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Some new Hermite-Hadamard type inequalities for *MT*-convex functions on differentiable coordinates

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ABSTRACT

In this paper, we introduce the notion of *MT*-convex functions on co-ordinates and establish some new integral inequalities of Hermite-Hadamard type for *MT*-convex functions on co-ordinates on a rectangle Δ in the plane $\mathbb{R}^2$.

1. Introduction

Let us recall some definitions of various convex functions that are known in the literature.

Definition 1.1 (*Guo et al., 2016; Sarikaya et al., 2016*). A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on the interval I, if for all $x, y \in I$ and $t \in (0,1)$ it satisfies the following inequality:

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y). \tag{1.1}$$

Definition 1.2 (*Tunç et al., 2013; Park, 2015*). A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be *MT*-convex on I, if it is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ it satisfies the following inequality:

$$f(tx + (1-t)y) \leqslant \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.2}$$

Example of such functions are:

(1) The functions $f, g : (1,\infty) \to \mathbb{R}$, where

$$f(x) = x^p \quad \text{and} \quad g(x) = (1+x)^p, \quad p \in \left(0, \frac{1}{1000}\right)$$

(2) The function $h : [1, \frac{3}{2}] \to \mathbb{R}$, where

$$h(x) = (1+x^2)^q, \quad q \in \left(0, \frac{1}{1000}\right).$$

Notice that these functions are not convex.

Definition 1.3 Guo et al., 2016. If $(X, \mathcal{A})$ is a measurable space, then $f : X \to \mathbb{R}$ is measurable if $f^{-1}(B) \in \mathcal{A}$ for every Borel set $B \in \mathcal{B}(\mathbb{R})$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable if $f^{-1}(B)$ is a Lebesgue measurable subset of $\mathbb{R}^n$ for every Borel subset B of $\mathbb{R}$.

Let us now consider a formal definition for co-ordinated convex functions:

Definition 1.4 (*Dragomir et al., 2000; Dragomir, 2001*). A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on $\Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2$ with $a < b$ and $c < d$ if for all $t, \lambda \in (0,1)$ and $(x,y), (z,w) \in \Delta$ satisfies the following inequality:

$$f(tx + (1-t)z, \lambda y + (1-\lambda)w) \leqslant t\lambda f(x,y) + t(1-\lambda)f(x,w) + (1-t)\lambda f(z,y) + (1-t)(1-\lambda)f(z,w). \tag{1.3}$$

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Definition 1.5 Samko et al., 1993. The incomplete beta function is defined by

$$B_x(a,b)=\int_0^x z^{a-1}(1-z)^{b-1}dz,\quad a,b>0.$$

For $z=1$, the incomplete beta function coincides with the complete beta function.

Throughout this paper we denote by $L_1(\Delta)$ the set of all Lebesgue integrable functions on Δ as indicated by the authors in Guo et al. (2016). Some integral inequalities of Hermite-Hadamard type for co-ordinated convex functions on the rectangle in the plane $\mathbb{R}^2$ may be recited as follows:

Theorem 1.1 (*Dragomir et al., 2000; Dragomir, 2001, Theorem 2.2*). Let $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$ be convex on the co-ordinates on Δ with $a<b$ and $c<d$. Then

$$\begin{aligned}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\leqslant\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\&\leqslant\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dy\,dx\\&\leqslant\frac{1}{4}\left[\frac{1}{b-a}\left(\int_a^b f(x,c)dx+\int_a^b f(x,d)dx\right)+\frac{1}{d-c}\int_c^d\left(\int_c^d f(a,y)dx+\int_c^d f(b,y)dx\right)\right]\\&\leqslant\frac{1}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)].\end{aligned}$$

Theorem 1.2 Guo et al., 2015, Theorem 2.1. Let $f:\Omega\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partial differentiable mapping on Ω^o (the interior of Ω) and let $\Delta=[a,b]\times[c,d]\subseteq\Omega^o$ with $a<b, c<d$ and $\frac{\partial^2 f}{\partial x\partial y}\in L_1(\Delta)$. If $\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q$ is convex on the co-ordinates on Δ and $q\geqslant 1$, then the following inequality holds:

$$|I(f)|\leqslant\frac{1}{4}\left(\frac{1}{9}\right)^{\frac{1}{q}}\{g_q(1,2,2,4)+g_q(4,2,2,1)+g_q(2,1,4,2)+g_q(2,4,1,2)\},$$

where

$$I(f)=\frac{16}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dx\,dy\right],$$

and

$$g_q(r_1,r_2,r_3,r_4)=\left[r_1|f_{xy}(a,c)|^q+r_2|f_{xy}(a,d)|^q+r_3|f_{xy}(b,c)|^q+r_4|f_{xy}(b,d)|^q\right]^{\frac{1}{q}}.$$

For more information on integral inequalities of the Hermite-Hadamard type for various kinds of convex functions, the reader is referred to the recently published papers (Park, 2013; Guo et al., 2016; Meftah and Boukerrioua, 2015; Xi and Qi, 2015; Bai et al., 2016), and the closely related references therein.

In this paper, we will establish more integral inequalities of the Hermite-Hadamard type for *MT*-convex functions on the co-ordinates on a rectangle Δ in the plane $\mathbb{R}^2$.

2. A definition and a lemma

Motivated by Definitions 1.1 and 1.3, we introduce the notion of "co-ordinated *MT*-convex function".

Definition 2.1. We say that a function $f:\Delta\to\mathbb{R}$ is *MT*-convex on the co-ordinates on $\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2$ with $a<b$ and $c<d$, if it is nonnegative and for all $t,\lambda\in(0,1)$ and $(x,y),(z,w)\in\Delta$ it satisfies the following inequality:

$$\begin{aligned}f(tx+(1-t)z,\lambda y+(1-\lambda)w)&\leqslant\frac{\sqrt{t\lambda}}{4\sqrt{(1-t)(1-\lambda)}}f(x,y)\\&+\frac{\sqrt{t(1-\lambda)}}{4\sqrt{\lambda(1-t)}}f(x,w)+\frac{\sqrt{\lambda(1-t)}}{4\sqrt{t(1-\lambda)}}f(z,y)+\frac{\sqrt{(1-t)(1-\lambda)}}{4\sqrt{t\lambda}}f(z,w).\end{aligned}\tag{2.1}$$

Now, we give an example to show that a function can be *MT*-convex on the co-ordinates on Δ without being convex on the co-ordinates on Δ. The function $f(x,y):(1,\infty)\times(1,\infty)\to\mathbb{R}$, where

$$f(x,y)=x^c+y^c\quad\text{for}\quad c\in\left(0,\frac{1}{1000}\right)$$

is *MT*-convex on the co-ordinates on $\Delta=(1,\infty)\times(1,\infty)$ while this is not convex on the co-ordinates on Δ.

In order to prove our main results, we need the following lemma.

Lemma 2.1. *Let $f:\Omega\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partial differentiable mapping on Ω^o and let $\Delta=[a,b]\times[c,d]\subseteq\Omega^o$ with $a<b, c<d$ and $\frac{\partial^2 f}{\partial x\partial y}\in L_1(\Delta)$. Then the following equality holds:*

$$\begin{aligned}I(f):&=\frac{16}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dx\,dy\right]\\&=\int_0^1\int_0^1 t\lambda f_{xy}\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b,\frac{\lambda}{2}c+\left(1-\frac{\lambda}{2}\right)d\right)dt\,d\lambda\\&+\int_0^1\int_0^1 t\lambda f_{xy}\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b,\left(1-\frac{\lambda}{2}\right)c+\frac{\lambda}{2}d\right)dt\,d\lambda\\&-\int_0^1\int_0^1 t\lambda f_{xy}\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b,\left(1-\frac{\lambda}{2}\right)c+\frac{\lambda}{2}d\right)dt\,d\lambda\\&-\int_0^1\int_0^1 t\lambda f_{xy}\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b,\frac{\lambda}{2}c+\left(1-\frac{\lambda}{2}\right)d\right)dt\,d\lambda.\end{aligned}\tag{2.2}$$

Proof. By integration by parts, we have

$$\begin{aligned}&\int_0^1\int_0^1 t\lambda f_{xy}\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b,\frac{\lambda}{2}c+\left(1-\frac{\lambda}{2}\right)d\right)dt\,d\lambda\\&=\frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\int_0^1 f\left(\frac{a+b}{2},\frac{\lambda}{2}c+\left(1-\frac{\lambda}{2}\right)d\right)d\lambda\right.\\&\left.-\int_0^1 f\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b,\frac{c+d}{2}\right)dt+\int_0^1\int_0^1 f\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b,\frac{\lambda}{2}c+\left(1-\frac{\lambda}{2}\right)d\right)dt\,d\lambda\right]\\&=\frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{2}{d-c}\int_{\frac{c+d}{2}}^d f\left(\frac{a+b}{2},y\right)dy\right.\\&\left.-\frac{2}{b-a}\int_{\frac{a+b}{2}}^b f\left(x,\frac{c+d}{2}\right)dx+\frac{4}{(b-a)(d-c)}\int_{\frac{c+d}{2}}^d\int_{\frac{a+b}{2}}^b f(x,y)dx\,dy\right].\end{aligned}$$

Similarly, we find

$$\int_0^1\int_0^1 t\lambda f_{xy}\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b,\left(1-\frac{\lambda}{2}\right)c+\frac{\lambda}{2}d\right)dt\,d\lambda$$
$$=\frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{2}{d-c}\int_c^{\frac{c+d}{2}}f\left(\frac{a+b}{2},y\right)dy\right.$$
$$\left.-\frac{2}{b-a}\int_a^{\frac{a+b}{2}}f\left(x,\frac{c+d}{2}\right)dx+\frac{4}{(b-a)(d-c)}\int_c^{\frac{c+d}{2}}\int_a^{\frac{a+b}{2}}f(x,y)dx\,dy\right],$$

$$\int_0^1\int_0^1 t\lambda f_{xy}\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b,\left(1-\frac{\lambda}{2}\right)c+\frac{\lambda}{2}d\right)dt\,d\lambda$$
$$=-\frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{2}{d-c}\int_c^{\frac{c+d}{2}}f\left(\frac{a+b}{2},y\right)dy\right.$$
$$\left.-\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}f\left(x,\frac{c+d}{2}\right)dx+\frac{4}{(b-a)(d-c)}\int_c^{\frac{c+d}{2}}\int_{\frac{a+b}{2}}^{b}f(x,y)dx\,dy\right],$$

and

$$\int_0^1\int_0^1 t\lambda f_{xy}\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b,\frac{\lambda}{2}c+\left(1-\frac{\lambda}{2}\right)d\right)dt\,d\lambda$$
$$=-\frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{2}{d-c}\int_{\frac{c+d}{2}}^{d}f\left(\frac{a+b}{2},y\right)dy\right.$$
$$\left.-\frac{2}{b-a}\int_a^{\frac{a+b}{2}}f\left(x,\frac{c+d}{2}\right)dx+\frac{4}{(b-a)(d-c)}\int_{\frac{c+d}{2}}^{d}\int_a^{\frac{a+b}{2}}f(x,y)dx\,dy\right].$$

This ends the proof. □

3. Some integral inequalities of the Hermite-Hadamard type

Now we start off to establish some integral inequalities of the Hermite-Hadamard type for the above-introduced *MT*-convex functions on the co-ordinates.

Theorem 3.1. *Let $f:\Omega\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partial differentiable mapping on Ω^o (the interior of Ω) and let $\Delta=[a,b]\times[c,d]\subseteq\Omega^o$ with $a<b, c<d$ and $\frac{\partial^2 f}{\partial x\partial y}\in L_1(\Delta)$. If $\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q$ is MT-convex on the co-ordinates on Δ and $q\geqslant 1$, then the following inequality holds:*

$$|I(f)|\leqslant\left(\frac{1}{4}\right)^{1-\frac{2}{q}}\{g_q(a_1,a_2,a_2,a_3)+g_q(a_3,a_2,a_2,a_1)$$
$$+g_q(a_2,a_1,a_3,a_2)+g_q(a_2,a_3,a_1,a_2)\},$$

where

$$g_q(r_1,r_2,r_3,r_4)=\left[r_1|f_{xy}(a,c)|^q+r_2|f_{xy}(a,d)|^q+r_3|f_{xy}(b,c)|^q\right.$$
$$\left.+r_4|f_{xy}(b,d)|^q\right]^{\frac{1}{q}},$$

and

$$a_1=B^2_{\frac{1}{2}}\left(\frac{5}{2},\frac{1}{2}\right),\quad a_2=B_{\frac{1}{2}}\left(\frac{5}{2},\frac{1}{2}\right)B_{\frac{1}{2}}\left(\frac{3}{2},\frac{3}{2}\right),\quad a_3=B^2_{\frac{1}{2}}\left(\frac{3}{2},\frac{3}{2}\right).$$

Proof. By using Lemma 2.1 and by changing the variables $u=t/2$ and $v=\lambda/2$, we have

$$|I_f|\leqslant\int_0^1\int_0^1 t\lambda\left|f_{xy}\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b,\frac{\lambda}{2}c+\left(1-\frac{\lambda}{2}\right)d\right)\right|dt\,d\lambda$$
$$+\int_0^1\int_0^1 t\lambda\left|f_{xy}\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b,\left(1-\frac{\lambda}{2}\right)c+\frac{\lambda}{2}d\right)\right|dt\,d\lambda$$
$$+\int_0^1\int_0^1 t\lambda\left|f_{xy}\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b,\left(1-\frac{\lambda}{2}\right)c+\frac{\lambda}{2}d\right)\right|dt\,d\lambda$$
$$+\int_0^1\int_0^1 t\lambda\left|f_{xy}\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b,\frac{\lambda}{2}c+\left(1-\frac{\lambda}{2}\right)d\right)\right|dt\,d\lambda$$
$$=16\left\{\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}uv|f_{xy}(ua+(1-u)b,vc+(1-v)d)|du\,dv\right.$$
$$+\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}uv|f_{xy}((1-u)a+ub,(1-v)c+vd)|du\,dv$$
$$+\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}uv|f_{xy}(ua+(1-u)b,(1-v)c+vd)|du\,dv$$
$$\left.+\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}uv|f_{xy}((1-u)a+ub,vc+(1-v)d)|du\,dv\right\}.$$

Using the *MT*-convexity of $\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q$ on the co-ordinates on Δ and the power-mean integral inequality, we have

$$|I_f|\leqslant 16\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}uv\,du\,dv\right)^{1-\frac{1}{q}}\left\{\left[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}uv\left(\frac{\sqrt{uv}}{4\sqrt{(1-u)(1-v)}}|f_{xy}(a,c)|^q\right.\right.\right.$$
$$\left.\left.+\frac{\sqrt{u(1-v)}}{4\sqrt{v(1-u)}}|f_{xy}(a,d)|^q+\frac{\sqrt{v(1-u)}}{4\sqrt{u(1-v)}}|f_{xy}(b,c)|^q+\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(b,d)|^q\right)du\,dv\right]^{\frac{1}{q}}$$
$$+\left[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}uv\left(\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(a,c)|^q+\frac{\sqrt{v(1-u)}}{4\sqrt{u(1-v)}}|f_{xy}(a,d)|^q\right.\right.$$
$$\left.\left.+\frac{\sqrt{v(1-u)}}{4\sqrt{u(1-v)}}|f_{xy}(b,c)|^q+\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(b,d)|^q\right)du\,dv\right]^{\frac{1}{q}}$$
$$+\left[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}uv\left(\frac{\sqrt{u(1-v)}}{4\sqrt{v(1-u)}}|f_{xy}(a,c)|^q+\frac{\sqrt{uv}}{4\sqrt{(1-u)(1-v)}}|f_{xy}(a,d)|^q\right.\right.$$
$$\left.\left.+\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(b,c)|^q+\frac{\sqrt{u(1-v)}}{4\sqrt{v(1-u)}}|f_{xy}(b,d)|^q\right)du\,dv\right]^{\frac{1}{q}}$$
$$+\left[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}uv\left(\frac{\sqrt{v(1-u)}}{4\sqrt{u(1-v)}}|f_{xy}(a,c)|^q+\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(a,d)|^q\right.\right.$$
$$\left.\left.\left.+\frac{\sqrt{uv}}{4\sqrt{(1-u)(1-v)}}|f_{xy}(b,c)|^q+\frac{\sqrt{u(1-v)}}{4\sqrt{v(1-u)}}|f_{xy}(b,d)|^q\right)du\,dv\right]^{\frac{1}{q}}\right\}$$

$$\leqslant\left(\frac{1}{4}\right)^{1-\frac{2}{q}}\{g_q(a_1,a_2,a_2,a_3)+g_q(a_3,a_2,a_2,a_1)+g_q(a_2,a_1,a_3,a_2)+g_q(a_2,a_3,a_1,a_2)\}.$$

This ends the proof. □

Remark 3.1. Under the assumptions of Theorem 3.1, when $q=1$, we have

$$|I(f)|\leqslant 4\{g_q(a_1,a_2,a_2,a_3)+g_q(a_3,a_2,a_2,a_1)+g_q(a_2,a_1,a_3,a_2)+g_q(a_2,a_3,a_1,a_2)\}.$$

Theorem 3.2. *Let $f:\Omega\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partial differentiable mapping on Ω^o and let $\Delta=[a,b]\times[c,d]\subseteq\Omega^o$ with $a<b, c<d$. If $\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q$ is MT- convex on the co-ordinates on Δ and $q>1$, then the following inequality holds:*

$$|I(f)|\leqslant\left(\frac{q-1}{2q-1}\right)^{2\left(1-\frac{1}{q}\right)}\{g_q(b_1,b_2,b_2,b_3)+g_q(b_3,b_2,b_2,b_1)$$
$$+g_q(b_2,b_1,b_3,b_2)+g_q(b_2,b_3,b_1,b_2)\},$$

where $g_q(r_1,r_2,r_3,r_4)$ is defined in Theorem 3.1 and

$$b_1 = B^2_{\frac{1}{2}}\left(\frac{3}{2},\frac{1}{2}\right),\quad b_2 = B_{\frac{1}{2}}\left(\frac{3}{2},\frac{1}{2}\right)B_{\frac{1}{2}}\left(\frac{1}{2},\frac{3}{2}\right),\quad b_3 = B^2_{\frac{1}{2}}\left(\frac{1}{2},\frac{3}{2}\right).$$

Proof. By using Lemma 2.1 and by changing the variables $u = t/2$ and $v = \lambda/2$, we have

$$\begin{aligned}|I_f| \leqslant 16\Bigg\{&\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} uv\left|f_{xy}(ua+(1-u)b,\, vc+(1-v)d)\right|du\,dv\\
&+\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} uv\left|f_{xy}((1-u)a+ub,\,(1-v)c+vd)\right|du\,dv\\
&+\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} uv\left|f_{xy}(ua+(1-u)b,\,(1-v)c+vd)\right|du\,dv\\
&+\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} uv\left|f_{xy}((1-u)a+ub,\, vc+(1-v)d)\right|du\,dv\Bigg\}.\end{aligned}$$

Now, by using the MT-convexity of $\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q$ on the co-ordinates on Δ and Hölder's inequality, we have

$$\begin{aligned}|I_f| \leqslant 16&\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}(uv)^{\frac{q}{q-1}}du\,dv\right)^{1-\frac{1}{q}}\Bigg\{\Bigg[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\Bigg(\frac{\sqrt{uv}}{4\sqrt{(1-u)(1-v)}}|f_{xy}(a,c)|^q\\
&+\frac{\sqrt{u(1-v)}}{4\sqrt{v(1-u)}}|f_{xy}(a,d)|^q+\frac{\sqrt{v(1-u)}}{4\sqrt{u(1-v)}}|f_{xy}(b,c)|^q+\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(b,d)|^q\Bigg)du\,dv\Bigg]^{\frac{1}{q}}\\
&+\Bigg[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\Bigg(\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(a,c)|^q+\frac{\sqrt{v(1-u)}}{4\sqrt{u(1-v)}}|f_{xy}(a,d)|^q\\
&+\frac{\sqrt{v(1-u)}}{4\sqrt{u(1-v)}}|f_{xy}(b,c)|^q+\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(b,d)|^q\Bigg)du\,dv\Bigg]^{\frac{1}{q}}\\
&+\Bigg[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\Bigg(\frac{\sqrt{u(1-v)}}{4\sqrt{v(1-u)}}|f_{xy}(a,c)|^q+\frac{\sqrt{uv}}{4\sqrt{(1-u)(1-v)}}|f_{xy}(a,d)|^q\\
&+\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(b,c)|^q+\frac{\sqrt{u(1-v)}}{4\sqrt{v(1-u)}}|f_{xy}(b,d)|^q\Bigg)du\,dv\Bigg]^{\frac{1}{q}}\\
&+\Bigg[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\Bigg(\frac{\sqrt{v(1-u)}}{4\sqrt{u(1-v)}}|f_{xy}(a,c)|^q+\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(a,d)|^q\\
&+\frac{\sqrt{uv}}{4\sqrt{(1-u)(1-v)}}|f_{xy}(b,c)|^q+\frac{\sqrt{u(1-v)}}{4\sqrt{v(1-u)}}|f_{xy}(b,d)|^q\Bigg)du\,dv\Bigg]^{\frac{1}{q}}\Bigg\}\end{aligned}$$

$$\leqslant\left(\frac{q-1}{2q-1}\right)^{2\left(1-\frac{1}{q}\right)}\{g_q(b_1,b_2,b_2,b_3)+g_q(b_3,b_2,b_2,b_1)+g_q(b_2,b_1,b_3,b_2)+g_q(b_2,b_3,b_1,b_2)\}.$$

This ends the proof. □

Remark 3.2. Under the assumptions of Theorem 3.2, when $q = 1$, we have

$$\begin{aligned}|I(f)|les g_q(b_1,b_2,b_2,b_3)&+g_q(b_3,b_2,b_2,b_1)+g_q(b_2,b_1,b_3,b_2)\\&+g_q(b_2,b_3,b_1,b_2).\end{aligned}$$

Theorem 3.3. *Let $f:\Omega\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partial differentiable mapping on Ω^o and let $\Delta=[a,b]\times[c,d]\subseteq\Omega^o$ with $a<b, c<d$. If $\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q$ is MT-convex on the co-ordinates on Δ and $q>1$, then the following inequality holds:*

$$\begin{aligned}|I(f)|\leqslant&\frac{q-1}{2(2q-1)}\left(\frac{4(2q-1)}{q-1}\right)^{\frac{1}{q}}\\
&\times\{g_q(c_1,c_2,c_3,c_4)+g_q(c_4,c_3,c_2,c_1)+g_q(c_2,c_1,c_4,c_3)+g_q(c_3,c_4,c_1,c_2)\},\end{aligned}$$

where $g_q(r_1,r_2,r_3,r_4)$ is defined in Theorem 3.1 and

$$\begin{aligned}c_1 &= B_{\frac{1}{2}}\left(\frac{5}{2},\frac{1}{2}\right)B_{\frac{1}{2}}\left(\frac{3}{2},\frac{1}{2}\right),\quad c_2 = B_{\frac{1}{2}}\left(\frac{5}{2},\frac{1}{2}\right)B_{\frac{1}{2}}\left(\frac{1}{2},\frac{3}{2}\right),\\
c_3 &= B_{\frac{1}{2}}\left(\frac{3}{2},\frac{3}{2}\right)B_{\frac{1}{2}}\left(\frac{3}{2},\frac{1}{2}\right),\quad c_4 = B_{\frac{3}{2}}\left(\frac{3}{2},\frac{3}{2}\right)B_{\frac{1}{2}}\left(\frac{1}{2},\frac{3}{2}\right).\end{aligned}$$

Proof. By using Lemma 2.1, the MT-convexity of $\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q$ on the co-ordinates on Δ, Hölder's inequality and changing the variables $u = t/2$ and $v = \lambda/2$, we have

$$\begin{aligned}|I_f| \leqslant 16&\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}u\,v^{\frac{q}{q-1}}du\,dv\right)^{1-\frac{1}{q}}\Bigg\{\Bigg[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}u\Bigg(\frac{\sqrt{uv}}{4\sqrt{(1-u)(1-v)}}|f_{xy}(a,c)|^q\\
&+\frac{\sqrt{u(1-v)}}{4\sqrt{v(1-u)}}|f_{xy}(a,d)|^q+\frac{\sqrt{v(1-u)}}{4\sqrt{u(1-v)}}|f_{xy}(b,c)|^q+\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(b,d)|^q\Bigg)du\,dv\Bigg]^{\frac{1}{q}}\\
&+\Bigg[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}u\Bigg(\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(a,c)|^q+\frac{\sqrt{v(1-u)}}{4\sqrt{u(1-v)}}|f_{xy}(a,d)|^q\\
&+\frac{\sqrt{v(1-u)}}{4\sqrt{u(1-v)}}|f_{xy}(b,c)|^q+\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(b,d)|^q\Bigg)du\,dv\Bigg]^{\frac{1}{q}}\\
&+\Bigg[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}u\Bigg(\frac{\sqrt{u(1-v)}}{4\sqrt{v(1-u)}}|f_{xy}(a,c)|^q+\frac{\sqrt{uv}}{4\sqrt{(1-u)(1-v)}}|f_{xy}(a,d)|^q\\
&+\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(b,c)|^q+\frac{\sqrt{u(1-v)}}{4\sqrt{v(1-u)}}|f_{xy}(b,d)|^q\Bigg)du\,dv\Bigg]^{\frac{1}{q}}\\
&+\Bigg[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}u\Bigg(\frac{\sqrt{v(1-u)}}{4\sqrt{u(1-v)}}|f_{xy}(a,c)|^q+\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(a,d)|^q\\
&+\frac{\sqrt{uv}}{4\sqrt{(1-u)(1-v)}}|f_{xy}(b,c)|^q+\frac{\sqrt{u(1-v)}}{4\sqrt{v(1-u)}}|f_{xy}(b,d)|^q\Bigg)du\,dv\Bigg]^{\frac{1}{q}}\Bigg\}\end{aligned}$$

$$\begin{aligned}\leqslant&\frac{q-1}{2(2q-1)}\left(\frac{4(2q-1)}{q-1}\right)^{\frac{1}{q}}\\
&\times\{g_q(c_1,c_2,c_3,c_4)+g_q(c_4,c_3,c_2,c_1)+g_q(c_2,c_1,c_4,c_3)+g_q(c_3,c_4,c_1,c_2)\}.\end{aligned}$$

Theorem 3.3 is thus proved. □

Theorem 3.4. *Let $f:\Omega\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partial differentiable mapping on Ω^o and let $\Delta=[a,b]\times[c,d]\subseteq\Omega^o$ with $a<b, c<d$. If $\left|\frac{\partial^2 f}{\partial x\,\partial y}\right|^q$ is an MT- convex on the co-ordinates on $\Delta, q\geqslant 1$ and $q\geqslant r,s>0$, then the following inequality holds:*

$$\begin{aligned}|I(f)|\leqslant&\left(\frac{1}{2}\right)^{\frac{-(r+s)}{q}}\left(\frac{(q-1)^2}{(2q-r-1)(2q-s-1)}\right)^{1-\frac{1}{q}}\\
&\times\{g_q(d_1,d_2,d_3,c_4)+g_q(d_4,d_3,d_2,d_1)+g_q(d_2,d_1,d_4,d_3)+g_q(d_3,d_4,d_1,d_2)\},\end{aligned}$$

where $g_q(r_1,r_2,r_3,r_4)$ is as defined in Theorem 3.1 and

$$\begin{aligned}d_1 &= B_{\frac{1}{2}}\left(r+\frac{1}{2},\frac{1}{2}\right)B_{\frac{1}{2}}\left(s+\frac{3}{2},\frac{1}{2}\right),\quad d_2 = B_{\frac{1}{2}}\left(r+\frac{3}{2},\frac{1}{2}\right)B_{\frac{1}{2}}\left(s+\frac{1}{2},\frac{3}{2}\right),\\
d_3 &= B_{\frac{1}{2}}\left(r+\frac{1}{2},\frac{3}{2}\right)B_{\frac{1}{2}}\left(s+\frac{3}{2},\frac{1}{2}\right),\quad d_4 = B_{\frac{3}{2}}\left(r+\frac{1}{2},\frac{3}{2}\right)B_{\frac{1}{2}}\left(s+\frac{1}{2},\frac{3}{2}\right).\end{aligned}$$

Proof. From Lemma 2.1, the MT-convexity of $\left|\frac{\partial^2 f}{\partial x\,\partial y}\right|^q$ on the co-ordinates on Δ, Hölder's inequality and changing the variables $u = t/2$ and $v = \lambda/2$, we have

$$\begin{aligned}|I_f| \leqslant 16&\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}u^{\frac{q-r}{q-1}}v^{\frac{q-s}{q-1}}du\,dv\right)^{1-\frac{1}{q}}\Bigg\{\Bigg[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}u^r v^s\Bigg(\frac{\sqrt{uv}}{4\sqrt{(1-u)(1-v)}}|f_{xy}(a,c)|^q\\
&+\frac{\sqrt{u(1-v)}}{4\sqrt{v(1-u)}}|f_{xy}(a,d)|^q+\frac{\sqrt{v(1-u)}}{4\sqrt{u(1-v)}}|f_{xy}(b,c)|^q+\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(b,d)|^q\Bigg)du\,dv\Bigg]^{\frac{1}{q}}\\
&+\Bigg[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}u^r v^s\Bigg(\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(a,c)|^q+\frac{\sqrt{v(1-u)}}{4\sqrt{u(1-v)}}|f_{xy}(a,d)|^q\\
&+\frac{\sqrt{v(1-u)}}{4\sqrt{u(1-v)}}|f_{xy}(b,c)|^q+\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(b,d)|^q\Bigg)du\,dv\Bigg]^{\frac{1}{q}}\\
&+\Bigg[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}u^r v^s\Bigg(\frac{\sqrt{u(1-v)}}{4\sqrt{v(1-u)}}|f_{xy}(a,c)|^q+\frac{\sqrt{uv}}{4\sqrt{(1-u)(1-v)}}|f_{xy}(a,d)|^q\\
&+\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(b,c)|^q+\frac{\sqrt{u(1-v)}}{4\sqrt{v(1-u)}}|f_{xy}(b,d)|^q\Bigg)du\,dv\Bigg]^{\frac{1}{q}}\\
&+\Bigg[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}u^r v^s\Bigg(\frac{\sqrt{v(1-u)}}{4\sqrt{u(1-v)}}|f_{xy}(a,c)|^q+\frac{\sqrt{(1-u)(1-v)}}{4\sqrt{uv}}|f_{xy}(a,d)|^q\\
&+\frac{\sqrt{uv}}{4\sqrt{(1-u)(1-v)}}|f_{xy}(b,c)|^q+\frac{\sqrt{u(1-v)}}{4\sqrt{v(1-u)}}|f_{xy}(b,d)|^q\Bigg)du\,dv\Bigg]^{\frac{1}{q}}\Bigg\}\end{aligned}$$

$$\leqslant \left(\frac{1}{2}\right)^{\frac{-(r+s)}{q}} \left(\frac{(q-1)^2}{(2q-r-1)(2q-s-1)}\right)^{1-\frac{1}{q}}$$
$$\times \{g_q(d_1,d_2,d_3,c_4) + g_q(d_4,d_3,d_2,d_1) + g_q(d_2,d_1,d_4,d_3) + g_q(d_3,d_4,d_1,d_2)\}.$$

Theorem 3.4 is thus proved. □

Remark 3.3. Under the assumptions of Theorem 3.4, when $r = s = q$, we have

$$|I(f)| \leqslant 4\{g_q(d_1,d_2,d_3,c_4) + g_q(d_4,d_3,d_2,d_1) + g_q(d_2,d_1,d_4,d_3) + g_q(d_3,d_4,d_1,d_2)\}.$$

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