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Exact solitary-wave solutions for the nonlinear dispersive $K(2,2,1)$ and $K(3,3,1)$ equations

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Abstract We implemented homotopy analysis method for approximating the solution to the nonlinear dispersive $K(m,p,1)$ type equations. By using this scheme, the explicit exact solution is calculated in the form of a quickly convergent series with easily computable components. To illustrate the application of this method, numerical results are derived by using the calculated components of the homotopy analysis series.

1. Introduction

The HAM is developed in 1992 by Liao (1992, 1995, 1997, 1999, 2003a,b, 2004), Liao and Campo (2002). This method has been successfully applied to solve many types of nonlinear problems in science and engineering by many authors Ayub et al. (2003), Hayat et al. (2004a,b), Abbasbandy (2007a,b,c), Bataineh et al., in press, and references therein. By the present method, numerical results can be obtained with using a few iterations. The HAM contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series for large values of t. Unlike, other numerical methods are given low degree of accuracy for large values of t. Therefore, the HAM handles linear and nonlinear problems without any assumption and restriction.

In the past decades, directly seeking for exact solutions of nonlinear partial differential equations has become one of the central themes of perpetual interest in Mathematical Physics. Nonlinear wave phenomena appear in many fields, such as fluid mechanics, plasma physics, biology, hydrodynamics, solid state physics, and optical fibers. These nonlinear phenomena are often related to nonlinear wave equations. In order to understand better these phenomena as well as further apply them in the practical life, it is important to seek their exact solutions. Many powerful methods had been developed such as Backlund transformation (Ablowitz and Clarkson, 1991; Miura, 1978), Darboux transformation (Gu, 1999), the inverse scattering transformation (Hirota, 1974), the bilinear method (Hirota, 1973), the tanh method (Malfliet, 1992; Inc and Fan, 2005), the sine–cosine method (Yan and Zhang, 1999; Inc and Evans, 2004), the homogeneous balance method (Wang, 1996), the Riccati method (Yan and Zhang, 2001), the Jacobi elliptic function method (Fu et al., 2001), the extended Jacobi elliptic function method (Yan, 2003), etc.

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In the well-known Korteweg–de Vries (KdV) equation

$$u_t - auu_x + u_{xxx} = 0, \tag{1}$$

the nonlinear term uu_x causes the steepening of the wave form. On the other hand, the dispersion term u_{xxx} in this equation makes the wave form to spread. Due to the balance between this weak nonlinearity and dispersion, solitons exist (Wazwaz and Helal, 2004).

Rosenau and Hyman (1993) presented a class of compactons of nonlinear $K(m,n)$ equation as follows:

$$u_t + a(u^m)_x + (u^n)_{xxx} = 0, \quad m,n > 1 \tag{2}$$

In Eq. (2), if we take $a = 1$ then this equation is referred to as the focusing $(+)$ branch. This focusing $(+)$ branch exhibits compacton solutions (Wazwaz and Helal, 2004). In Eq. (1), if we take $a = -1$ then the equation is referred to as the defocusing $(-)$ branch. This defocusing $(-)$ branch exhibits solitary pattern solutions (Wazwaz, 2001). Compacton is a soliton solution which has finite wavelength or is free of exponential wings. Unlike solitons that narrow as the amplitude increases, the compacton's width is independent of its amplitude. Compacton solutions have been used in many fields of scientific applications such as in super-deformed nuclei, phonon, photon, in the fission of liquid drops (nuclear physics), pre-formation of cluster in hydrodynamic models, and inertial fusion as was also indicated by Wazwaz and Helal (2004) and Wazwaz (2001).

Reecntly, there are also some researchers studying the numerical solutions of the nonlinear dispersive $K(m,p,1)$ equations. Zhu et al. (2007) obtained some numerical solutions of the nonlinear dispersive $K(m,p,1)$ equation by using Adomian decomposition method. Also Inc (2008) used Variational iteration method for solving the nonlinear dispersive $K(m,p,1)$ equations.

The aim of this paper is to extend the homotopy analysis method to derive the numerical and exact compacton solutions to the nonlinear dispersive $K(m,p,1)$ equation subject to the initial condition:

$$u_t + (u^m)_x - (u^p)_{xxx} + u_{5x} = 0, \quad m > 1, \quad 1 \leqslant p \leqslant 3, \tag{3a}$$

$$u(x,0) = f(x). \tag{3b}$$

Particularly, we have found some new special exact solutions to $K(2,2,1)$ and $K(3,3,1)$ equations by this scheme.

2. The homotopy analysis method (HAM)

We apply the HAM (Liao, 1992, 1995, 1997, 1999, 2003a,b, 2004; Liao and Campo, 2002) to the nonlinear dispersive $K(m,p,1)$ Eqs. (3a–b). We consider the following differential equation

$$N[u(x,t)] = 0, \tag{4}$$

where N is a nonlinear operator for this problem, x and t denote an independent variables, $u(x,t)$ is an unknown function.

In the frame of HAM (Liao, 1992, 1995, 1997, 1999, 2003a,b, 2004; Liao and Campo, 2002), we can construct the following zeroth-order deformation:

$$(1-q)L(U(x,t;q) - u_0(x,t)) = q\hbar H(x,t)N(U(x,t;q)), \tag{5}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, $H(x,t) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $U(x,t;q)$ is an unknown function on the independent variables x, t and q.

Obviously, when $q = 0$ and $q = 1$, it holds

$$U(x,t;0) = u_0(x,t), \quad U(x,t;1) = u(x,t), \tag{6}$$

respectively. Using the parameter q, we expand $U(x,t;q)$ in Taylor series as follows:

$$U(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m, \tag{7}$$

where

$$u_m = \frac{1}{m!}\frac{\partial^m U(x,t;q)}{\partial^m q}\bigg|_{q=0}. \tag{8}$$

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function $H(x,t)$ are selected such that the series (7) is convergent at $q = 1$, then due to (6) we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{9}$$

Let us define the vector

$$\vec{u}_n(x,t) = \{u_0(x,t), u_1(x,t), \ldots, u_n(x,t)\}. \tag{10}$$

Differentiating (5) m times with respect to the embedding parameter q, then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t)R_m(\vec{u}_{m-1}), \tag{11}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N(U(x,t;q))}{\partial^{m-1}q}\bigg|_{q=0}, \tag{12}$$

and

$$\chi_m = \begin{cases} 0 & m \leqslant 1, \\ 1 & m > 1. \end{cases} \tag{13}$$

Finally, for the purpose of computation, we will approximate the HAM solution (9) by the following truncated series:

$$\phi_m(x,t) = \sum_{k=0}^{m-1} u_k(x,t). \tag{14}$$

3. Applications

3.1. The $K(2,2,1)$ equation

We first consider the following initial value problem of the $K(2,2,1)$ equation (Zhu et al., 2007):

$$u_t + (u^2)_x - (u^2)_{xxx} + u_{5x} = 0, \tag{15a}$$

$$u(x,0) = \frac{16c-1}{12}\cosh^2\left(\frac{1}{4}x\right), \tag{15b}$$

where c is an arbitrary constant.

According to (5), the zeroth-order deformation can be given by

$$(1-q)L(U(x,t;q) - u_0(x,t)) = q\hbar H(x,t)(U_t + (U^2)_x - (U^2)_{xxx} + U_{5x}). \tag{16}$$

We can start with an initial approximation $u_0(x,t) = \frac{16c-1}{12}\cosh^2\left(\frac{1}{4}x\right)$, and we choose the auxiliary linear operator

$$L(U(x,t;q)) = \frac{\partial U(x,t;q)}{\partial t},$$

with the property

$$L(C) = 0,$$

where C is an integral constant. We also choose the auxiliary function to be

$$H(x,t) = 1.$$

Hence, the mth-order deformation can be given by

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t) R_m(\vec{u}_{m-1}),$$

where

$$R_m(\vec{u}_{m-1}) = \frac{\partial(u_{m-1})}{\partial t} + \frac{\partial}{\partial x}\left(\sum_{i=0}^{m-1} u_i u_{m-1-i}\right) - \frac{\partial^3}{\partial x^3}\left(\sum_{i=0}^{m-1} u_i u_{m-1-i}\right) + \frac{\partial^5(u_{m-1})}{\partial x^5}. \quad (17)$$

Now the solution of the mth-order deformation Eq. (17) for $m \geqslant 1$ become

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[R_m(\vec{u}_{m-1})]. \quad (18)$$

Consequently, the first few terms of the HAM series solution are as follows:

$$u_0(x,t) = \frac{16c-1}{12}\cosh^2\left(\frac{1}{4}x\right),$$

$$u_1(x,t) = \frac{\hbar c(16c-1)t}{24}\sinh\left(\frac{x}{2}\right),$$

$$u_2(x,t) = \frac{\hbar c(16c-1)t}{24}\sinh\left(\frac{x}{2}\right) + \frac{\hbar^2 c(16c-1)t}{24}\sinh\left(\frac{x}{2}\right) + \left(\frac{\hbar^2 c^3 t^2}{12} - \frac{\hbar^2 c^2 t^2}{192}\right)\cosh\left(\frac{x}{2}\right),$$

and so on. Hence, the HAM series solution (for $\hbar = -1$) is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \dots = \frac{16c-1}{12}\cosh^2\left(\frac{1}{4}x\right) - \frac{c(16c-1)t}{24}\sinh\left(\frac{x}{2}\right) + \left(\frac{c^3t^2}{12} - \frac{c^2t^2}{192}\right)\cosh\left(\frac{x}{2}\right) + \frac{c^3t^3}{1152}\sinh\left(\frac{x}{2}\right) - \frac{c^4t^3}{72}\sinh\left(\frac{x}{2}\right) + \dots \quad (19)$$

Using Taylor series into (19), we find the closed form solution

$$u(x,t) = \frac{16c-1}{12}\cosh^2\left(\frac{ct-x}{4}\right), \quad (20)$$

which is an exact solitary solution for the nonlinear $K(2,2,1)$ equation.

We now consider another initial condition as

$$u(x,0) = Ae^{\left(\frac{1-16c}{4}\right)\left|\frac{x+x_0}{3}\right|} + c_0, \quad (21)$$

where A, x_0 *and* c_0 are arbitrary constants. We can then obtain the solution $u(x,t)$ in a closed form as

$$u(x,t) = Ae^{\left(\frac{1-16c}{4}\right)\left|\frac{x+x_0-ct}{3}\right|} + c_0. \quad (22)$$

Thus, we can obtain a new solitary solution called peakon solitary pattern solution which can be written in the form

$$u(x,t) = A\,sign(x-ct)e^{\left(\frac{1-16c}{4}\right)\left|\frac{x+x_0-ct}{3}\right|} + c_0. \quad (23)$$

In the same manner, we get another peakon solitary pattern solution to Eq. (15a):

$$u(x,t) = sign(x-ct)\left\{Ae^{\left(\frac{1-16c}{4}\right)\left|\frac{x-ct}{4}\right|} + Be^{\sqrt{\frac{81c-1}{54}}}\left|\frac{x-ct}{3}\right|\right\} + c_0, \quad (24)$$

where A, B and c_0 are arbitrary constants. We assume that the $K(2,2,1)$ equation has the solution of the form

$$u(x,t) = \frac{16c-1}{12}\left[M\cosh^2\left(\frac{ct-x}{4}\right) + N\sinh^2\left(\frac{ct-x}{4}\right)\right], \quad (25)$$

where M and N are constants to be determined. Substituting (25) into (15a), it is easy to see that if M and N satisfy

$$M = N \quad \text{and} \quad M = 1 - N, \quad (26)$$

then (25) is a solution to the $K(2,2,1)$ equation. Figs. 1–4

3.2. The K(3,3,1) equation

We now consider the initial value problem in the following form (Zhu et al., 2007):

$$u_t + (u^3)_x - (u^3)_{xxx} + u_{5x} = 0, \quad (27a)$$

$$u(x,0) = \sqrt{\frac{81c-1}{54}}\cosh\left(\frac{1}{3}x\right), \quad (27b)$$

where c is an arbitrary constant.

According to (5), the zeroth-order deformation can be given by

$$(1-q)L(U(x,t;q) - u_0(x,t)) = q\hbar H(x,t)(U_t + (U^3)_x - (U^3)_{xxx} + U_{5x}). \quad (28)$$

We can start with an initial approximation $u_0(x,t) = \sqrt{\frac{81c-1}{54}}\cosh\left(\frac{1}{3}x\right)$, and we choose the auxiliary linear operator

$$L(U(x,t;q)) = \frac{\partial U(x,t;q)}{\partial t},$$

with the property

$$L(C) = 0,$$

where C is an integral constant. We also choose the auxiliary function to be

$$H(x,t) = 1.$$

Hence, the mth-order deformation can be given by

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t) R_m(\vec{u}_{m-1}),$$

where

$$R_m(\vec{u}_{m-1}) = \frac{\partial(u_{m-1})}{\partial t} + \frac{\partial}{\partial x}\left(\sum_{i=0}^{m-1} u_i\left(\sum_{k=0}^{m-1-i} u_k u_{m-1-i-k}\right)\right) - \frac{\partial^3}{\partial x^3}\left(\sum_{i=0}^{m-1} u_i\left(\sum_{k=0}^{m-1-i} u_k u_{m-1-i-k}\right)\right) + \frac{\partial^5(u_{m-1})}{\partial x^5}. \quad (29)$$

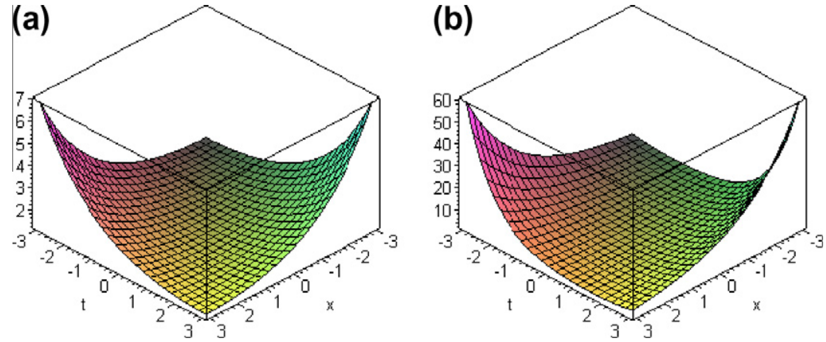

Figure 1 The surface shows the solution $u(x, t)$ for Eqs. (15a and b): (a) exact solution for $c = 1$, and (b) exact solution for $c = 2$.

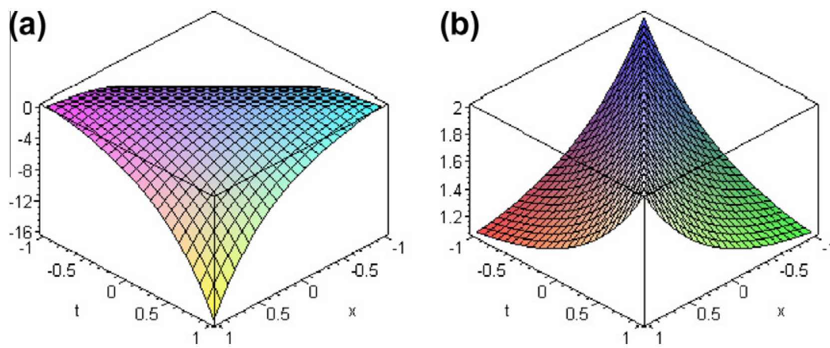

Figure 2 The surface shows the peakon pattern solution to Eq. (23) with fixed values: (a) $A = c = -1$, $x_0 = 0$ and $c_0 = 1$, and (b) $A = c = 1$, $x_0 = 0$ and $c_0 = 1$.

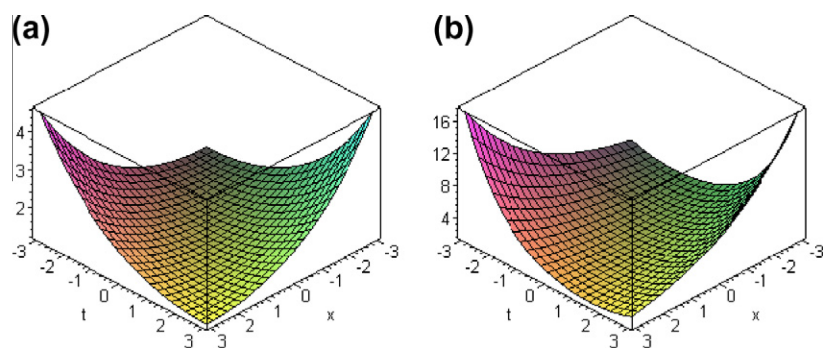

Figure 3 The surface shows the solution $u(x, t)$ for Eqs. (27a and b): (a) exact solution for $c = 1$, and (b) exact solution for $c = 2$.

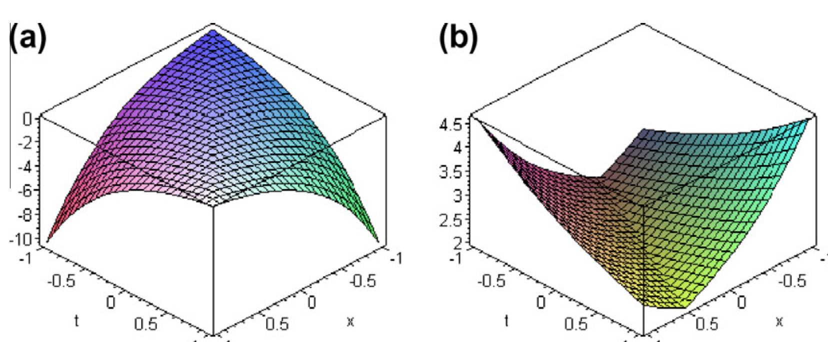

Figure 4 The surface shows the peakon pattern solution to Eq. (32) with fixed values: (a) $A = -1$, $c = 1$, $x_0 = 0$ and $c_0 = 1$, and (b) $A = 1$, $c = 1/2$, $x_0 = 0$ and $c_0 = 1$.

Now the solution of the mth-order deformation Eq. (29) for $m \geqslant 1$ become

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[R_m(\vec{u}_{m-1})]. \tag{30}$$

Consequently, the first few terms of the HAM series solution are as follows:

$$u_0(x,t) = \sqrt{\frac{81c-1}{54}}\cosh\left(\frac{1}{3}x\right),$$
$$u_1(x,t) = \frac{1}{54}\hbar ct\sqrt{486c-6}\sinh\left(\frac{1}{3}x\right),$$
$$u_2(x,t) = \frac{1}{54}\hbar ct\sqrt{486c-6}\sinh\left(\frac{1}{3}x\right) + \frac{1}{54}\hbar^2 ct\sqrt{486c-6}\sinh\left(\frac{1}{3}x\right) + \frac{\sqrt{486c-6}}{324}\hbar^2c^2t^2\cosh\left(\frac{1}{3}x\right),$$

and so on. Hence, the HAM series solution (for $\hbar = -1$) is

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \dots \\ &= \sqrt{\frac{81c-1}{54}}\cosh\left(\frac{1}{3}x\right) - \frac{1}{54}ct\sqrt{486c-6} \\ &\quad \times \sinh\left(\frac{1}{3}x\right) + \frac{\sqrt{486c-6}}{324}c^2t^2\cosh\left(\frac{1}{3}x\right) \\ &\quad - \frac{\sqrt{486c-6}}{2916}c^3t^3\sinh\left(\frac{1}{3}x\right) + \dots \end{aligned} \tag{31}$$

Using Taylor series into (31), we find the closed form solution

$$u(x,t) = \sqrt{\frac{81c-1}{54}}\cosh\left(\frac{ct-x}{3}\right), \tag{32}$$

which is an exact solitary solution for the nonlinear $K(3,3,1)$ equation.

We now consider another initial condition as

$$u(x,0) = Ae^{\sqrt{\frac{81c-1}{54}}|x+x_0|} + c_0, \tag{33}$$

where A, x_0 *and* c_0 are arbitrary constants. We can then obtain the solution $u(x,\ t)$ in a closed form as

$$u(x,t) = Ae^{\sqrt{\frac{81c-1}{54}}|x+x_0-ct|} + c_0. \tag{34}$$

Thus, we can obtain a new solitary solution called peakon solitary pattern solution which can be written in the form

$$u(x,t) = A sign(x-ct)e^{\sqrt{\frac{81c-1}{54}}|x+x_0-ct|} + c_0, \tag{35}$$

In the same manner, we get another peakon solitary pattern solution to the Eq. (27a):

$$u(x,t) = sign(x-ct)\left\{Ae^{-\sqrt{\frac{81c-1}{54}}\left|\frac{x-ct}{3}+x_0\right|} + Be^{\sqrt{\frac{81c-1}{54}}\left|\frac{x-ct}{3}+x_0\right|}\right\} + c_0, \tag{36}$$

where A, B *and* c_0 are arbitrary constants. We assume that the $K(3,3,1)$ equation has the solution of the form

$$u(x,t) = \sqrt{\frac{81c-1}{54}}\left[A\cosh\left(\frac{ct-x}{3}\right) + B\sinh\left(\frac{ct-x}{3}\right)\right], \tag{37}$$

where A and B are constants to be determined. Substituting (37) into (27a), it is easy to see that if A and B satisfy

$$A = B, \quad A = 1 + B, \quad A = B - 1, \tag{38}$$

then (37) is a solitary pattern solution to the $K(3,3,1)$ equation.

Eqs. (25) and (35) can be used to exhibit other solutions to the $K(2,2,1)$ and $K(3,3,1)$ equations, respectively, by adding a phase shift, thus we obtain

$$u(x,t) = \frac{16c-1}{12}\left[M\cosh^2\left(\frac{ct-x}{4}+r\pi\right) + N\sinh^2\left(\frac{ct-x}{4}+r\pi\right)\right], \tag{39}$$

$$u(x,t) = \sqrt{\frac{81c-1}{54}}\left[A\cosh\left(\frac{ct-x}{3}+r\pi\right) + B\sinh\left(\frac{ct-x}{3}+r\pi\right)\right], \tag{40}$$

where r is an arbitrary constant.

4. Conclusion

In this paper, we have presented a scheme used to obtain exact solitary pattern solutions to the nonlinear dispersive $K(2,2,1)$ and $K(3,3,1)$ equations with initial conditions using the homotopy analysis method. The results show that the present method is a powerful mathematical tool for finding other solitary pattern solutions to many nonlinear dispersive equations with initial conditions. HAM does not require discretization of the variables, i.e. time and space, it is not affected by computational round-off errors and one is not faced with the necessity of large computer memory and time. It is worth noting that unlike the traditional numerical techniques, the solution here is given in a closed form and by using the initial condition only. An important advantage of the HAM is that it attacks the problem directly in a straightforward manner without any need for transformation formulae or restrictions on boundary conditions. In closing HAM avoids the difficulties and massive computational work by determining the analytic solutions. The efficiency of the variational iteration scheme gives it much wider applicability.

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