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Generalizations of rough set concepts

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Abstract In this paper, we generalized the notions of rough set concepts using two topological structures generated by general binary relation defined on the universe of discourse. New types of topological rough sets are initiated and studied using new types of topological sets. Some properties of topological rough approximations are studied by many propositions.

1. Introduction

The theory of rough set, proposed by Pawlak (1982), is an effective tool for data analysis (Greco et al., 2001, 2002; Hu and Cercone, 1995; Kryszkiewicz, 1998, 1999; Pawlak et al., 1994; Pawlak, 1991, 1997, 1998, 2001; Lin and Yao, 1996; Lingras and Yao, 1998; Leung and Li, 2003; Orlowska, 1986; Orlowska and Pawlak, 1987; Skowron and Rauszer, 1992; Skowron, 1993, 1995; Stepaniuk, 1998, 2000; Zhang et al., 2003). It can be used in the attribute-value representation model to describe the dependencies among attributes and evaluate the significance of attributes and derive decision rules. Classical rough set philosophy is based on an assumption that every object in the universe of discourse is associated with some information. Objects characterized by the same information are indiscernible with the available information about them. The indiscernibility relation generated in this way is the mathematical basis for the rough set theory. Classical rough set theory has used successfully in the analysis of data in complete information systems.

The indiscernibility relation is reflexive, symmetric and transitive. The set of all indiscernible objects is called an elementary set or equivalent class. Any set of objects, being a union of some elementary sets, is refereed to as crisp set, otherwise is called rough set. A rough set can be described by a pair of crisp sets, called the lower and upper approximations. By relaxing indiscernibility relation to more general binary relation, classical rough set can be extended to a more general model. Slowinski and Vanderpooten (1997) discussed rough approximation based on the reflexive and transitive binary relation. Skowron and Stepaniuk (1995) and Yao and Wong (1995) discussed generalized approximation space based on the reflexive and symmetry binary relation. Slowinski and Vanderpooten (2000) proposed generalized definition of rough approximation based on reflexive binary relation and

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compared with other definitions. Lin (1988), Lin and Yao (1996), Yao and Lin (1996), and Yao (1996, 1998) study the approximation operators defined by different neighborhood operators. Skowron and Stepaniuk (1995, 1996) and Stepaniuk (1998, 2000) defined generalized approximation space by using uncertain function and rough inclusion function and described its construction. Also they used the proposed techniques to investigate the problem of object selection.

In practice, tolerance relation (reflective, symmetric) and preference relation (reflective, anti-symmetric, transitive) are important relations. Greco et al. (2001, 2002) proposed rough approximations based on preference relation and applied it to multi-criteria decision analysis; rough approximation based on tolerance relation has been used successfully to compute attribute reducts and derive decision rules in incomplete information systems (Skowron and Rauszer, 1992; Skowron, 1993; Slowinski and Stepaniuk, 1989; Kryszkiewicz, 1998, 1999; Leung and Li, 2003). For example, Kryszkiewicz (1998, 1999) defined tolerance relation in incomplete information systems and proposed the concepts of generalized decision and relative reduct for an object. By using discernibility function and Boolean reasoning techniques, one can obtain the relative reduct of every object and the optimal decision rules supported by the object. Leung and Li (2003) gave a computational approach of relative reduct of each object by using maximal consistent block techniques.

This paper is organized as follows: Section 2 discuss the topological transition to rough generalizations. In Section 3 we initiated five types of rough generalizations using any binary relation and using the topological structure generated by this relation. The properties of the new five types of generalizations and some approximations are studied in Section 4. The conclusion of this work is discussed in Section 5.

2. Rough sets topological view

The reference space in rough set theory is the approximation space whose topological structure is generated by the equivalence relation R. This topology has the property that every open set in it is closed. This topology is called quasi-discrete topology and it is a kind of approximations that are transitive.

We will express rough set properties in terms of topological concepts. Let X be a subset of U. Let $cl(X)$; $\text{int}(X)$ and $b(X)$ be closure, interior, and boundary points respectively. X is exact if $b(X) = U$, otherwise X is rough. It is clear X is exact iff $cl(X) = \text{int}(X)$. In Pawlak space a subset $X \subseteq U$ has two possibilities rough or exact. For a general topological space, $X \subseteq U$ has the following types of definability:

(1) X is totally definable if X is exact set "$cl(X) = X = \text{int}(X)$",
(2) X is internally definable if $X = \text{int}(X)$, $X \neq cl(X)$,
(3) X is externally definable if $X \neq \text{int}(X)$, $X = cl(X)$,
(4) X is undefinable if $X \neq \text{int}(X)$, $X \neq cl(X)$.

Original rough membership function is defined using equivalence classes. We will extend it to topological spaces. If τ is a topology on a finite set U, where its base is β, then the rough membership function is

$$\mu_X^{\tau}(x) = \frac{|\{\cap\beta_x\} \cap X|}{|\{\cap\beta_x\}|}, \quad \beta_x \in \beta, \quad x \in U,$$

where β_x is any member of β containing x. It can be shown that this number is independent of the choice of bases. Since, the intersection of all members of the topology containing X concedes with the intersection of all members of a base containing x.

3. Rough topological approximations

In this section we introduce the basic notations to topological lower and topological upper approximations. Here we define two topologies generated by any binary general relation R. The subbase of the first topology τ_{xR} (right topology) is the right neighborhood xR. Also, the topology τ_{Rx} (left topology) is the left neighborhood Rx where, $xR = \{y \in X : xRy\}$ and $Rx = \{y \in X : yRx\}$.

The topological lower and the topological upper approximations of a subset X of U are defined using the topologies τ_{xR} and τ_{Rx} as follows:

$$\underline{R}_{\tau_{xR}}(X) = \cup\{xR : xR \subseteq X\} \quad \text{and}$$
$$\overline{R}_{\tau_{xR}}(X) = \cup\{xR : xR \cap X \neq \phi\},$$

$$\underline{R}_{\tau_{Rx}}(X) = \cup\{Rx : Rx \subseteq X\} \quad \text{and}$$
$$\overline{R}_{\tau_{Rx}}(X) = \cup\{Rx : Rx \cap X \neq \phi\}.$$

Some types of topological rough sets are initiated in the following definition.

Definition 3.1. Let (U, R) be a generalized approximation space. Let τ_{xR} and τ_{Rx} be the two topologies generated using the relation R. Then the subset $X \subseteq U$ is called:

(i) Semi rough (briefly S_{12}-rough) if $X \subseteq \overline{R}_{\tau_{Rx}}(\underline{R}_{\tau_{xR}}(X))$.
(ii) Pre rough (briefly P_{12}-rough) if $X \subseteq \overline{R}_{\tau_{xR}}(\underline{R}_{\tau_{Rx}}(A))$.
(iii) Semi-pre rough (briefly β_{12}-rough) if $X \subseteq \overline{R}_{\tau_{Rx}}(\underline{R}_{\tau_{xR}}(\overline{R}_{\tau_{Rx}}(X)))$.
(iv) α-Rough (briefly α_{12}-rough) if $X \subseteq \underline{R}_{\tau_{xR}}(\overline{R}_{\tau_{Rx}}(\underline{R}_{\tau_{xR}}(X)))$.
(v) γ-Rough (briefly γ_{12}-rough) if $X \subseteq \overline{R}_{\tau_{Rx}}(\underline{R}_{\tau_{xR}}(X)) \cup \underline{R}_{\tau_{xR}}(\overline{R}_{\tau_{Rx}}(X))$.

The family of all S_{12}-rough (resp. P_{12}-rough, β_{12}-rough, α_{12}-rough and γ_{12}-rough) sets in (U, R) is denoted by $FS_{12}(U)$ (resp. $FP_{12}(U)$, $F\beta_{12}(U)$, $F\alpha_{12}(U)$ and $F\gamma_{12}(U)$).

The complement of S_{12}-rough (resp. P_{12}-rough, β_{12}-rough, α_{12}-rough and γ_{12}-rough) set is called S_{12}^c-rough (resp. P_{12}^c-rough, β_{12}^c-rough, α_{12}^c-rough and γ_{12}^c-rough).

The family of all S_{12}^c-rough (resp. P_{12}^c-rough, β_{12}^c-rough, α_{12}^c-rough and γ_{12}^c-rough) sets of (U, R) is denoted by $FS_{12}^c(U)$ (resp. $FP_{12}^c(U)$, $F\beta_{12}^c(U)$, $F\alpha_{12}^c(U)$ and $F\gamma_{12}^c(U)$).

Proposition 3.1. *In the generalized approximation space (U, R), we can prove that:*

(i) $F\alpha_{12}(U) < FS_{12}(U) < F\gamma_{12}(U) < F\beta_{12}(U)$.
(ii) $F\alpha_{12}(U) < FP_{12}(U) < F\gamma_{12}(U) < F\beta_{12}(U)$.

Proof. Obvious. □

The following example illustrates the above definition.

Example 3.1. Let R be any binary relation defined on a nonempty set $U = \{a, b, c, d\}$ defined by $R = \{(a, a), (a, c), (a, d), (b, b), (b, d), (c, a), (c, b), (c, d), (d, a)\}$. Hence the subbase of τ_{xR} is $\{\{a,c,d\}, \{b,d\}, \{a,b,d\}, \{a\}\}$ and the subbase of τ_{Rx} is $\{\{a,c,d\}, \{b,c\}, \{a\}, \{a,b,c\}\}$. Then

$$\tau_{xR} = \{U, \phi, \{a, c, d\}, \{b, d\}, \{a, b, d\}, \{a\}, \{d\}, \{a, d\}\},$$

$$\tau_{Rx} = \{U, \phi, \{a, c, d\}, \{a, c\}, \{a, b, c\}, \{a\}, \{c\}, \{b, c\}\}.$$

Consequently,

$$F\alpha_{12}(U) = FS_{12}(U) = \{U, \phi, \{a, c, d\}, \{b, d\}, \{a, b, d\}, \{a\}, \{d\}, \{a, d\}\},$$

$$F\gamma_{12}(U) = F\beta_{12}(U) = P_{rl}O(U) = \{U, \phi, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, c\}, \{a, b, c\}, \times \{b, d\}, \{a, b, d\}, \{a, c, d\}\}.$$

Definition 3.2. Let (U,R) be a generalized approximation space and $X \subseteq U$. Then the general lower (briefly λ_{12} lower) of X denoted by $\underline{\lambda}_{12}(X)$ for all $\lambda_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$ is defined by: $\underline{\lambda}_{12}(X) = \cup\{G \in F\lambda_{12}(U), G \subseteq X\}$.

Definition 3.3. Let (U,R) be a generalized approximation space and $X \subseteq U$. Then the general upper of X denoted by $\overline{\lambda}_{12}(X)$ for all $\lambda_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$ is defined by $\overline{\lambda}_{12}(X) = \cap\{H \in F\lambda^c_{12}(U), H \supseteq X\}$.

Definition 3.4. Let (U,R) be a generalized approximation space. Then for all $\lambda_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$ the topological general lower and topological general upper approximations of any subset $X \subseteq U$ are defined as: $\underline{R}_{\lambda_{12}}(X) = \underline{\lambda}_{12}(X)$; $\overline{R}_{\lambda_{12}}(X) = \overline{\lambda}_{12}(X)$.

Proposition 3.2. *Let (U,R) be a generalized approximation space generated by any binary relation R. Then for any subset $X \subseteq U$:*

(i) $\underline{R}_{\tau_{xR}}(X) \subseteq \underline{R}_{\alpha_{12}}(X) \subseteq \underline{R}_{S_{12}}(X) \subseteq \underline{R}_{\gamma_{12}}(X) \subseteq \underline{R}_{\beta_{12}}(X) \subseteq X \subseteq \overline{R}_{\beta_{12}}(X) \subseteq \overline{R}_{\gamma_{12}}(X) \subseteq \overline{R}_{S_{12}}(X) \subseteq \overline{R}_{\alpha_{12}}(X) \subseteq \overline{R}_{\tau_{xR}}(X)$.

(ii) $\underline{R}_{\tau_{Rx}}(X) \subseteq \underline{R}_{\alpha_{12}}(X) \subseteq \underline{R}_{P_{12}}(X) \subseteq \underline{R}_{\gamma_{12}}(X) \subseteq \underline{R}_{\beta_{12}}(X) \subseteq X \subseteq \overline{R}_{\beta_{12}}(X) \subseteq \overline{R}_{\gamma_{12}}(X) \subseteq \overline{R}_{P_{12}}(X) \subseteq \overline{R}_{\alpha_{12}}(X) \subseteq \overline{R}_{\tau_{Rx}}(X)$.

Proof.

(i) $\underline{R}_{\tau_{xR}}(X) = \cup\{G \in \tau_{xR} : G \subseteq X\} \subseteq \cup\{G \in F\alpha_{12}(U) : G \subseteq X\}$
$\subseteq \cup\{G \in FS_{12}(U) : G \subseteq X\} \subseteq \cup\{G \in F\gamma_{12}(U) : G \subseteq X\}$
$\subseteq \cup\{G \in F\beta_{12}(U) : G \subseteq X\} \subseteq X \subseteq \cap\{H \in F\beta^c_{12}(U) : X \subseteq H\}$
$\subseteq \cap\{H \in F\gamma^c_{12}(U) : X \subseteq H\} \subseteq \cap\{H \in FS^c_{12}(U) : X \subseteq H\}$
$\subseteq \cap\{H \in F\alpha^c_{12}(U) : X \subseteq H\} \subseteq \cap\{H \in \tau^c_{xR} : X \subseteq H\}$.

Hence, $\underline{R}_{\tau_{xR}}(X) \subseteq \underline{R}_{\alpha_{12}}(X) \subseteq \underline{R}_{S_{12}}(X) \subseteq \underline{R}_{\gamma_{12}}(X) \subseteq \underline{R}_{\beta_{12}}(X) \subseteq X \subseteq \overline{R}_{\beta_{12}}(X) \subseteq \overline{R}_{\gamma_{12}}(X) \subseteq \overline{R}_{S_{12}}(X) \subseteq \overline{R}_{\alpha_{12}}(X) \subseteq \overline{R}_{\tau_{xR}}(X)$.

(ii) By the same manner as (i). □

Example 3.2. According to Example 3.1, if $X = \{a,b\}$ and $Y = \{d\}$. Then $\underline{R}_{\alpha_{12}}(X) = \{a\}, \underline{R}_{P_{12}}(X) = \{a,b\}, \overline{R}_{\alpha_{12}}(Y) = \{b,c,d\}$ and $\overline{R}_{P_{12}}(Y) = \{d\}$. So $\underline{R}_{\alpha_{12}}(X) \subset \underline{R}_{P_{12}}(X)$ and $\overline{R}_{P_{12}}(Y) \subset \overline{R}_{\alpha_{12}}(Y)$.

Proposition 3.3. *Let (U, R) be a generalized approximation space generated by any binary relation R. Then for any two subsets $X, Y \subseteq U$ we have for all $\lambda_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$:*

(i) $\underline{R}_{\lambda_{12}}(\phi) = \overline{R}_{\lambda_{12}}(\phi) = \phi$ *and* $\underline{R}_{\lambda_{12}}(U) = \overline{R}_{\lambda_{12}}(U) = U$.
(ii) *If* $X \subseteq Y$, *then* $\underline{R}_{\lambda_{12}}(X) \subseteq \underline{R}_{\lambda_{12}}(Y)$.
(iii) *If* $X \subseteq Y$, *then* $\overline{R}_{\lambda_{12}}(X) \subseteq \overline{R}_{\lambda_{12}}(Y)$.
(iv) $\underline{R}_{\lambda_{12}}(X \cup Y) \supseteq \underline{R}_{\lambda_{12}}(X) \cup \underline{R}_{\lambda_{12}}(Y)$.
(v) $\overline{R}_{\lambda_{12}}(X \cup Y) \supseteq \overline{R}_{\lambda_{12}}(X) \cup \overline{R}_{\lambda_{12}}(Y)$.
(vi) $\underline{R}_{\lambda_{12}}(X \cap Y) \subseteq \underline{R}_{\lambda_{12}}(X) \cap \underline{R}_{\lambda_{12}}(Y)$.
(vii) $\overline{R}_{\lambda_{12}}(X \cap Y) \subseteq \overline{R}_{\lambda_{12}}(X) \cap \overline{R}_{\lambda_{12}}(Y)$.
(viii) $\underline{R}_{\lambda_{12}}(X^c) = (\overline{R}_{\lambda_{12}}(X))^c$.
(ix) $\overline{R}_{\lambda_{12}}(X^c) = (\underline{R}_{\lambda_{12}}(X))^c$.

Proof. By using the properties of $\underline{\lambda}_{12}$ and $\overline{\lambda}_{12}$ for all $\lambda_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$ the proof is complete. □

The following example, at $\lambda_{12} = \alpha_{12}$ illustrates that the inverse of Property (iv) in the above proposition in general does not hold for all $\lambda_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$.

Example 3.3. According to Example 3.1 if $X = \{a\}$ and $Y = \{c,d\}$, then $\underline{R}_{\alpha_{12}}(X) = \{a\}, \underline{R}_{\alpha_{12}}(Y) = \{d\}$, and $\underline{R}_{\alpha_{12}}(X \cup Y) \neq \underline{R}_{\alpha_{12}}(X) \cup \underline{R}_{\alpha_{12}}(Y)$.

The following example shows that the inverse of the Properties (v) and (vi) in Proposition 3.2, in general are not true for all $\lambda_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$, we consider $\lambda_{12} = \beta_{12}$.

Example 3.4. According to Example 3.1, if $X_1 = \{a\}, X_2 = \{d\}, Y_1 = \{a,b\}$ and $Y_2 = \{b,d\}$, then $\overline{R}_{\beta_{12}}(X_1) = \{a,c\}, \overline{R}_{\beta_{12}}(X_2) = \{d\}, \overline{R}_{\beta_{12}}(X_1 \cup X_2) = U, \underline{R}_{\beta_{12}}(Y_1) = \{a,b\}, \underline{R}_{\beta_{12}}(Y_2) = \{b,d\}$, and $\underline{R}_{\beta_{12}}(Y_1 \cap Y_2) = \phi$, hence $\overline{R}_{\beta_{12}}(X_1 \cup X_2) \neq \overline{R}_{\beta_{12}}(X_1) \cup \overline{R}_{\beta_{12}}(X_2)$, and $\underline{R}_{\beta_{12}}(Y_1 \cap Y_2) \neq \underline{R}_{\beta_{12}}(Y_1) \cap \underline{R}_{\beta_{12}}(Y_2)$.

The following example shows that the Property (vii) in Proposition 3.2, in general are not true for all $\lambda_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$, we consider here $\lambda_{12} = P_{12}$.

Example 3.5. According to Example 3.1, if $X = \{a,c,d\}$ and $Y = \{b,c,d\}$. Then $\overline{R}_{P_{12}}(X) = U, \overline{R}_{P_{12}}(Y) = \{b,c,d\}$ and $\overline{R}_{P_{12}}(X \cap Y) = \{c,d\}$, hence $\overline{R}_{P_{12}}(X \cap Y) \neq \overline{R}_{P_{12}}(X) \cap \overline{R}_{P_{12}}(Y)$.

Proposition 3.4. *Let (X,R) be a generalized approximation space defined on any binary relation R. Then for all $\lambda_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$,* and for any $X \subseteq U$, *the following properties do not hold:*

(i) $\underline{R}_{\lambda_{12}}(\underline{R}_{\lambda_{12}}(X)) = \overline{R}_{\lambda_{12}}(\underline{R}_{\lambda_{12}}(X)) = \underline{R}_{\lambda_{12}}(X)$.
(ii) $\overline{R}_{\lambda_{12}}(\overline{R}_{\lambda_{12}}(X)) = \underline{R}_{\lambda_{12}}(\overline{R}_{\lambda_{12}}(X)) = \overline{R}_{\lambda_{12}}(X)$.

The following example illustrates the above proposition, using $\lambda_{12} = \beta_{12}$.

Example 3.6. According to Example 3.1, if $X = \{a\}$ and $Y = \{c,d\}$. Then $\underline{R}_{\beta_{12}}(X) = \{a\}, \underline{R}_{\beta_{rl}}\underline{R}_{\beta_{rl}}(A) = \{a\}, \overline{R}_{\beta_{rl}}\underline{R}_{\beta_{rl}}(A) = \{a,c\}, \overline{R}_{\beta_{rl}}(B) = \{c,d\}, \overline{R}_{\beta_{rl}}\overline{R}_{\beta_{rl}}(B) = \{c,d\}$, and $\underline{R}_{\beta_{12}}(\overline{R}_{\beta_{12}}(Y)) = \{d\}$, hence $\underline{R}_{\beta_{12}}(\underline{R}_{\beta_{12}}(X)) \neq \overline{R}_{\beta_{12}}(\underline{R}_{\beta_{12}}(X))$, and $\overline{R}_{\beta_{12}}(\overline{R}_{\beta_{12}}(Y)) \neq \underline{R}_{\beta_{12}}(\overline{R}_{\beta_{12}}(Y))$.

Lemma 3.1. *Let (U, R) be a generalized approximation space, and for any $X \subseteq U$, then $(Cl_{\lambda_{12}}(X))^c = int_{\lambda_{12}}(X^c)$ for all $\lambda_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$.*

Proof. Let $X \subseteq U$, then for all $\lambda_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$, we get:

$$(Cl_{\lambda_{12}}(X))^c = U - Cl_{\lambda_{12}}(X)$$
$$= U - \cap\{F \subseteq U : F \text{ is } \lambda_{12} \text{ upper set and } X \subseteq F\}$$
$$= \cup\{(U - F) \subseteq U : (U - F) \text{ is } \lambda_{12} \text{ lower set, } (U - F) \subseteq (U - X) = int_{\lambda_{12}}(U - X).$$

Thus $(Cl_{\lambda_{12}}(X))^c = int_{\lambda_{12}}(X^c)$. □

Proposition 3.5. *Let (U, R) be a generalized approximation space defined on any binary relation R for any two subsets $X, Y \subseteq U$ we have: $\underline{R}_{\lambda_{12}}(X - Y) \subseteq \underline{R}_{\lambda_{12}}(X) - \underline{R}_{\lambda_{12}}(Y)$, for all $\lambda_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$.*

Proof. As $X - Y = X \cap Y^c$, then:

$$\underline{R}_{\lambda_{12}}(X - Y) = int_{\lambda_{12}}(X - Y) = int_{\lambda_{12}}(X \cap Y^c) \subseteq int_{\lambda_{12}}(X) \cap int_{\lambda_{12}}(Y^c).$$

By Lemma 3.1, we have:

$$\underline{R}_{\lambda_{12}}(X - Y) \subseteq int_{\lambda_{12}}(X) \cap (Cl_{\lambda_{12}}(Y))^c = int_{\lambda_{12}}(X) - Cl_{\lambda_{12}}(Y) \subseteq int_{\lambda_{12}}(X) - int_{\lambda_{12}}(Y)$$

Thus $\underline{R}_{\lambda_{12}}(X - Y) \subseteq \underline{R}_{\lambda_{12}}(X) - \underline{R}_{\lambda_{12}}(Y)$.

The next example illustrates that the inverse of Proposition 3.5, in general does not hold with respect to $\lambda_{12} = \beta_{12}$. □

Example 3.7. According to Example 3.1, if $X = \{a,b\}$ and $Y = \{a\}$, then $\underline{R}_{\beta_{12}}(Y) = \{a\}, \underline{R}_{\beta_{12}}(X) = \{a,b\}$, and $\underline{R}_{\beta_{12}}(X - Y) = \phi$, thus $\underline{R}_{\beta_{12}}(X - Y) \neq \underline{R}_{\beta_{12}}(X) - \underline{R}_{\beta_{12}}(Y)$.

4. Topological generalizations of rough concepts

In this section we introduce and study some topological generalizations for some concepts of the rough set theory by using the λ_{12} lower and λ_{12} upper approximations.

Definition 4.1. Let (U, R) be a generalized approximation space defined on any binary relation R. Then for all $\lambda_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$ and for any $X \subseteq U$ we define:

(i) X is totally topologicalλ_{12}-definable (λ_{12}-exact) set if $\underline{R}_{\lambda_{12}}(X) = \overline{R}_{\lambda_{12}}(X) = X$.
(ii) X is internally topological λ_{12}-definable set if $\underline{R}_{\lambda_{12}}(X) = X$, and $\overline{R}_{\lambda_{12}}(X) \neq X$.
(iii) X is externally topological λ_{12}-definable set if $\underline{R}_{\lambda_{12}}(X) \neq X$, and $\overline{R}_{\lambda_{12}}(X) = X$.
(iv) X is topologically λ_{12}-indefinable (λ_{12}-rough) set if $\underline{R}_{\lambda_{12}} \neq (X)$ A, and $\overline{R}_{\lambda_{12}}(X) \neq X$.

Example 4.1. According to Example 3.1, for subsets $X = \{d\}$ and $Y = \{c, d\}$, X is topologicallyβ_{12}-exact set, X is topologically internally α_{12}-definable set, Y is topologically S_{12}-rough set, and Y is topologically externally P_{12}-definable set.

Definition 4.2. Let (U, R) be a generalized approximation space defined on any binary relation R. Then we can introduce the generalized accuracy measure for any set $X \subseteq U$ as the following:

$$Acc_{\lambda_{12}}(X) = \frac{|\underline{R}_{\lambda_{12}}(X)|}{|\overline{R}_{\lambda_{12}}(X)|}, \quad \overline{R}_{\lambda_{12}}(X) \neq \phi,$$

where $\lambda_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$,and $|X|$ denoted the cardinality of the set X.

The number $Acc_{\lambda_{12}}$of the above definition is a measure of the degree of exactness of any subset $X \subseteq U$. So by this measure we will determine, what is the best of our definitions for the λ_{12} lower and λ_{12} upper approximations. We can notice that:

(i) $0 \leqslant Acc_{\tau_{xR}}(X) \leqslant Acc_{\alpha_{12}}(X) \leqslant Acc_{S_{12}}(X) \leqslant Acc_{\gamma_{12}}(X) \leqslant Acc_{\beta_{12}}(X) \leqslant 1$.
(ii) $0 \leqslant Acc_{\tau_{xR}}(X) \leqslant Acc_{\alpha_{12}}(X) \leqslant Acc_{P_{12}}(X) \leqslant Acc_{\gamma_{12}}(X) \leqslant Acc_{\beta_{12}}(X) \leqslant 1$.

So the best definition here is $\lambda_{12} = \beta_{12}$.

The next example studies the comparison between β_{12} and S_{12}.

Example 4.2. According to Example 3.1, we have the following table:

Set X	$Acc_{S_{12}}$	$Acc_{\beta_{12}}$
$\{d\}$	1/3	1
$\{a,c\}$	1/2	1
$\{b,d\}$	2/3	1
$\{a,b,c\}$	1/3	1

By using the definitions of rough concepts at $\lambda_{12} = \beta_{12}$ we can tends to exactness of many sets. This will lead to accurate results in many data reduction applications using new topological approaches. Next works shall deal with more types of applications in data reductions, data processing, image processing and rule extraction.

Definition 4.3. Let (U, R) be a generalized approximation space defined on any binary relation R. Then for all $\lambda_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$ and for any $X, Y \subseteq U$ we define:

(i) $X \underset{\approx\lambda_{12}}{\subset} Y$ if $\underline{R}_{\lambda_{12}}(X) \subseteq \underline{R}_{\lambda_{12}}(Y)$.
(ii) $X \tilde{\subset}_{\lambda_{12}} Y$ if $\overline{R}_{\lambda_{12}}(X) \subseteq \overline{R}_{\lambda_{12}}(Y)$.The illustration of the facts of the above definition are given as below example.

Example 4.3. According to Example 3.1, let $X_1 = \{a,c,d\}$, $X_2 = \{b,c,d\}$, $X_3 = \{b,d\}$ and $X_4 = \{c,d\}$, then we have: $X_4 \underset{\approx p_{12}}{\subset} X_3, X_2 \tilde{\subset}_{s_{12}} X_1$ and $X_2 \tilde{\subset}_{\beta_{12}} X_1$.

Definition 4.4. Let (U, R) be a generalized approximation space defined on any binary relation R. For any subset $X \subseteq U$ and any element $x \in U$, for all $\lambda_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$, we define:

(i) $x \underset{\approx\lambda_{12}}{\in} X$ if $x \in \underline{R}_{\lambda_{12}}(X)$.
(ii) $x \tilde{\in}_{\lambda_{12}} X$ if $x \in \overline{R}_{\lambda_{12}}(X)$.

Proposition 4.1. *Let (U, R) be a generalized approximation space defined on any binary relation R. For any subset $X \subseteq U$ and any element $x \in U$, for all $\lambda_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$, we have:*

(i) *if $x \underset{\approx\lambda_{12}}{\in} X$ then $x \in X$.*
(ii) *if $x \tilde{\notin}_{\lambda_{12}} X$ then $x \notin X$.*

Proof. The proof is direct from definitions. □

The following example shows that the inverse of Proposition 4.1, in general does not hold.

Example 4.4. According to Example 3.1, let $X = \{a, b, c\}$ and $Y = \{a, d\}$, then we have $b \in X$, but $b \underset{\approx s_{12}}{\notin} X$ and $b \underset{\approx \alpha_{12}}{\notin} X$. Also, we have $b \notin Y$, but $b \tilde{\notin}_{s_{12}} Y, b \tilde{\notin}_{\alpha_{12}} Y, b \tilde{\notin}_{p_{rL}} D, b \tilde{\notin}_{\gamma_{12}} Y$ and $b \tilde{\notin}_{\beta_{12}} Y$.

5. Conclusions

One of the main contributions of this paper is in the area of topological classifications. Based on topological space, we presented an underlying theory to explain how classifications of rough sets topologically may be performed.

We conclude that the intermingling of topology in the construction of some approximation space concepts will help to get results with abundant logical statements. That is discovering hidden relationships among data and, moreover, probably helps in producing accurate programs (Duntsch et al., 2001; Lipski, 1981).

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