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Resonance states completeness for relativistic particle on a sphere with two semi-infinite lines attached

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ABSTRACT

The paper is devoted to resonances playing an important role in direct and inverse scattering problems. A model of a relativistic particle on hybrid manifold consisting of a sphere with two semi-infinite wires attached is considered. The model is based on the theory of self-adjoint extensions of symmetric operators. Completeness of resonance states in the space of square integrable functions on the sphere is proved. The proof uses the relation between the completeness and the factorization of the characteristic function in Sz.-Nagy functional model.

1. Introduction

The problem of resonances is very important in scattering theory. Resonance as a phenomenon is a strong variation of transmission or reflection in the scattering system. It plays an important role in scattering description (see, e.g., Geyler et al., 2003; Edward, 2002; Geyler and Popov, 1996; Exner et al., 2016; Boitsev et al., 2018; Duclos et al., 2001) Resonance as mathematical object is a quasi-eigenvalue of the Hamiltonian. It can be treated as an eigenvalue of some dissipative operator. The resonance effect is related to closeness of this eigenvalue to the real axis (Lax and Phillips, 1967; Lax and Phillips, 1976). This operator view allowed one to develop a few models and asymptotic approaches to the problem (see, e.g., Hislop and Martinez, 1991; Gadylshin, 1997; Popov, 1992) and references therein). There is an important unsolved problem: which is a maximal domain Ω that ensures the completeness of the resonance states in $L_2(\Omega)$? There are only some examples of solved particular problems (Shushkov, 1985; Vorobiev and Popov, 2015). Recently, there appeared a few works concerning to the problem for the Schrödinger operator on the graph (Popov and Popov, 2017a; Popov et al., 2017) and on a simple hybrid manifold (Popov and Popov, 2017b). The Schrödinger operator corresponds to non-relativistic particle. It is interesting to investigate the completeness property for the relativistic particle on the same manifolds. In the present paper we consider the completeness of resonance states for relativistic particle on a hybrid manifold consisting of a sphere with attached wires. As for technique of the proof, it is based on using the functional model (Sz.-Nagy et al., 2010; Nikol'skii, 2012; Khrushchev et al., 1981). There is an interesting relation between the scattering problem and functional model discovered in Adamyan and Arov (1965). Namely, the scattering matrix is the same as the characteristic function of the functional model. This approach allows one to reduce the completeness/incompleteness of the system of resonance states to the question about absence/presence of singular inner function in factorization of the characteristic function for the functional model. In cases when there are only finite number of one-dimensional "ways to infinity" (infinite edges of the quantum graph) the problem reduces to scalar factorization problems which has an effective criterion.

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2. Dirac operator on a hybrid manifold

Through the present paper we consider the sphere S with two attached wires $\mathbb{R}_+$ and $\mathbb{R}_-$. The behavior of a relativistic spinless particle is described by the Dirac operator. It has the following form (see, e.g. Gesztesy and Seba, 1987; Abrikosov, 2002; Blinova and Popov, 2018):

$$H_- = H_+ = ic\hbar\frac{d}{dx}\otimes\sigma_1 + \frac{Mc^2}{2}\otimes\sigma_3,$$

on the straight line and

$$H_S = -i\hbar c\sigma_1\left(\frac{\partial}{\partial\theta}+\frac{\cot\theta}{2}\right) - \frac{i\hbar c\sigma_2}{\sin\theta}\frac{\partial}{\partial\varphi} + Mc^2\sigma_3,$$

on the sphere. Here $\sigma_1,\ \sigma_2,\ \sigma_3$ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$\varphi,\ \theta$ are the spherical coordinates, M is the particle mass, c is the speed of light, $\hbar$ is the Plank's constant. The domains of these operators are as follows:

$$\mathscr{D}(H_\pm) = W_2^1(\mathbb{R}_\pm)\otimes\mathbb{C}^2, \mathscr{D}(H_S) = W_2^1(S)\otimes\mathbb{C}^2.$$

The starting operator for our model is the direct sum of the operators described above

$$H = H_- \oplus H_S \oplus H_+ \tag{1}$$

with the domain

$$\mathscr{D}(H) = \left(W_2^1(\mathbb{R}_-)\otimes\mathbb{C}^2\right)\oplus\left(W_2^1(S)\otimes\mathbb{C}^2\right)\oplus\left(W_2^1(\mathbb{R}_+)\otimes\mathbb{C}^2\right).$$

The spectral (or scattering) problem on the line reduces to the equation

$$\begin{pmatrix} mc^2 & -i\hbar c\frac{d}{dx} \\ -i\hbar c\frac{d}{dx} & -mc^2 \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \lambda\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

The solution of the equation has the form

$$\begin{cases} \psi_1 = C_1e^{ikx} + C_2e^{-ikx} \\ \psi_2 = \beta\left(C_1e^{ikx} - C_2e^{-ikx}\right) \end{cases} \tag{2}$$

Here and throughout below in the text, $k = \frac{\sqrt{\lambda^2-m^2c^4}}{\hbar c}$ is the wave number and $\beta = \mathrm{sign}(\lambda+mc^2)\sqrt{\frac{\lambda-mc^2}{\lambda+mc^2}}$.

There is a method of switching coupling between manifolds of different dimensions. It is based on the theory of self-adjoint extensions of symmetric operators (Geyler et al., 2003; Bruning and Geyler, 2003). Namely, we use so-called "restriction-extension" procedure (see, e.g., Grishanov et al., 2016; Eremin et al., 2012; Mikhailova et al., 2002). One starts with the restriction of the initial self-adjoint operator on the set of functions vanishing at coupling points. Let $\widetilde{H}_\pm,\ \widetilde{H}_S$ be such restrictions. The Hamiltonian of the system with coupling between the wires and the sphere is constructed as a self-adjoint extension H of the operator

$$\widetilde{H} = \widetilde{H}_- \oplus \widetilde{H}_S \oplus \widetilde{H}_+.$$

It is more convenient to describe the resolvent instead of the operator. For this purpose, one can use the Krein resolvent formula:

$$R(z) = R^0(z) - \Gamma(z)[Q(z)+A]^{-1}\Gamma^*(\bar{z}). \tag{3}$$

Here $R^0(z)$ and $R(z)$ are the resolvents of the initial self-adjoint operator and the extension H, respectively, A is a Hermitian matrix, which parameterizes the self-adjoint extension of $\widetilde{H}, \Gamma(z)$ is the Krein γ-function, $Q(z)$ is the Krein Q-function (see, e.g., Geyler and Popov, 1996). To find $Q(z)$, we should compute the Green's function.

Green's function for the Dirac operator on the half-line is known (Benvegnu and Dabrowski, 1994)

$$G_\pm(x,y;z) = \frac{i}{2\hbar c}\left[\begin{pmatrix}\zeta(z) & \varepsilon(x-y)\\ \varepsilon(x-y) & \zeta^{-1}(z)\end{pmatrix}e^{ik|x-y|} + \begin{pmatrix}\zeta(z) & \varepsilon(x+y)\\ \varepsilon(x+y) & \zeta^{-1}(z)\end{pmatrix}e^{\pm ik|x+y|}\right],$$

where

$$\zeta(z) = \frac{z+Mc^2}{\sqrt{z^2-\left(Mc^2\right)^2}}, \varepsilon(x) = \begin{cases} 1, & x\geqslant 0,\\ -1, & x<0.\end{cases}$$

Green's function on the sphere can be represented in the form of the conventional eigenfunctions expansion for the resolvent:

$$G_S\left(\varphi,\theta;\tilde{\varphi},\tilde{\theta};z\right) = \sum_{n=0}^{+\infty}\sum_m \frac{1}{z-\lambda_{mn}}\begin{pmatrix}\alpha_\lambda(\varphi,\theta)\overline{\alpha_\lambda\left(\tilde{\varphi},\tilde{\theta}\right)} & \alpha_\lambda(\varphi,\theta)\overline{\beta_\lambda\left(\tilde{\varphi},\tilde{\theta}\right)}\\ \beta_\lambda(\varphi,\theta)\overline{\alpha_\lambda\left(\tilde{\varphi},\tilde{\theta}\right)} & \beta_\lambda(\varphi,\theta)\overline{\beta_\lambda\left(\tilde{\varphi},\tilde{\theta}\right)}\end{pmatrix}. \tag{4}$$

We introduce new variable $x,\ x=\cos\theta$, and new functions

$$\begin{pmatrix}\alpha_{\lambda m}(x)\\ \beta_{\lambda m}(x)\end{pmatrix} = \begin{pmatrix}(1-x)^{\frac{1}{2}|m-\frac{1}{2}|}(1+x)^{\frac{1}{2}|m+\frac{1}{2}|}\xi_{\lambda m}(x)\\ (1-x)^{\frac{1}{2}|m+\frac{1}{2}|}(1+x)^{\frac{1}{2}|m-\frac{1}{2}|}\eta_{\lambda m}(x)\end{pmatrix}.$$

The eigenfunctions corresponding to the eigenvalues λ_{mn},

$$\lambda = \lambda_{mn} = \sqrt{n(n+|m|+1)+m(m+1)+1/2}, \tag{5}$$

have the form

$$\begin{pmatrix}\xi_{\lambda m}(x)\\ \eta_{\lambda m}(x)\end{pmatrix} = \begin{pmatrix}C_\alpha^{mn}P_n^{\left(|m-\frac{1}{2}|,|m+\frac{1}{2}|\right)}(x)\\ C_\beta^{mn}P_n^{\left(|m+\frac{1}{2}|,|m-\frac{1}{2}|\right)}(x)\end{pmatrix},$$

where $P_n^{(\alpha,\beta)}(z)$ are the Jacobi polynomials

$$P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)}\sum_{k=0}^n\binom{n}{k}\frac{\Gamma(\alpha+\beta+n+k+1)}{\Gamma(\alpha+k+1)},$$

where

$$C_\alpha^{mn} = \frac{\sqrt{n!(n+2|m|)!}}{2^{|m|+\frac{1}{2}}\Gamma\left(n+|m|+\frac{1}{2}\right)},\quad C_\beta^{mn} = i\,\mathrm{sign}(m\lambda)C_\alpha^{mn}$$

are the normalization constants.

To construct the resolvent of the extended operator by formula (3), we choose the matrix A (of size 8×8) in the following form

$$A = \begin{pmatrix} M_- & A_- & O & O\\ A_- & O & O & O\\ O & O & O & A_+\\ O & O & A_+ & M_+\end{pmatrix},$$

where

$$A_\pm = \begin{pmatrix}0 & \alpha_\pm\\ \alpha_\pm & 0\end{pmatrix},\ M_\pm = \begin{pmatrix}0 & \mu_\pm\\ \mu_\pm & 0\end{pmatrix},\ O = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

The Krein Q-matrix for the model operator has the form

$$Q(z) = \begin{pmatrix} Q_-(z) & O & O & O\\ O & Q_{11}(z) & Q_{12}(z) & O\\ O & Q_{21}(z) & Q_{22}(z) & O\\ O & O & O & Q_+(z)\end{pmatrix},$$

where

$$Q_{\pm}(z) = \frac{i}{\hbar c}\mathrm{diag}\left(\frac{z+Mc^2}{\sqrt{z^2-\left(Mc^2\right)^2}}, \frac{\sqrt{z^2-\left(Mc^2\right)^2}}{z+Mc^2}\right) \quad (6)$$

is the Krein Q-matrix for the half-line, $\{Q_{ij}\}_{i,j=1}^{2}$ is the Q-matrix for the sphere (see (4))

$$Q_{11} = Q_{22} = \left(G_S\left(\varphi_1,\theta_1;\tilde{\varphi}_2,\tilde{\theta}_2;z\right) - G_S\left(\varphi_1,\theta_1;\tilde{\varphi}_2,\tilde{\theta}_2;z_0\right)\right)|_{(\varphi_1,\theta_1)=(\tilde{\varphi}_2,\tilde{\theta}_2)}$$
$$= \sum_{n=0}^{+\infty}\sum_m \frac{z_0-z}{(z-\lambda_{mn})(z_0-\lambda_{mn})}\begin{pmatrix}\alpha_\lambda(\varphi,\theta)\overline{\alpha_\lambda\left(\tilde{\varphi},\tilde{\theta}\right)} & \alpha_\lambda(\varphi,\theta)\overline{\beta_\lambda\left(\tilde{\varphi},\tilde{\theta}\right)} \\ \beta_\lambda(\varphi,\theta)\overline{\alpha_\lambda\left(\tilde{\varphi},\tilde{\theta}\right)} & \beta_\lambda(\varphi,\theta)\overline{\beta_\lambda\left(\tilde{\varphi},\tilde{\theta}\right)}\end{pmatrix}. \quad (7)$$

$$Q_{12} = \overline{Q_{21}} = G_S\left(\varphi_1,\theta_1;\tilde{\varphi}_1,\tilde{\theta}_1;z\right). \quad (8)$$

3. Scattering problem

3.1. Lax-Phillips approach and functional model

For our purposes, it is convenient to consider the scattering in the framework of the Lax-Phillips approach (Lax and Phillips, 1967; Lax and Phillips, 1976). Let us briefly describe the method for the case of the simplest manifold structure (Γ): Sphere with two wires attached.

Consider the Cauchy problem for the time-dependent Dirac equation:

$$\begin{cases} i\hbar u\prime_t = Hu, \\ u(x,0) = u^0(x),\ x\in\Gamma. \end{cases} \quad (9)$$

Here $u(x,t) = \begin{pmatrix}u_1(x,t)\\u_2(x,t)\end{pmatrix}$. Let $\mathscr{E}$ be the Hilbert space of two-component functions u on the manifold Γ with the following norm

$$\left\|\begin{pmatrix}u_1(x,t)\\u_2(x,t)\end{pmatrix}\right\|_{\mathscr{E}}^2 = \int_\Gamma\left(|u_1(x,t)|^2+|u_2(x,t)|^2\right)dx.$$

The solution for non-stationary problem (9) is given by a continuous, one parameter, evolution unitary group $U(t)|_{t\in\mathbb{R}}$ of operators in $\mathscr{E}$:

$$U(t)\begin{pmatrix}u_1(x,0)\\u_2(x,0)\end{pmatrix} = \begin{pmatrix}u_1(x,t)\\u_2(x,t)\end{pmatrix}.$$

There exist two orthogonal subspaces D_- and D_+ in $\mathscr{E}$, called, correspondingly, the incoming and outgoing subspaces, with the following properties.

Definition 3.1. The outgoing subspace D_+ is a subspace of $\mathscr{E}$ having the following properties:

(a) $U(t)D_+ \subset D_+$ for $t > 0$,
(b) $\cap_{t>0}U(t)D_+ = \{0\}$,
(c) $\overline{\cup_{t<0}U(t)D_+} = \mathscr{E}$.

Remark 3.2. D_- is defined analogously (with the natural replacement $t<0$ instead of $t>0$). The subspace D_- corresponds to incoming waves which do not interact with the target (scatterer) prior to $t=0$ while the subspace D_+ corresponds to outgoing waves which do not interact with the target after $t=0$.

Let P_- be the orthogonal projection of $\mathscr{E}$ onto the orthogonal complement of D_- and P_+ be the orthogonal projection of $\mathscr{E}$ onto the orthogonal complement of D_+. Consider the family $\{Z(t)\}|_{t\geqslant 0}$ of operators on $\mathscr{E}$ (known as the Lax-Phillips semigroup) defined by

$$Z(t) = P_+U(t)P_-,\ t\geqslant 0.$$

Lax and Phillips (1967) proved the following theorem.

Theorem 3.3. *The operators $\{Z(t)\}|_{t\geqslant 0}$ annihilate D_+ and D_-, map the orthogonal complement subspace $K = \mathscr{E}\ominus(D_-\oplus D_+)$ into itself and form a strongly continuous semigroup (i.e., $Z(t_1)Z(t_2) = Z(t_1+t_2)$ for $t_1,t_2\geqslant 0$) of contraction operators on K. Furthermore, we have $s\text{-}\lim_{t\to\infty}Z(t) = 0$.*

$\mathscr{E}$ can be represented isometrically as the Hilbert space of functions $L_2(\mathbb{R},N)$ for some Hilbert space N (called the auxiliary Hilbert space) in such a way that $U(t)$ goes to translation to the right by t units and D_+ is mapped onto $L_2(\mathbb{R}_+,N)$. This representation is unique up to an isomorphism of N.

Such a representation is called an outgoing translation representation. Analogously, one can obtain an incoming translation representation related to D_-.

The Lax-Phillips scattering operator $\widetilde{S}$ is defined as follows. Suppose $W_+ : \mathscr{E}\to L_2(\mathbb{R},N)$ and $W_- : \mathscr{E}\to L_2(\mathbb{R},N)$ are the mappings of $\mathscr{E}$ onto the outgoing and incoming translation representations, respectively. The map $\widetilde{S} : L_2(\mathbb{R},N)\to L_2(\mathbb{R},N)$ is defined by the formula (which is equivalent to the standard definition of the scattering operator)

$$\widetilde{S} = W_+(W_-)^{-1}.$$

It is more convenient to work with the Fourier transforms F of the incoming and outgoing translation representations, respectively, called the incoming spectral representation and outgoing spectral representation. According to the Paley-Wiener theorem, in the incoming spectral representation, D_- is represented by $H_+^2(\mathbb{R},N)$, i.e., by the space of boundary values on $\mathbb{R}$ of functions in the Hardy space $H^2(\mathbb{C}^+,N)$ of vector-valued functions (with values in N) defined in the upper half-plane $\mathbb{C}^+$. Correspondingly, the same theorem gives one a symmetric result concerning to the outgoing spectral representation. Accordingly, the scattering operator $\widetilde{S}$ in the spectral representation is transformed to

$$S = F\widetilde{S}F^{-1}.$$

The operator S is realized in the spectral representation as the operator of multiplication by the operator-valued function $S(\cdot):\mathbb{R}\to B(N)$, where $B(N)$ is the space of all bounded linear operators on N. $S(\cdot)$ is called the Lax-Phillips S-matrix. The following theorem (Lax and Phillips, 1967) presents the main properties of S.

Theorem 3.4.

(a) *$S(\cdot)$ is the boundary value on $\mathbb{R}$ of an operator-valued function $S(\cdot):\mathbb{C}^+\to B(N)$ analytic in $\mathbb{C}^+$,*
(b) *$\|S(z)\|\leqslant 1$ for every $z\in\mathbb{C}^+$,*
(c) *$S(E), E\in\mathbb{R}$, is, pointwise, a unitary operator on N.*

The analytic continuation of $S(\cdot)$ from the upper half-plane to the lower half-plane is constructed in a conventional manner:

$$S(z) = (S^*(\bar{z}))^{-1},\ \Im z<0.$$

Thus, $S(\cdot)$ is a meromorphic operator-valued function on the whole complex plane. Let B be the generator of the semigroup $Z(t) : Z(t) = \exp i\mathrm{B}t, t>0$. The eigenvalues of B are called resonances and the corresponding eigenvectors are the resonance states. There is a relation between the eigenvalues of B and the poles of the S-matrix. It is described in the following theorem from Lax and Phillips (1967).

Theorem 3.5. *If $\Im k < 0$, then k belongs to the point spectrum of B if and only if $S^*\left(\bar{k}\right)$ has a non-trivial null space.*

Remark 3.6. The theorem shows that a pole of the Lax-Phillips S-matrix at a point k in the lower half-plane is associated with an eigenvalue k of the generator of the Lax-Phillips semigroup. In other words, resonance poles of the Lax-Phillips S-matrix correspond to eigenvalues of the Lax-Phillips semigroup with well defined eigenvectors belonging to the subspace $K = \mathscr{E} \ominus (D_- \oplus D_+)$, which is called the resonance subspace.

Let us return to the problem of the Dirac quantum graph. In this case, analogously to the Schrödinger graph, one can construct $D_\pm$ and the spectral representations explicitly. Accordingly, the following lemma take place analogously to the corresponding lemmas in Popov and Popov (2017).

Lemma 3.7. *There is a pair of isometric maps $T_\pm : \mathscr{E} \to L_2\left(\mathbb{R}, \mathbb{C}^2\right)$ (the outgoing and incoming spectral representations) having the following properties:*

$$T_\pm U(t) = e^{ikt} T_\pm,\ T_+ D_+ = H^2_+\left(\mathbb{C}^2\right),\ T_- D_- = H^2_-\left(\mathbb{C}^2\right),\ T_- D_+ = S H^2_+\left(\mathbb{C}^2\right),$$

where $H^2_\pm$ is the Hardy space of the upper (lower) half-plane. the matrix-function S is an inner function in $\mathbb{C}_+$, and

$$K_- = T_- K = H^2_+ \ominus S H^2_+, \quad T_- Z(t)|_K = P_{K_-} e^{ikt} T_-|_{K_-}.$$

As an inner operator-function, S can be represented in the form $S = \Pi\Theta$, where Π is a Blaschke-Potapov product (it is a generalization of scalar Blaschke product) and Θ is a singular inner operator-function having no zeros inside upper half-plane (Sz.-Nagy et al., 2010; Nikol'skii, 2012; Khrushchev et al., 1981). The completeness of the system of resonance states is related to the factorization of the scattering matrix. The next theorem shows this relation (we use here the notations described above).

Definition 3.8. The operator is said to be complete if it has a complete set of the root vectors.

Theorem 3.9. (Completeness criterion from Nikol'skii (2012))*The following statements are equivalent:*

1. *The operator B is complete.*
2. *The operator B^* is complete.*
3. *S is a Blaschke-Potapov product.*

Remark 3.10. The auxiliary space N in our case is $\mathbb{C}^2$.

There is a simple criterion for the absence of the singular inner factor in the case $\dim N < \infty$ (for the general operator case one has no such simple criterion). Initially, the criterion was for unit disc, but, of course, we can easily transform it to the case of upper half-plane.

Theorem 3.11. (Nikol'skii, 2012)*Let $\dim N < \infty$. The following statements are equivalent:*

1. *S is a Blaschke-Potapov product;*
2. $$\lim_{r\to 1-0} \int_{C_r} \ln|\det S(k)| \frac{2i}{(k+i)^2} dk = 0, \tag{10}$$

where C_r is the image of $|\zeta| = r, r < 1$, under the inverse Cayley transform.

The integration curve can be parameterized as $C_r = \left\{\mathrm{R}(r)e^{it} + i\mathrm{C}(r) | t \in [0, 2\pi)\right\}$ (see (12) below). For brevity, we define

$$s(k) = |\det S(k)|,$$

and after throwing away constants which are irrelevant for convergence, we obtain the final form of the criterion (10), which is convenient for us and will be used afterwards:

$$\lim_{r\to 1} \int_0^{2\pi} \frac{\mathrm{R}(r)\ln\left(s\left(\mathrm{R}(r)e^{it} + i\mathrm{C}(r)\right)\right)}{\left(\mathrm{R}(r)e^{it} + i\mathrm{C}(r) + i\right)^2} dt = 0, \tag{11}$$

where

$$\mathrm{C}(r) = \frac{1+r^2}{1-r^2}, \quad \mathrm{R}(r) = \frac{2r}{1-r^2}. \tag{12}$$

It should be noted that $\mathrm{R} \to \infty$ corresponds to $r \to 1 - 0$.

3.2. Scattering matrix

Further, we consider the particular scattering problem. Let us take the incoming wave in $\mathbb{R}_-$ in the form

$$\psi_{in} = \left(\begin{array}{c} 1 \\ \left(\frac{\sqrt{z^2 - \left(Mc^2\right)^2}}{\frac{Mc^2}{2} - z}\right)\end{array}\right) e^{ikx}.$$

The corresponding outgoing wave in $\mathbb{R}_+$ has the form

$$\psi_{out} = t(z)\left(\begin{array}{c} 1 \\ \left(-\frac{\sqrt{z^2 - \left(Mc^2\right)^2}}{\frac{Mc^2}{2} - z}\right)\end{array}\right) e^{ikx}.$$

After straightforward algebraic manipulations, one obtains the following formulas for the transmission and reflection coefficients $T(z) = |t(z)|, R(z) = |r(z)|$

$$t(z) = \frac{1}{\det[Q(z)+A]} \frac{i}{\hbar c}\left(\frac{z+Mc^2}{\sqrt{z^2-\left(Mc^2\right)^2}}\left(B_{71} - \frac{\sqrt{z^2-\left(Mc^2\right)^2}}{\frac{Mc^2}{2}-z} B_{72}\right) + \frac{\frac{Mc^2}{2}-z}{z+Mc^2}\left(\frac{\sqrt{z^2-\left(Mc^2\right)^2}}{\frac{Mc^2}{2}-z} B_{81} - B_{82}\right)\right), \tag{13}$$

$$r(z) = \frac{1}{\det[Q(z)+A]} \frac{i}{\hbar c}\left(\frac{z+Mc^2}{\sqrt{z^2-\left(Mc^2\right)^2}}\left(B_{11} - \frac{\sqrt{z^2-\left(Mc^2\right)^2}}{\frac{Mc^2}{2}-z} B_{12}\right) + \frac{\frac{Mc^2}{2}-z}{z+Mc^2}\left(\frac{\sqrt{z^2-\left(Mc^2\right)^2}}{\frac{Mc^2}{2}-z} B_{21} - B_{22}\right)\right), \tag{14}$$

where B_{jp} are the corresponding minors of the matrix $Q(z) + A$. We are interested in $\det S(k) = t^2 - r^2,\ k^2 = z$.

Roots and poles of the scattering matrix (correspondingly, of $\det S$) are symmetric in respect to the real axis. Taking into account (4) and (5) one concludes that B_{ij} are analytic in the resolvent set. Correspondingly, poles are given by roots of the denominator $\det(Q + A)$. This determinant is an analytic function of the spectral parameter. Hence, the set of roots has no accumulation points in the complex plane.

Theorem 3.12. *If an eigenvalue $\lambda_{mn} = k_{mn}^2$ of the Dirac operator on the sphere is such that the corresponding eigenfunction vanishes at the both points of wires coupling then the model operator for the resonator with attached wires has the same eigenvalue $\lambda_{mn} = k_{mn}^2$. In alternative case, λ_{mn} is not an eigenvalue of the model operator but a neighborhood of the point λ_{mn} contains a resonance $\widetilde{k}_{mn}^2$.*

Due to (5) the theorem shows that the resonances form a sequence which tends to infinity along the real axis and has no accumulation points.

Consider the condition of singular inner factor absence (11). Recall that integral (11) is evaluated along a circle (an image of the circle $|z| = r < 1$ under the transformation $\eta = -i\frac{z+1}{z-1}$). Singularities (roots of $\det S$) which can appear at the integration path are integrable. We divide the integration curve into two parts: neighborhood ($L_{r,n}$) of singularity $\widetilde{k}_n^2$ and the rest of the curve. Singularities are separated and pose near the real axis. Taking into account ()()()(6)–(8) one can see that $\det S$ has no exponential growth at infinity in $\mathbb{C}$. The growth of logarithm near zero is slower than any inverse power. Correspondingly,

$$|\int_{L_{r,n}} \ln|\det S(k)| \frac{2\imath}{(k+\imath)^2} dk| \leqslant \frac{C}{|k_n|^\delta}, \quad \delta > 0, \tag{15}$$

where C does not depend on n. Note that if $r \to 1-0$, i.e. $R(r) \to \infty$, then $k_n \to \infty$ where $\widetilde{k}_n^2$ is that resonance which belongs to the integration curve. Hence, the limit of such part of the integral is zero.

As for the rest of the curve, the following estimation takes place:

$$|\ln|\det S(k)|\left(k^2+1\right)^{-1}| \leqslant \frac{C}{(|k|+1)^{1+\varepsilon}}, \quad \varepsilon > 0. \tag{16}$$

The length of the path is linear in respect to the path diameter (i.e. in $|k|$ for large $|k|$). Note that the diameter of the curve tends to infinity if $r \to 1-0$. One can see (due to (15) and (16)) that the integral tends to zero if $r \to 1-0$. It means that there is no singular inner factor in $\det S(k)$ and, hence, we come to the main theorem:

Theorem 3.13. (Main theorem)*The system of resonance states of the operator H is complete in $L_2(S)$.*

Remark 3.14. Eigenstates of the model operator (see Theorem 3.12) which correspond to eigenfunctions of the unperturbed operator vanishing at the both contact points "wire-sphere" are added to the system of resonance states to obtain the completeness.

4. Conclusion

The obtained result shows that the sphere S gives one a domain for which one has a completeness of resonance states in $L_2(S)$. It is simple to show that this is the maximal domain ensuring the completeness. Technically, the completeness is related to the definition of the scattering matrix and incoming and outgoing subspaces. If one considers a wider domain, i.e. adds some segment $[0, a]$ from semi-infinite edges attached, then the incoming and outgoing subspaces become narrower. As a result, there appears exponential factor $\exp(ika)$ in the scattering matrix. It leads to incompleteness of the resonance states in $L_2(S \cup [0, a])$ in accordance with the completeness criterion.

The same completeness result was obtained earlier for the Schrödinger operator on the sphere with wires attached (Popov and Popov, 2017b). The key point for this correlation is given by a similarity of the properties of the incoming and outgoing subspaces for the Schrödinger and the Dirac cases. It is evident, that one obtains the same completeness result for the case of any finite number of semi-infinite wires attached. One can see also that the spherical form of the 2D manifold is not essential. The requirement is that the 2D manifold is smooth and bounded.

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