p(O) \subset O^{\backprime}$. Therefore, $p(O) \subset G^{\backprime}$ and $O \subset p^{-1}(G^{\backprime})$ implies $p^{-1}(G^{\backprime}) \in \mathfrak{g}_{\psi}$. □

There is a GP-continuous map produced by the neighbourhood system, which is not GPN-continuous; the following example makes that clear.

Example 25. Let $(T,\mathfrak{g},\mathcal{P})$ be a GPT space, where $T = \{t_1,t_2,t_3\}$, $\mathfrak{g} = \{\phi,\{t_1\},\{t_2\},\{t_1,t_2\},T\}$, and $\mathcal{P} = 2^T \setminus T$. Suppose that $(T^{\backprime},\mathfrak{g}^{\backprime},\mathcal{P}^{\backprime})$ is a GPT space, where $T^{\backprime} = T$, $\mathfrak{g}^{\backprime} = \{\phi,\{t_1\},\{t_3\},\{t_1,t_3\},T^{\backprime}\}$, and $\mathcal{P}^{\backprime} = 2^{T^{\backprime}} \setminus T^{\backprime}$. Consider $\psi = \psi(cl^{\diamond},\mathfrak{g})$ and $\psi^{\backprime} = \psi(cl^{\diamond},\mathfrak{g}^{\backprime})$. Since if $O \in \mathfrak{g}$ and $t_1 \in O$, then $\{t_1\} \in \mathfrak{g}$ and $cl^{\diamond}\{t_1\} = \{t_1,t_3\} \notin \mathfrak{g}$. If $O \in \mathfrak{g}$ and $t_2 \in O$, then $\{t_2\} \in \mathfrak{g}$ and $cl^{\diamond}\{t_2\} = \{t_2,t_3\} \notin \mathfrak{g}$. If $O \in \mathfrak{g}$ and $t_3 \in O$, then $T \in \mathfrak{g}$ and $cl^{\diamond}(T) = T \in \mathfrak{g}$. Hence, $\mathfrak{g}_{\psi} = \{\phi,T\}$. By the same way, we have $\mathfrak{g}_{\psi^{\backprime}} = \{\phi,T\}$. Let $p : T \to T^{\backprime}$ be the identity map. Thus, p is a GP-continuous. However, p is not GPN-continuous since $p(t_1) \in \{t_1\} \in \mathfrak{g}^{\backprime}$ and $cl^{\diamond}(\{t_1\}) = \{t_1,t_2\}$ implies $\{t_1,t_2\} \in \psi^{\backprime}\{t_1\}$ if $O \in \psi(\{t_1\})$ and $cl^{\diamond}(\{t_1\}) = \{t_1,t_3\}$ and $\{t_1,t_3\} \nsubseteq \{t_1,t_2\}$.

The extra condition that makes the inverse of Theorem 24 true is described in the following.

Theorem 26. *Every GP-continuous function is GPN-continuous, whenever $\psi^{\backprime} = \psi^{\backprime}_{\mathfrak{g}^{\backprime}}$, where $\mathfrak{g}$ produced by the neighbourhood system.*

Proof. Consider p as a function from $(T,\mathfrak{g}_{\psi},\mathcal{P})$ to $(T^{\backprime},\mathfrak{g}^{\backprime}_{\psi},\mathcal{P}^{\backprime})$. Put $t \in T$ and $O^{\backprime} \in \psi^{\backprime}(p(t))$. Thus, $p(t) \in O^{\backprime}$ and $O^{\backprime} \in \mathfrak{g}^{\backprime}_{\psi^{\backprime}}$. Hence, $t \in p^{-1}(O^{\backprime}) \in \mathfrak{g}_{\psi}$, then there exists $O \in \psi(t)$ satisfies $O \subset p^{-1}(O^{\backprime})$ and implies $p(O) \subset O^{\backprime}$. □

The map η_{ψ} in Definition 20 is a useful tool to define the GPN-continuous in a different way as follows:

Theorem 27. *Consider $p : T \to T^{\backprime}, \psi \in \Psi(T), \psi^{\backprime} \in \Psi(T^{\backprime}), M \subset T, N \subset T^{\backprime}$. Then, the next are equivalent:*

(i) p is GPN-*continuous;*
(ii) $p(\eta_{\psi}(M)) \subset \eta_{\psi^{\backprime}}(p(M))$;
(iii) $\eta_{\psi}(p^{-1}(N)) \subset p^{-1}(\eta_{\psi^{\backprime}}(N))$.

Proof. $(i) \Rightarrow (ii)$. Let $t \in \eta_{\psi}(M)$. Thus $p(t) \in \eta_{\psi^{\backprime}}(p(M))$. Because if that does not hold, then $\exists\ O \in \psi^{\backprime}(p(t))$ satisfies $O^c \cup (p(M))^c \notin \mathcal{P}$. We get, $\exists\ S \in \psi(t)$ such that $p(S) \subset O$ and $(p(S))^c \cup (p(M))^c \notin \mathcal{P}$. Hence, $S^c \cup M^c \notin \mathcal{P}$, which is a contradiction.

$(ii) \Rightarrow (iii)$. Consider $M = p^{-1}(N)$. Thus, from (ii), we get

$$p(\eta_{\psi}(M)) \subset \eta_{\psi^{\backprime}}(p(M)) = \eta_{\psi^{\backprime}}(p(p^{-1}(N))) \subset \eta_{\psi^{\backprime}}(N).$$

Therefore, $\eta_{\psi}(p^{-1}(N)) \subset p^{-1}(\eta_{\psi^{\backprime}}(N))$.

$(iii) \Rightarrow (i)$. Consider $O \in \psi^{\backprime}(p(t))$. Put $N = (T^{\backprime} \setminus O)$ and $p(t) \notin \eta_{\psi^{\backprime}}(N)$. Thus, $t \notin p^{-1}(\eta_{\psi^{\backprime}}(N))$. From (iii) $t \notin \eta_{\psi} p^{-1}(N)$ we get $\exists\ S \in \psi(t)$ satisfies $S^c \cup (p^{-1}(N))^c \notin \mathcal{P}$. Thus, $(p(S))^c \cup N^c \notin \mathcal{P}$. Hence, $p(S) \subset O$. ☐

2.2. GP-θ-continuous function

By using a generalized primal neighbourhood system $\psi(cl^{\diamond}, \mathfrak{g}) \in \Psi(T)$ that is mentioned in Definition 23, we can present a new kind of continuous function as follows:

Definition 28. Let $(T, \mathfrak{g}, \mathcal{P})$ and $(T^{\backprime}, \mathfrak{g}^{\backprime}, \mathcal{P}^{\backprime})$ be two GPT spaces. Let $\psi = \psi(cl^{\diamond}, \mathfrak{g})$ and $\psi^{\backprime} = \psi^{\backprime}(cl^{\diamond}, \mathfrak{g}^{\backprime})$. Hence $p : T \to T^{\backprime}$ is named GP-θ-Continuous if it is GPN-Continuous.

In anther word if $t \in T$, $O^{\backprime} \in \mathfrak{g}^{\backprime}$, and $p(t) \in O^{\backprime}$, then $\exists\ O \in \mathfrak{g} : t \in O$ satisfies $p(cl^{\diamond}(O)) \subset cl^{\diamond}(O^{\backprime})$, where $cl^{\diamond}(O), cl^{\diamond}(O^{\backprime})$ restricted on $(T, \mathfrak{g}, \mathcal{P})$ and $(T^{\backprime}, \mathfrak{g}^{\backprime}, \mathcal{P}^{\backprime})$ respectively.

Theorem 29. *Let $(T, \mathfrak{g}, \mathcal{P})$ and $(T^{\backprime}, \mathfrak{g}^{\backprime}, \mathcal{P}^{\backprime})$ be two* GPT *spaces $M \subset T, N \subset T^{\backprime}$, and $p : T \to T^{\backprime}$. If $O \in \psi(cl^{\diamond}, \mathfrak{g})$ and $O^{\backprime} \in \psi^{\backprime}(cl^{\diamond}, \mathfrak{g}^{\backprime})$ satisfy the next two conditions:*

(1) $t \in \eta_{\psi}(M)$ iff $t \in G \in \mathfrak{g}$ implies $(cl^{\diamond}(G))^c \cup M^c \in \mathcal{P}$.

(2) $t^{\backprime} \in \eta^{\backprime}_{\psi}(N)$ iff $t^{\backprime} \in G^{\backprime} \in \mathfrak{g}^{\backprime}$ implies $(cl^{\diamond}(G^{\backprime}))^c \cup N \in \mathcal{P}$, then the next are equivalent:

(i) p *is* GP-θ-*continuous;*
(ii) $p(\eta_{\psi}(M)) \subset \eta_{\psi^{\backprime}}(p(M))$;
(iii) $\eta_{\psi}(p^{-1}(N)) \subset p^{-1}(\eta_{\psi^{\backprime}}(N))$.

2.3. Almost GP-continuity

This part is a discussion of the notion of "almost continuous" in the generalized field. Min (2009b) presented this notion between two generalized topological spaces. Here, we will give it with consideration for the primal set.

Definition 30. Let $(T, \mathfrak{g}, \mathcal{P})$ and $(T^{\backprime}, \mathfrak{g}^{\backprime}, \mathcal{P}^{\backprime})$ be two GPT spaces. Consider $p : T \to T^{\backprime}$. Then, p is named almost GP-continuous at $t \in T$ if for every $(\mathfrak{g}^{\backprime}, \mathcal{P}^{\backprime})$-open set $G^{\backprime} : p(t) \in G^{\backprime}$, there existsa $(\mathfrak{g}, \mathcal{P})$-open set $G : t \in G$ satisfies $p(G) \subset i_{\mathfrak{g}^{\backprime}}(cl^{\diamond}(G^{\backprime}))$. Moreover, p is named almost GP-continuous if it is almost GP-continuous at every $t \in T$.

The more general concept between GP-continuity and almost GP-continuity is qualified in the following.

Theorem 31. *Every* GP-*continuous function is almost* GP-*continuous.*

Proof. For $t \in T$ let $G^{\backprime}$ be a $(\mathfrak{g}^{\backprime}, \mathcal{P}^{\backprime})$-open set such that $p(t) \in G^{\backprime}$. Since p is GP-continuous, $p^{-1}(G^{\backprime})$ is $(\mathfrak{g}, \mathcal{P})$-open set. Put $p^{-1}(G^{\backprime}) = G$. Thus $t \in G$. Now, $p(G) = G^{\backprime} \subset i_{\mathfrak{g}^{\backprime}}(cl^{\diamond}(G^{\backprime}))$. Hence, p is almost GP-continuous at $t \in T$. Since t is arbitrary, p is almost GP-continuous. ☐

To say these two notions coincide, we need to discuss the technique that Császár presented in Császár (2008a). This technique changes the generalized topology of other generalized topologies, which is smaller than it. Here we will do the same, but this time with consideration of a primal set as follows:

Definition 32. Let $(T, \mathfrak{g}, \mathcal{P})$ be a GPT space. Then, $\vartheta(\mathfrak{g})$ and $\sigma(\mathfrak{g})$ are two GPT spaces, given as follows for each $M \subseteq T$:

(i) $\vartheta(\mathfrak{g}) = \{M \subseteq T : \forall\ t \in M, \exists\ (\mathfrak{g}, \mathcal{P}){-}closedset\ N satisfies\ t \in i_{\mathfrak{g}}(N) \subset M\}$.

(ii) $\sigma(\mathfrak{g}) = \{M \subseteq T : \forall\ t \in M, \exists\ (\mathfrak{g}, \mathcal{P}){-}openset\ N satisfies\ t \in cl^{\diamond}(N) \subset M\}$.

Note that the member of $\vartheta(\mathfrak{g})$ coincides with the union of all $(\mathfrak{g}, \mathcal{P})$-regular open set. By another way the family of all $(\mathfrak{g}, \mathcal{P})$- regular open sets $\mathcal{R}\mathfrak{g}_{\mathcal{P}}$ is a based for the GPT space $\vartheta(\mathfrak{g})$.

So, the concept of almost GP-continuous coincides with the notion of GP-continuity if we make a suitable change to the GPT space as follows:

Proposition 33. *$p : (T, \mathfrak{g}, \mathcal{P}) \to (T^{\backprime}, \mathfrak{g}^{\backprime}, \mathcal{P}^{\backprime})$ is almost* GP-*continuous iff $p : (T, \mathfrak{g}, \mathcal{P}) \to (T^{\backprime}, \vartheta(\mathfrak{g}^{\backprime}), \mathcal{P}^{\backprime})$ is* GP-*continuity.*

Proof. Since $p : (T, \mathfrak{g}, \mathcal{P}) \to (T^{\backprime}, \mathfrak{g}^{\backprime}, \mathcal{P}^{\backprime})$ is almost GP-continuous, then for every $t \in T$ and for every $(\mathfrak{g}^{\backprime}, \mathcal{P}^{\backprime})$-open set $G^{\backprime} : p(t) \in G^{\backprime}$, there existsa $(\mathfrak{g}, \mathcal{P})$-openset $G : t \in G$ satisfies $p(G) \subset i_{\mathfrak{g}^{\backprime}}(cl^{\diamond}(G^{\backprime}))$. Hence,

$$p(G) \subset i_{\vartheta(\mathfrak{g}^{\backprime})}(cl^{\diamond}(G^{\backprime})) \subseteq p(G),$$

from the definition of $\vartheta(\mathfrak{g}^{\backprime})$. By regularity of $G^{\backprime}$ we have $i_{\vartheta(\mathfrak{g}^{\backprime})}(cl^{\diamond}(G^{\backprime})) = G^{\backprime}$. Hence $p^{-1}(G^{\backprime}) = G$. Therefore, $p : (T, \mathfrak{g}, \mathcal{P}) \to (T^{\backprime}, \vartheta(\mathfrak{g}^{\backprime}), \mathcal{P}^{\backprime})$ is GP-continuity. ☐

3. Strong GP-continuity

In this part, special types of GP-continuity will be introduced. In fact, the concepts of super $(\mathfrak{g}, \mathfrak{g}^{\backprime})$-continuous functions and strongly $(\mathfrak{g}, \mathfrak{g}^{\backprime})$-$\theta$-continuous functions are given in Min and Kim (2011). Also, the notion of strong $(\mathfrak{g}, \mathfrak{g}^{\backprime})$-continuous functions is presented by Kim and Min (2012). All these notions are described according to GT space features. Here, we will study these types of continuity in light of the primal set. First, let us define the set $\omega_{\mathfrak{g}} = \bigcup\{M \subset T : M is a (\mathfrak{g}, \mathcal{P}){-}open\}$.

Definition 34. Let $(T, \mathfrak{g}, \mathcal{P})$ and $(T^{\backprime}, \mathfrak{g}^{\backprime}, \mathcal{P}^{\backprime})$ be two GPT spaces. Consider $p : T \to T^{\backprime}$. Then, p is named a strong GP-continuity if for every $t \in T$ and $(\mathfrak{g}^{\backprime}, \mathcal{P}^{\backprime})$-open set $G^{\backprime} : p(t) \in G^{\backprime}$, there is a $(\mathfrak{g}, \mathcal{P})$-open set $G : t \in G$ satisfies $cl^{\diamond}(p(G)) \cap \omega_{\mathfrak{g}^{\backprime}} \subset G^{\backprime}$.

Theorem 35. *Consider $p : T \to T^{\backprime}$ as strong* GP-*continuous function. If $p(\omega_{\mathfrak{g}}) \subseteq \omega_{\mathfrak{g}^{\backprime}}$, then $p(cl^{\diamond}(H)) \subseteq cl^{\diamond}(p(H))$, for all $(\mathfrak{g}, \mathcal{P})$-open set $H \subseteq T$.*

Proof. Assume $H \subseteq T$ is $(\mathfrak{g}, \mathcal{P})$-open. Let $t \in c_{\mathfrak{g}}(H)$. Let $G^{\backprime} \subseteq T^{\backprime}$ be $(\mathfrak{g}^{\backprime}, \mathcal{P}^{\backprime})$-open such that $p(t) \in G^{\backprime}$. As p is a strong GP-continuity, there existsa $(\mathfrak{g}, \mathcal{P})$-openset $G \subseteq T : t \in G$ and $cl^{\diamond}(p(G)) \cap \omega_{\mathfrak{g}^{\backprime}} \subset G^{\backprime}$. Moreover, as $t \in cl^{\diamond}(H)$ and $G \subseteq T : t \in G$, $H \cap G \neq \phi$. But $p(\omega_{\mathfrak{g}}) \subseteq \omega_{\mathfrak{g}^{\backprime}}$. Hence,

$$\begin{aligned}\phi \neq p(G \cap H) &\subseteq p(G) \cap p(H),\\ &\subseteq p(H) \cap cl^{\diamond}(p(G)),\\ &= (p(H) \cap \omega_{\mathfrak{g}^{\backprime}}) \cap cl^{\diamond}(p(G)),\\ &\subseteq p(H) \cap G^{\backprime}.\end{aligned}$$

Thus, $p(H) \cap G^{\backprime} \neq \phi$ and $p(t) \in cl^{\diamond}(p(H))$. Hence, $p(cl^{\diamond}(H)) \subseteq cl^{\diamond}(p(H))$. ☐

Theorem 36. *Consider $p : T \to T^{\backprime}$. If $p(\omega_{\mathfrak{g}}) \subseteq \omega_{\mathfrak{g}^{\backprime}}$, then the next are identical:*

(i) p is a strong GP-*continuity;*
(ii) For every $t \in T$ and a $(\mathfrak{g}^{\backprime}, \mathcal{P}^{\backprime})$-open set $G^{\backprime} : p(t) \in G^{\backprime}$, there is a $(\mathfrak{g}, \mathcal{P})$-open set $G : t \in G$ satisfies $cl^{\diamond}(p(cl^{\diamond}(G))) \cap \omega_{\mathfrak{g}^{\backprime}} \subseteq G^{\backprime}$;
(iii) For every $t \in T$ and a $(\mathfrak{g}^{\backprime}, \mathcal{P}^{\backprime})$-closed set $H^{\backprime} : p(t) \notin H^{\backprime}$, there is a $(\mathfrak{g}, \mathcal{P})$-open set $G : t \in G$ and a $(\mathfrak{g}^{\backprime}, \mathcal{P}^{\backprime})$-open set $G^{\backprime}$ satisfies $H^{\backprime} \cap \omega_{\mathfrak{g}^{\backprime}} \subseteq G^{\backprime}$, and $p(cl^{\diamond}(G)) \cap G^{\backprime} = \phi$;

(iv) *For every* $t \in T$ *and a* $(\mathfrak{g}', \mathcal{P}')$*-closed set* H' : $p(t) \notin H'$*, there is a* $(\mathfrak{g}, \mathcal{P})$*-open set* G : $t \in G$ *and a* $(\mathfrak{g}', \mathcal{P}')$*-open set* G' *satisfies* $H' \cap \omega_{\mathfrak{g}'} \subseteq G'$*, and* $p(G) \cap G' = \phi$.

Proof. $(i) \Rightarrow (ii)$. For $t \in T$, let G' be a $(\mathfrak{g}', \mathcal{P}')$-open set such that $p(t) \in G'$. So, there is a $(\mathfrak{g}, \mathcal{P})$-open set $G : t \in G$ satisfies $cl^{\diamond}(p(G)) \cap \omega_{\mathfrak{g}'} \subseteq G'$. From Theorem 35, we get $p(cl^{\diamond}(G)) \subseteq cl^{\diamond}(p(G))$. Hence, $p(cl^{\diamond}(G)) \cap \omega_{\mathfrak{g}'} \subseteq G'$.

$(ii) \Rightarrow (iii)$. For $t \in T$, let H' be a $(\mathfrak{g}', \mathcal{P}')$-closed set such that $p(t) \notin H'$. Since $p(t) \in (X' \setminus H')$ and $(X' \setminus H')$ is a $(\mathfrak{g}', \mathcal{P}')$-open set, there is a $(\mathfrak{g}, \mathcal{P})$-open set $G : t \in G$ satisfies $cl^{\diamond}(p(cl^{\diamond}(G))) \cap \omega_{\mathfrak{g}'} \subseteq (X' \setminus H')$. Put $G' = (X' \setminus cl^{\diamond}(p(cl^{\diamond}(G))))$. Hence, G' is a $(\mathfrak{g}', \mathcal{P}')$-open set satisfies $H' \cap \omega_{\mathfrak{g}'} \subseteq G'$, and $p(cl^{\diamond}(G)) \cap G' = \phi$.

$(iii) \Rightarrow (iv)$. It is clear.

$(iv) \Rightarrow (i)$. Let $t \in T$ and G' be a $(\mathfrak{g}', \mathcal{P}')$-open set with $p(t) \in G'$. Hence, $(X' \setminus G')$ is a $(\mathfrak{g}', \mathcal{P}')$-closed set with $p(t) \notin (X' \setminus G')$. So, there is a $(\mathfrak{g}, \mathcal{P})$-open set $G : t \in G$ and $(\mathfrak{g}', \mathcal{P}')$-open set W satisfy $(X' \setminus G') \cap \omega_{\mathfrak{g}'} \subseteq W$ and $p(G) \cap W = \phi$. Hence,

$$\begin{aligned} cl^{\diamond}(p(G)) \cap \omega_{\mathfrak{g}'} &\subseteq cl^{\diamond}((X' \setminus G')) \cap \omega_{\mathfrak{g}'} \\ &= (X' \setminus G') \cap \omega_{\mathfrak{g}'} \subseteq G'. \end{aligned}$$

Therefore, p is a strong GP-continuous. □

Corollary 37. *Consider* $p : T \to T'$. *If* T' *is strong, then the next are identical:*

(i) *p is a strong* GP*-continuity;*
(ii) *For every* $t \in T$ *and a* $(\mathfrak{g}', \mathcal{P}')$*-open set* G' : $p(t) \in G'$*, there is a* $(\mathfrak{g}, \mathcal{P})$*-open set* $G : t \in G$ *satisfies* $cl^{\diamond}(p(cl^{\diamond}(G))) \subseteq G'$;
(iii) *For every* $t \in T$ *and a* $(\mathfrak{g}', \mathcal{P}')$*-closed set* H' : $p(t) \notin H'$*, there is a* $(\mathfrak{g}, \mathcal{P})$*-open set* $G : t \in G$ *and a* $(\mathfrak{g}', \mathcal{P}')$*-open set* G' *satisfies* $H' \subseteq G'$ *and* $p(cl^{\diamond}(G)) \cap G' = \phi$;
(iv) *For every* $t \in T$ *and a* $(\mathfrak{g}', \mathcal{P}')$*-closed set* H' : $p(t) \notin H'$*, there is a* $(\mathfrak{g}, \mathcal{P})$*-open set* $G : t \in G$ *and a* $(\mathfrak{g}', \mathcal{P}')$*-open set* G' *satisfies* $H' \subseteq G'$ *and* $p(G) \cap G' = \phi$*. Hence,* G' *is satisfies* $H' \cap \omega_{\mathfrak{g}'} \subseteq G'$*, and* $p(cl^{\diamond}(G)) \cap G' = \phi$.

Definition 38. Let $(T, \mathfrak{g}, \mathcal{P})$ and $(T', \mathfrak{g}', \mathcal{P}')$ be two GPT spaces. Then, $p : T \to T'$ is named strongly GP-θ-continuous, if for every $t \in T$ and every $(\mathfrak{g}', \mathcal{P}')$-open set G' : $p(t) \in G'$, there existsa $(\mathfrak{g}, \mathcal{P})$-openset $G : t \in G$ satisfies $p(cl^{\diamond}(G)) \subseteq G'$.

The next result studies the relationship between the notions of a strong GP-continuous function and a strongly GP-θ-continuous.

Theorem 39. *If* $p : T \to T'$ *is a strong* GP*-continuous and* T' *is strong, then p is strongly* GP*-θ-continuous.*

Proof. It automatically from Corollary 37. □

There is a strongly GP-θ-continuous function, which is not strong GP-continuous; the following example makes that clear.

Example 40. Let $(T, \mathfrak{g}, \mathcal{P})$ be a GPT space, where $T = \{t_1, t_2, t_3\}$, $\mathfrak{g} = \{\phi, \{t_1\}\}$, and $\mathcal{P} = \{\phi, \{t_1\}, \{t_2\}, \{t_1, t_2\}\}$. Suppose that $(T', \mathfrak{g}', \mathcal{P}')$ is a GPT space, where $T' = \{a, b, c\}$, $\mathfrak{g}' = \{\phi, \{a\}, T'\}$, and $\mathcal{P}' = 2^{T'} \setminus T'$. Define $p : T \to T'$ by

$$p(t_1) = p(t_2) = p(t_3) = a.$$

Thus, p is strongly GP-θ-continuous function.

However, it is not strong GP-continuous since $cl^{\diamond}(p(\{t_1\})) = T' \nsubseteq \{a\}$.

Definition 41. Let $(T, \mathfrak{g}, \mathcal{P})$ and $(T', \mathfrak{g}', \mathcal{P}')$ be two GPT spaces. Then, $p : T \to T'$ is named super GP-continuous, if for every $t \in T$ and every $(\mathfrak{g}', \mathcal{P}')$-open set G' : $p(t) \in G'$, there existsa $(\mathfrak{g}, \mathcal{P})$-open set $G : t \in G$ satisfies $p(i_{\mathfrak{g}}(cl^{\diamond}(G))) \subseteq G'$.

The connection between the previous types of continuous functions and the GP-continuous function is qualified in the following.

Theorem 42. *Every strongly* GP*-θ-continuous function is super* GP*-continuous.*

Proof. Let $p : (T, \mathfrak{g}, \mathcal{P}) \to (T', \mathfrak{g}', \mathcal{P}')$ be strongly GP-θ-continuous. Then, for every $t \in T$ and every $(\mathfrak{g}', \mathcal{P}')$-open set G' : $p(t) \in G'$, there existsa $(\mathfrak{g}, \mathcal{P})$-open set $G : t \in G$ satisfies $p(cl^{\diamond}(G)) \subseteq G'$. But $i_{\mathfrak{g}}(cl^{\diamond}(G)) \subseteq cl^{\diamond}(G)$. Thus, $p(i_{\mathfrak{g}}(cl^{\diamond}(G))) \subseteq G'$. Therefore, p is super GP-continuous. □

Theorem 43. *Every super* GP*-continuous function is* GP*-continuous.*

Proof. Let $p : (T, \mathfrak{g}, \mathcal{P}) \to (T', \mathfrak{g}', \mathcal{P}')$ be super GP-continuous. Then, for every $t \in T$ and every $(\mathfrak{g}', \mathcal{P}')$-open set G' : $p(t) \in G'$, there existsa $(\mathfrak{g}, \mathcal{P})$-open set $G : t \in G$ satisfies $p(i_{\mathfrak{g}}(cl^{\diamond}(G))) \subseteq G'$. Hence, $p(i_{\mathfrak{g}}(G)) \subseteq G'$. Since $t \in p^{-1}(G')$, $p^{-1}(G') \subseteq G$. Thus, $i_{\mathfrak{g}}(G) \subseteq p^{-1}(G') \subseteq G$. Therefore, $p^{-1}(G')$ is $(\mathfrak{g}, \mathcal{P})$-open. Hence, p is GP-continuous. □

Example 44. Let $(T, \mathfrak{g}, \mathcal{P})$ be a GPT space, where $T = \{t_1, t_2, t_3\}$, $\mathfrak{g} = \{\phi, \{t_1, t_3\}, T\}$, and $\mathcal{P} = 2^T \setminus T$. Suppose that $(T', \mathfrak{g}', \mathcal{P}')$ is a GPT space, where $T' = \{a, b, c\}$, $\mathfrak{g}' = \{\phi, \{a\}\}$, and $\mathcal{P}' = 2^{T'} \setminus T'$. Define $p : T \to T'$ by

$$p(t_1) = p(t_3) = a, \text{and } p(t_3) = b.$$

Thus, p is GP-continuous function.

However, it is not super GP-continuous since $p(i_{\mathfrak{g}}(cl^{\diamond}(\{t_1, t_3\}))) = p(T) \nsubseteq \{a\}$.

3.1. Weakly GP-closed and GP-open functions

In this part, we introduce the notions of “weakly GP-closed function” and “GP-regular space”. These two concepts are useful tools to show the converse of the relationship that we gave in Theorem 39. This result studies the connection between the notions of “strong GP-continuous function” and “strongly GP-θ-continuous”.

Definition 45. Let $(T, \mathfrak{g}, \mathcal{P})$ and $(T', \mathfrak{g}', \mathcal{P}')$ be two GPT spaces. Then, $p : T \to T'$ is named weakly GP-closed, if for every $(\mathfrak{g}, \mathcal{P})$-closed set F, we have $cl^{\diamond}(p(i_{\mathfrak{g}}(F))) \subseteq p(F)$.

Proposition 46. *If p is a weakly* GP*-closed, then for each* $(\mathfrak{g}, \mathcal{P})$*-open set G, we have* $cl^{\diamond}(p(i_{\mathfrak{g}}(G))) \subseteq p(cl^{\diamond}(G))$.

Proof. Since $cl^{\diamond}(G)$ is $(\mathfrak{g}, \mathcal{P})$-closed set, $G \subseteq cl^{\diamond}(G)$, and p is a weakly GP-closed. Then, $cl^{\diamond}(p(i_{\mathfrak{g}}(G))) \subseteq p(cl^{\diamond}(G))$. □

Theorem 47. *If p is a weakly* GP*-closed and strongly* GP*-θ-continuous, then p is strong* GP*-continuous.*

Proof. For $t \in T$, let G' be a $(\mathfrak{g}', \mathcal{P}')$-open : $p(t) \in G'$. Thus, there existsa $(\mathfrak{g}, \mathcal{P})$-open : $t \in G$ satisfies $p(cl^{\diamond}(G)) \subseteq G'$. By Proposition 46, we get

$$cl^{\diamond}(p(G)) \cap \omega_{\mathfrak{g}'} \subseteq p(cl^{\diamond}(G)) \cap \omega_{\mathfrak{g}'} \subseteq G'.$$

Hence, by Theorem 36 (ii), p is strong GP-continuous. □

Definition 48. Let $(T, \mathfrak{g}, \mathcal{P})$ be a GPT space. Then, T is called GP-regular on $\omega_{\mathfrak{g}}$ if for $t \in \omega_{\mathfrak{g}}$ and a $(\mathfrak{g}, \mathcal{P})$-closed set $F : t \notin F$, there exist $G_1, G_2 \in \mathfrak{g}$ satisfies $t \in G_1$, $F \cap \omega_{\mathfrak{g}} \subseteq G_2$ and $G_1 \cap G_2 = \phi$.

Proposition 49. *Let* $(T, \mathfrak{g}, \mathcal{P})$ *be a* GPT *space. Then, T is* GP*-regular iff for* $t \in \omega_{\mathfrak{g}}$ *and a* $(\mathfrak{g}, \mathcal{P})$*-open set* $G : t \in G$*, there exists a* $(\mathfrak{g}, \mathcal{P})$*-open set* $H : t \in H$ *satisfies* $t \in H \subseteq cl^{\diamond}(H) \cap \omega_{\mathfrak{g}} \subseteq G$.

Proof. It is direct from Definition 48 and considering $H = (T \setminus F)$. □

Theorem 50. *If p is a strongly* GP-θ*-continuous and $T^\backprime$ is* GP*-regular, then p is strong* GP*-continuous.*

Proof. For $t \in T$, let $G^\backprime$ be a $(\mathfrak{g}^\backprime, \mathcal{P}^\backprime)$-open set : $p(t) \in G_1^\backprime$. Since $T^\backprime$ is GP-regular, there is a $(\mathfrak{g}^\backprime, \mathcal{P}^\backprime)$-open set $G_2^\backprime : p(t) \in G_2^\backprime$ satisfies

$p(t) \in G_2^\backprime \subseteq cl^\diamond(G_2^\backprime) \cap \omega_{\mathfrak{g}^\backprime} \subseteq G_1^\backprime$.

Since p is a strongly GP-θ-continuous, there existsa $(\mathfrak{g}, \mathcal{P})$-open $G : t \in G$ satisfy $p(cl^\diamond(G)) \subseteq G_2^\backprime$. Hence,

$cl^\diamond(p(cl^\diamond(G))) \cap \omega_{\mathfrak{g}^\backprime} \subseteq cl^\diamond(G_2) \cap \omega_{\mathfrak{g}^\backprime} \subseteq G_1^\backprime$.

By using Theorem 36 (ii), p is strong GP-continuous. □

Corollary 51. *Let $(T, \mathfrak{g}, \mathcal{P})$ and $(T^\backprime, \mathfrak{g}^\backprime, \mathcal{P}^\backprime)$ be two* GPT *spaces. Consider $p : T \to T^\backprime$. If $T^\backprime$ is* GP*-regular and strong, then the next are identical:*

(i) p is a strong GP*-continuous;*
(ii) p is a strongly GP-θ*-continuous;*
(iii) p is a GP*-continuous.*

Proof. $(i) \Rightarrow (ii)$. It is directly from Theorem 39.
$(ii) \Rightarrow (iii)$. It comes directly from Theorems 42 and 43.
$(iii) \Rightarrow (i)$. It comes from the GP-regularity of the $T^\backprime$ Theorem 36 (iv). □

Definition 52. Let $(T, \mathfrak{g}, \mathcal{P})$ and $(T^\backprime, \mathfrak{g}^\backprime, \mathcal{P}^\backprime)$ be two GPT spaces. Consider $p : T \to T^\backprime$. Then, p is named GP-open function, if for all $(\mathfrak{g}, \mathcal{P})$-open set G in T, we have $p(G)$ is $(\mathfrak{g}^\backprime, \mathcal{P}^\backprime)$-open.

Theorem 53. *If $p : T \to T^\backprime$ is* GP*-open and strong* GP*-continuous with $p(\omega_{\mathfrak{g}}) = \omega_{\mathfrak{g}^\backprime}$, then $T^\backprime$ is* GP*-regular.*

Proof. Suppose $s \in \omega_{\mathfrak{g}^\backprime}$ and $G^\backprime$ is any $(\mathfrak{g}^\backprime, \mathcal{P}^\backprime)$-open such that $s \in G^\backprime$. Put $p(t) = s$ for $t \in T$. But p is strong GP-continuous. Thus, there existsa $(\mathfrak{g}, \mathcal{P})$-open set $G : t \in G$ satisfies $cl^\diamond(p(G)) \cap \omega_{\mathfrak{g}^\backprime} \subseteq G^\backprime$. Also, p is GP-open, and hence $p(G)$ is a $(\mathfrak{g}^\backprime, \mathcal{P}^\backprime)$-open such that $s \in p(G)$. Therefore,

$p(G) = p(G) \cap \omega_{\mathfrak{g}^\backprime} \subseteq cl^\diamond(p(G)) \cap \omega_{\mathfrak{g}^\backprime} \subseteq G^\backprime$.

Thus, by Proposition 49, we have $T^\backprime$ is GP-regular. □

4. GP-connected and GP-hyperconnected spaces

This part is an application of the notions studied in previous sections. We will study the preservation of the notions of "GP-connected" and "GP-hyperconnected" by different types of generalized primal continuous functions.

Definition 54. Let $(T, \mathfrak{g}, \mathcal{P})$ be a GPT space. Then, T is named GP-connected space if there are no non-empty disjoint $(\mathfrak{g}, \mathcal{P})$-open sets G and H satisfies $T = G \cup H$.

Definition 55. Let $(T, \mathfrak{g}, \mathcal{P})$ be a GPT space. Then, T is named:

(i) GP-α-connected space if there are no non-empty disjoint $(\mathfrak{g}, \mathcal{P})$-$\alpha$-open sets G and H satisfies $T = G \cup H$.
(ii) GP-β-connected space if there are no non-empty disjoint $(\mathfrak{g}, \mathcal{P})$-$\beta$-open sets G and H satisfies $T = G \cup H$.
(iii) GP-semi-connected space if there are no non-empty disjoint $(\mathfrak{g}, \mathcal{P})$-semi-open sets G and H satisfies $T = G \cup H$.
(iv) GP-pre-connected space if there are no non-empty disjoint $(\mathfrak{g}, \mathcal{P})$-pre-open sets G and H satisfies $T = G \cup H$.

Definition 56. Let $(T, \mathfrak{g}, \mathcal{P})$ and $(T^\backprime, \mathfrak{g}^\backprime, \mathcal{P}^\backprime)$ be two GPT spaces. Then, $p : T \to T^\backprime$ is named:

(i) Contra GP-continuous if for every $(\mathfrak{g}^\backprime, \mathcal{P}^\backprime)$-open set $G^\backprime \subseteq T^\backprime$, $p^{-1}(G^\backprime)$ is $(\mathfrak{g}, \mathcal{P})$-closed in T.
(ii) Contra GP-α-continuous if for every $(\mathfrak{g}^\backprime, \mathcal{P}^\backprime)$-open set $G^\backprime \subseteq T^\backprime$, $p^{-1}(G^\backprime)$ is $(\mathfrak{g}, \mathcal{P})$-$\alpha$-closed in T.
(iii) Contra GP-semi-continuous if for every $(\mathfrak{g}^\backprime, \mathcal{P}^\backprime)$-open set $G^\backprime \subseteq T^\backprime$, $p^{-1}(G^\backprime)$ is $(\mathfrak{g}, \mathcal{P})$-semi-closed in T.
(iv) Contra GP-pre-continuous if for every $(\mathfrak{g}^\backprime, \mathcal{P}^\backprime)$-open set $G^\backprime \subseteq T^\backprime$, $p^{-1}(G^\backprime)$ is $(\mathfrak{g}, \mathcal{P})$-pre-closed in T.
(v) Contra GP-β-continuous if for every $(\mathfrak{g}^\backprime, \mathcal{P}^\backprime)$-open set $G^\backprime \subseteq T^\backprime$, $p^{-1}(G^\backprime)$ is $(\mathfrak{g}, \mathcal{P})$-$\beta$-closed in T.

Theorem 57. *Let $p : (T, \mathfrak{g}, \mathcal{P}) \to (T^\backprime, \mathfrak{g}^\backprime, \mathcal{P}^\backprime)$ be a contra* GP-α*-continuous onto function. Consider T as* GP-α*-connected space. Then, $T^\backprime$ is* GP*-connected space.*

Proof. Consider $p : (T, \mathfrak{g}, \mathcal{P}) \to (T^\backprime, \mathfrak{g}^\backprime, \mathcal{P}^\backprime)$ as a contra GP-α-continuous onto function. Let T be GP-α-connected space. Suppose that $T^\backprime$ is not GP-connected space. Thus, there exist non-empty disjoint $(\mathfrak{g}^\backprime, \mathcal{P}^\backprime)$-open sets $G^\backprime$ and $H^\backprime$ satisfies $G^\backprime \cup H^\backprime = T^\backprime$. Therefore, we can say that $G^\backprime, H^\backprime$ are $(\mathfrak{g}^\backprime, \mathcal{P}^\backprime)$-closed sets. Hence, $p^{-1}(G^\backprime)$ and $p^{-1}(H^\backprime)$ are $(\mathfrak{g}, \mathcal{P})$-$\alpha$-open since p is contra GP-α-continuous. But $p^{-1}(H^\backprime) \cap p^{-1}(G^\backprime) = \phi$, and $p^{-1}(H^\backprime) \cup p^{-1}(G^\backprime) = T$. Thus, T is not GP-α-connected space, which is a contradiction. Hence, $T^\backprime$ is GP-connected space. □

Corollary 58. *Let $p : (T, \mathfrak{g}, \mathcal{P}) \to (T^\backprime, \mathfrak{g}^\backprime, \mathcal{P}^\backprime)$ be a contra* GP-β*-continuous (resp.* GP*-semi-continuous,* GP*-pre-continuous) onto function. Consider T as* GP-β*-connected (resp.* GP*-semi-connected,* GP*-pre-connected) space. Then, $T^\backprime$ is* GP*-connected space.*

According to Theorem 13 and Corollary 14, we conclude that:

(i) $\mathfrak{g}$-open $\Rightarrow$ $(\mathfrak{g}, \mathcal{P})$-$\alpha$-open $\Rightarrow$ $(\mathfrak{g}, \mathcal{P})$-pre-open $\Rightarrow$ $(\mathfrak{g}, \mathcal{P})$-$\beta$-open.
(ii) $(\mathfrak{g}, \mathcal{P})$-$\alpha$-open $\Rightarrow$ $(\mathfrak{g}, \mathcal{P})$-semi-open $\Rightarrow$ $(\mathfrak{g}, \mathcal{P})$-$\beta$-open.

However, the relationship between the types of GP-connected spaces takes the following path:

Theorem 59. *Let $(T, \mathfrak{g}, \mathcal{P})$ be a* GPT *space. Then, we have:*

(i) T is GP-β*-connected $\Rightarrow$* GP*-semi-connected $\Rightarrow$* GP-α*-connected $\Rightarrow$* GP*-connected.*
(ii) T is GP-β*-connected $\Rightarrow$* GP*-pre-connected $\Rightarrow$* GP-α*-connected.*

Proof. (i) (1) GP-β-connected $\Rightarrow$ GP-semi-connected: Let T be GP-β-connected, but not GP-semi-connected. Hence, there are non-empty disjoint $(\mathfrak{g}, \mathcal{P})$-semi-open sets G and H in T satisfies $G \cup H = T$ which implies that there are non-empty disjoint $(\mathfrak{g}, \mathcal{P})$-$\beta$-open sets G and H in T satisfies $G \cup H = T$. Hence, T is not GP-β-connected, which is contradiction. Therefore, T is GP-semi-connected.

(2) GP-semi-connected $\Rightarrow$ GP-α-connected: Let T be GP-semi-connected, but not GP-α-connected. Hence, there are non-empty disjoint $(\mathfrak{g}, \mathcal{P})$-$\alpha$-open sets G and H in T satisfies $G \cup H = T$ which implies that there are non-empty disjoint $(\mathfrak{g}, \mathcal{P})$-semi-open sets G and H in T satisfies $G \cup H = T$. Hence, T is not GP-semi-connected, which is contradiction. Therefore, T is GP-α-connected.

(3) GP-α-connected $\Rightarrow$ GP-connected: Let T be GP-α-connected, but not GP-connected. Hence, there are non-empty disjoint $(\mathfrak{g}, \mathcal{P})$-open sets G and H in T satisfies $G \cup H = T$ which implies that there are non-empty disjoint $(\mathfrak{g}, \mathcal{P})$-$\alpha$-open sets G and H in T satisfies $G \cup H = T$. Hence, T is not GP-α-connected, which is contradiction. Therefore, T is GP-connected.

By using the same technique, we can show the implications in (ii). □

Definition 60. Let $(T, \mathfrak{g}, \mathcal{P})$ be GPT space. Then, T is named:

(i) $(\mathfrak{g}, \mathcal{P})$-hyperconnected if each $(\mathfrak{g}, \mathcal{P})$-open set G of T is $(\mathfrak{g}, \mathcal{P})$-dense.
(ii) $(\mathfrak{g}, \mathcal{P})$-$\alpha$-hyperconnected if each $(\mathfrak{g}, \mathcal{P})$-$\alpha$-open set G of T is $(\mathfrak{g}, \mathcal{P})$-dense.

(iii) $(\mathfrak{g},\mathcal{P})$-semi-hyperconnected if each $(\mathfrak{g},\mathcal{P})$-semi-open set G of T is $(\mathfrak{g},\mathcal{P})$-dense.
(iv) $(\mathfrak{g},\mathcal{P})$-pre-hyperconnected if each $(\mathfrak{g},\mathcal{P})$-pre-open set G of T is $(\mathfrak{g},\mathcal{P})$-dense.
(v) $(\mathfrak{g},\mathcal{P})$-$\beta$-hyperconnected if each $(\mathfrak{g},\mathcal{P})$-$\beta$-open set G of T is $(\mathfrak{g},\mathcal{P})$-dense.

Theorem 61. *Let $(T,\mathfrak{g},\mathcal{P})$ be a* GPT *space. Then, we have:*

(i) T is GP-*β-hyperconnected $\Rightarrow$* GP-*semi-hyperconnected $\Rightarrow$* GP-*α-hyperconnected $\Rightarrow$* GP-*hyperconnected.*
(ii) T is GP-*β-hyperconnected $\Rightarrow$* GP-*pre-hyperconnected $\Rightarrow$* GP-*α-hyperconnected.*

Proof. (i) (1) GP-β-hyperconnected $\Rightarrow$ GP-semi-hyperconnected: Let T be GP-β-hyperconnected, but not GP-semi-hyperconnected. Hence, there is non-empty $(\mathfrak{g},\mathcal{P})$-semi-open set M in T satisfies $cl^{\diamond}(M) \neq T$ which implies that there is non-empty $(\mathfrak{g},\mathcal{P})$-$\beta$-open set M in T satisfies $cl^{\diamond}(M) \neq T$. Hence, T is not GP-β-hyperconnected, which is contradiction. Therefore, T is GP-semi-hyperconnected.

(2) GP-semi-hyperconnected $\Rightarrow$ GP-α-hyperconnected: Let T be GP-semi-hyperconnected, but not GP-α-hyperconnected. Hence, there is non-empty $(\mathfrak{g},\mathcal{P})$-$\alpha$-open set M in T satisfies $cl^{\diamond}(M) \neq T$ which implies that there is non-empty $(\mathfrak{g},\mathcal{P})$-semi-open set M in T satisfies $cl^{\diamond}(M) \neq T$. Hence, T is not GP-semi-hyperconnected, which is contradiction. Therefore, T is GP-α-hyperconnected.

(3) GP-α-hyperconnected $\Rightarrow$ GP-hyperconnected: Let T be GP-α-hyperconnected, but not GP-hyperconnected. Hence, there is non-empty $(\mathfrak{g},\mathcal{P})$-open set M in T satisfies $cl^{\diamond}(M) \neq T$ which implies that there is non-empty $(\mathfrak{g},\mathcal{P})$-$\alpha$-open sets M in T satisfy $cl^{\diamond}(M) \neq T$. Hence, T is not GP-α-hyperconnected, which is contradiction. Therefore, T is GP-hyperconnected.

By using the same technique, we can show the implications in (ii). □

5. Conclusion

Recently, Al-Saadi and Al-Malki (Al-Saadi and Al-Malki, 2023) presented a new generalized space. This space is characterized by all the nice features of GT spaces in the sense of primal sets. Some operators are defined in this space. Their behaviours are studied carefully.

In this article, we give attention to the notion of "continuity". So, we defined some type of generalized primal continuous function. We can summarize the results as follows:

If $\mathfrak{g}$ and $\mathfrak{g}^{\backslash}$ are generated by the neighbourhood system, then every GPN-continuous function is GP-continuous; see Theorem 24. The opposite of this is not always true; see Example 25. However, when we consider $\psi^{\backslash} = \psi^{\backslash}_{\mathfrak{g}^{\backslash}}$, we can show that every GP-continuous function is GPN-continuous; see Theorem 26. Moreover, by using a generalized primal neighbourhood system, $\psi(cl^{\diamond},\mathfrak{g}) \in \Psi(T)$, we conclude that every GPN-continuous is GP-θ-continuous; see Theorems 27 and 29. Also, every GP-continuous function is almost GP-continuous; see Theorem 31. To make the opposite of this relation true, we use the technique of Császár, which changes the generalized topology of other generalized topologies, which is weaker than it. So, Proposition 33 shows that the two notions coincide. All these results are presented in Section 2.

Section 3 discusses the relationship between a strong GP-continuous function and several different types, as follows: Every strong GP-continuous function is strongly GP-θ-continuous when the space $T^{\backslash}$ is strong; see Theorem 39. Example 40 shows that the converse is not true. Moreover, every strongly GP-θ-continuous is super GP-continuous, see Theorem 42 and every super GP-continuous is GP-continuous, see Theorem 43. Example 44 shows that the converse need not be true. To prove the converse of Theorem 39, we presented the notion of "weakly GP-closed function" or the notion of "GP-regular". So, Theorem 47 shows that every strongly GP-θ-continuous is strong GP-continuous if p is a weakly GP-closed function, while Theorem 50 shows that every strongly GP-θ-continuous is strong GP-continuous if the GPT space $T^{\backslash}$ is GP-regular space.

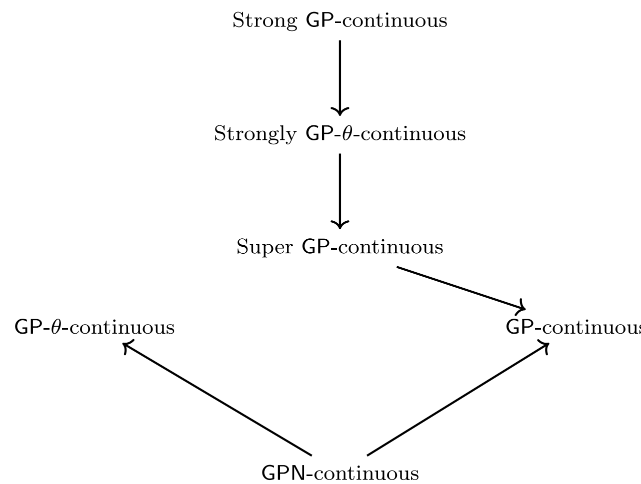

Fig. 1. The relationship among several kinds of GP-continuous function.

To make it easier for the reader, we present these results in Fig. 1 as follows:

Finally, Corollary 51 shows that the notion of "strong GP-continuous function" coincides with the notion "GP-continuous" when the space $T^{\backslash}$ is both strong and GP-regular.

CRediT authorship contribution statement

Hanan Al-Saadi: Methodology, Writing – original draft, Writing – review & editing. **Huda Al-Malki:** Methodology, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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