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Approximation by Stancu variant of λ-Bernstein shifted knots operators associated by Bézier basis function

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A B S T R A C T

The current paper presents the λ-Bernstein operators through the use of newly developed variant of Stancu-type shifted knots polynomials associated by Bézier basis functions. Initially, we design the proposed Stancu generated λ-Bernstein operators by means of Bézier basis functions then investigate the local and global approximation results by using the Ditzian–Totik uniform modulus of smoothness of step weight function. Finally we establish convergence theorem for Lipschitz generated maximal continuous functions and obtain some direct theorems of Peetre's K-functional. In addition, we establish a quantitative Voronovskaja-type approximation theorem.

1. Introduction and preliminaries

One of the most well-known mathematicians in the world, S. N. Bernstein, provided the quickest and most elegant demonstration of one of the most well-known Weierstrass approximation theorems. Bernstein also devised the series of positive linear operators implied by $\{B_s\}_{s\geq 1}$. The famous Bernstein polynomial, defined in Bernstein (2012), was found to be a function that uniformly approximates on $[0,1]$ for all $f \in C[0,1]$ (the class of all continuous functions). This finding was made in Bernstein's study. Thus, for any $y \in [0,1]$, the well-known Bernstein polynomial has the following results.

$$B_s(g;y) = \sum_{i=0}^{s} g\left(\frac{i}{s}\right) b_{s,i}(y),$$

where $b_{s,i}(y)$ are the Bernstein polynomials with a maximum degree of s and $s \in \mathbb{N}$ (the positive integers), which defined by

$$b_{s,i}(y) = \begin{cases} \binom{s}{i} y^i (1-y)^{s-i} & \text{for } s, y \in [0,1] \text{ and } i = 0,1,\ldots \\ 0 & \text{for any } i > s \text{ or } i < 0. \end{cases} \tag{1.1}$$

Testing the Bernstein-polynomials' recursive relation is not too difficult. The recursive relationship for Bernstein-polynomials $b_{s,i}(y)$ is quite simple to test.

$$b_{s,i}(y) = (1-y) b_{s-1,i}(y) + y b_{s-1,i-1}(y).$$

In 2010, Cai and colleagues introduced $\lambda \in [-1,1]$ is the shape parameter for the new Bézier bases, which they called λ-Bernstein operators. This definition of the Bernstein-polynomials is defined as follows:

$$B_{s,\lambda}(g;y) = \sum_{i=0}^{s} g\left(\frac{i}{s}\right) \tilde{b}_{s,i}(\lambda;y), \tag{1.2}$$

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where the Bernstein polynomial $b_{s,i}(y)$ expressed regarding the latest Bernstein basis function (Ye et al., 2010), $\tilde{b}_{s,i}(\lambda; y)$ defined as follows by Ye et al.:

$$\begin{aligned}
\tilde{b}_{s,0}(\lambda; y) &= b_{s,0}(y) - \frac{\lambda}{s+1} b_{s+1,1}(y),\\
\tilde{b}_{s,i}(\lambda; y) &= b_{s,i}(y) + \lambda\bigg(\frac{s-2i+1}{s^2-1} b_{s+1,i}(y)\\
&\qquad - \frac{s-2i-1}{s^2-1} b_{s+1,i+1}(y)\bigg), \text{ for } i \in [1, s-1]\\
\tilde{b}_{s,s}(\lambda; y) &= b_{s,s}(y) - \frac{\lambda}{s+1} b_{s+1,s}(y),
\end{aligned}$$

where if hen $\lambda = 0$, then $\tilde{b}_{s,i}(\lambda; y)$ reduce to classical Bernstein polynomial $b_{s,i}(y)$ by (1.1). It should be noted that adding the form parameter λ gives us more modeling flexibility. Using shifted knots polynomials, Gadjiev and Ghorbanalizadeh (2010), introduced the Bernstein-polynomials, which include the Stancu type polynomial by:

$$S^{\nu_1,\nu_2}_{s,\mu_1,\mu_2}(g; y) = \left(\frac{s+\nu_2}{s}\right)^s \sum_{i=0}^{s} \binom{s}{i} \left(y - \frac{\mu_2}{s+\nu_2}\right)^i \left(\frac{s+\mu_2}{s+\nu_2} - y\right)^{s-i} g\left(\frac{i+\mu_1}{s+\nu_1}\right) \tag{1.3}$$

where for all $i = 1, 2$ the positive real numbers are μ_i, ν_i such that $0 \le \mu_2 \le \mu_1 \le \nu_1 \le \nu_2$ and $y \in [\frac{\mu_2}{s+\nu_2}, \frac{s+\mu_2}{s+\nu_2}]$.

Through approximation procedures, researchers have recently developed Bernstein type operators. Among them are the new type λ-Bernstein polynomials (Aslan and Mursaleen, 2022), blending type Bernstein– Schurer–Kantorovich Operators (Ayman-Mursaleen et al., 2023), the q-Bernstein–Stancu–Kantorovich operators (Ayman-Mursaleen et al., 2022), λ-Bernstein operators via power series summability (Braha et al., 2020), λ-Hybrid type operators (Heshamuddin et al., 2023), α-Stancu–Schurer–Kantorovich operators (Heshamuddin et al., 2022), Bernstein–Kantorovich operators in Stancu variation (Mohiuddine and Özger, 2020), the Bernstein–Durrmeyer operators in Genuine modified form Mohiuddine et al. (2018), the new family of Bernstein–Kantorovich operators (Mohiuddine et al., 2017), Bézier bases with Schurer polynomials (Özger, 2020), λ-Bernstein–Kantorovich operators (Rahman et al., 2019), Bernstein operators based on Bézier bases (Srivastava et al., 2019), Error estimates using Higher modulus of smoothness (Srivastava and Bisht, 2022), approximation of Bernstein type operators (Zeng and Cheng, 2001) and reference there in. Most recent in article (Ayman-Mursaleen et al., 2024), λ-Bernstein-polynomial of shifted knots type operators which are associated by Bézier bases have been constructed and obtained Lemma 1.1 as follows:

$$B^{\varkappa_1,\varkappa_2}_{s,\lambda}(g; y) = \left(\frac{s+\varkappa_2}{s}\right)^s \sum_{i=0}^{s} \tilde{b}^{\varkappa_1,\varkappa_2}_{s,i}(\lambda; y) g\left(\frac{i}{s}\right), \tag{1.4}$$

Lemma 1.1. *For the test function $g(t) = 1, t, t^2$, the operators $B^{\varkappa_1,\varkappa_2}_{s,\lambda}$ having:*

$$B^{\varkappa_1,\varkappa_2}_{s,\lambda}(1; y) = 1;$$

$$\begin{aligned}
B^{\varkappa_1,\varkappa_2}_{s,\lambda}(t; y) &= \frac{1}{(s-1)s}((s+\nu)(s-1) - 2\lambda)\left(y - \frac{\mu}{s+\varkappa_2}\right)\\
&+ \frac{\lambda}{(s-1)s}\left(\frac{s+\varkappa_2}{s}\right)^s \left[\left(y - \frac{\varkappa_1}{s+\varkappa_2}\right)^{s+1} - \left(\frac{s+\varkappa_1}{s+\varkappa_2} - y\right)^{s+1} + \left(\frac{s+\varkappa_2}{s}\right)^{1-s}\right];\\
B^{\varkappa_1,\varkappa_2}_{s,\lambda}(t^2; y) &= \frac{1}{s}\left[\left(\frac{s+\varkappa_2}{s}\right)^s + \frac{2\lambda}{s-1}\right]\left(y - \frac{\varkappa_1}{s+\varkappa_2}\right)\\
&+ \frac{1}{s^2}\left(\frac{s+\varkappa_2}{s}\right)\Big((s+\varkappa_2)(s-1) - 4\lambda\Big)\left(y - \frac{\varkappa_1}{s+\varkappa_2}\right)^2\\
&+ \lambda\left(\frac{s+\varkappa_2}{s}\right)^s \left[\frac{1}{s+1} + \frac{(s+1)^2}{s^2(s-1)}\right]\left(y - \frac{\varkappa_1}{s+\varkappa_2}\right)^{s+1}\\
&+ \frac{\lambda}{s^2(s-1)}\left(\frac{s+\varkappa_2}{s}\right)^s \left(\frac{s+\varkappa_1}{s+\varkappa_2} - y\right)^{s+1} - \frac{\lambda}{s^2(s-1)}\left(\frac{s+\varkappa_2}{s}\right).
\end{aligned}$$

2. Operators using Bézier bases and calculating fundamental moments

Our goal in this section is to form the Stancu type of λ-Bernstein shifted knots operators enhanced by Bézier bases function. Our new construction of these types operators give a valuable research path to researcher in the field of approximation theory. Researcher can use the Bézier bases function, shifted knots properties as well as Stancu type construction in various research article and explore it to get a better observation and more flexibility. For more related results and its applications we would rather have operators of the Bernstein type (Cai et al., 2018; Gadjiev and Ghorbanalizadeh, 2010; Ye et al., 2010). By using the shifted knots properties (see Gadjiev and Ghorbanalizadeh (2010)), we design the Bernstein basis function $b^{\varkappa_1,\varkappa_2}_{s,i}$ as follows:

$$b^{\varkappa_1,\varkappa_2}_{s,i}(y) = \binom{s}{i}\left(y - \frac{\varkappa_1}{s+\varkappa_2}\right)^i \left(\frac{s+\varkappa_1}{s+\varkappa_2} - y\right)^{s-i}. \tag{2.1}$$

In addition, we design the Bézier bases function $\tilde{b}^{\varkappa_1,\varkappa_2}_{s,i}$ by means of Bernstein basis function $b^{\varkappa_1,\varkappa_2}_{s,i}$ (see Ye et al. (2010)) such as follows:

$$\begin{aligned}
\tilde{b}^{\varkappa_1,\varkappa_2}_{s,0}(\lambda; y) &= b^{\varkappa_1,\varkappa_2}_{s,0}(y) - \frac{\lambda}{s+1} b^{\varkappa_1,\varkappa_2}_{s+1,1}(y),\\
\tilde{b}^{\varkappa_1,\varkappa_2}_{s,i}(\lambda; y) &= b^{\varkappa_1,\varkappa_2}_{s,i}(y) + \lambda\bigg(\frac{s-2i+1}{s^2-1} b^{\varkappa_1,\varkappa_2}_{s+1,i}(y)
\end{aligned}$$

$$-\frac{s-2i-1}{s^2-1}b_{s+1,i+1}^{\varkappa_1,\varkappa_2}(y)\Bigg),\ \text{for } 1\le i\le s-1$$

$$\tilde{b}_{s,s}^{\varkappa_1,\varkappa_2}(\lambda;y)=b_{s,s}^{\varkappa_1,\varkappa_2}(y)-\frac{\lambda}{s+1}b_{s+1,s}^{\varkappa_1,\varkappa_2}(y).$$

Thus for all $\frac{\varkappa_1}{s+\varkappa_2}\le y\le\frac{s+\varkappa_1}{s+\varkappa_2}$ and the real number $0\le\varkappa_1\le\xi_1\le\xi_2\le\varkappa_2$, we define the Stancu variant of λ-Bernstein shifted knots operators associated by Bézier bases function $\tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}$ as follows:

$$S_{s,\lambda}^{\varkappa_1,\varkappa_2}(g;y)=\left(\frac{s+\varkappa_2}{s}\right)^s\sum_{i=0}^{s}\tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y)g\left(\frac{i+\xi_1}{s+\xi_2}\right), \tag{2.2}$$

Remark 2.1. For operators $S_{s,\lambda}^{\varkappa_1,\varkappa_2}$, we have the following observations:

(1) if take $\xi_1=\xi_2=0$, operators $S_{s,\lambda}^{\varkappa_1,\varkappa_2}$ having the polynomials and moments raised in Ayman-Mursaleen et al. (2024) by equality (1.4);
(2) for the choice of $\xi_1=\xi_2=0$ and $\varkappa_1=\varkappa_2=0$, operators $S_{s,\lambda}^{\varkappa_1,\varkappa_2}$ having the polynomials and moments raised in Cai et al. (2018) by Cai et al. (2018);
(3) for the choice of $\xi_1=\xi_2=0$ and $\varkappa_1=\varkappa_2=0$, with $\lambda=0$, operators $S_{s,\lambda}^{\varkappa_1,\varkappa_2}$ converted to classical Bernstein operators (Bernstein, 2012);
(4) Clearly, instead of Ayman-Mursaleen et al. (2024), Bernstein (2012) and Cai et al. (2018), we might say that our new operators are the most recent approximation results that are generalized.

This paper's primary structure is as follows: We examine our new operators, (2.2), moments and central moments. We explore an approximation theorem of Korovkin, establish a theorem of local approximation, offer a theorem of convergence for Lipschitz maximal space and generate an asymptotic convergence results of Voronovskaja type.

Lemma 2.2. *Take $g(t)=1,t,t^2$ the test function, then for any $s\in\{2,3,4,\dots\}$, the operators $S_{s,\lambda}^{\varkappa_1,\varkappa_2}$ defined by (2.2), having the following equalities:*

$$\begin{aligned}
S_{s,\lambda}^{\varkappa_1,\varkappa_2}(1;y)&=1;\\
S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y)&=\left(\frac{s+\varkappa_2}{s+\xi_2}-\frac{2\lambda}{(s+\xi_2)(s-1)}\right)\left(y-\frac{\varkappa_1}{s+\varkappa_2}\right)\\
&\quad+\frac{\lambda}{(s-1)(s+\xi_2)}\left(\frac{s+\varkappa_2}{s}\right)^s\left[\left(y-\frac{\varkappa_1}{s+\varkappa_2}\right)^{s+1}-\left(\frac{s+\varkappa_1}{s+\varkappa_2}-y\right)^{s+1}\right]\\
&\quad+\frac{\lambda}{(s-1)(s+\xi_2)}\left(\frac{s+\varkappa_2}{s}\right)+\frac{1}{(s+\xi_2)};
\end{aligned}$$

$$\begin{aligned}
S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t^2;y)&=\frac{s}{(s+\xi_2)^2}\left[\left(\frac{s+\varkappa_2}{s}\right)^s+\frac{2\lambda}{s-1}\right]\left(y-\frac{\varkappa_1}{s+\varkappa_2}\right)\\
&\quad+\frac{s(s+\varkappa_2)}{(s+\xi_2)^2}\left[\frac{s-1}{s}\frac{s+\varkappa_2}{s}-\frac{4\lambda}{s^2}\right]\left(y-\frac{\varkappa_1}{s+\varkappa_2}\right)^2\\
&\quad+\lambda\frac{s^2}{(s+\xi_2)^2}\left(\frac{s+\varkappa_2}{s}\right)^s\left[\frac{1}{s+1}+\frac{(s+1)^2}{(s-1)s^2}\right]\left(y-\frac{\varkappa_1}{s+\varkappa_2}\right)^{s+1}\\
&\quad+\frac{\lambda}{(s-1)(s+\xi_2)^2}\left(\frac{s+\varkappa_2}{s}\right)\left[\left(\frac{s+\varkappa_2}{s}\right)^{s-1}\left(\frac{s+\varkappa_1}{s+\varkappa_2}-y\right)^{s+1}-1\right]\\
&\quad+\frac{2s\xi_1}{(s+\xi_2)^2}\left(y-\frac{\varkappa_1}{s+\varkappa_2}\right)\left(\frac{s+\varkappa_2}{s}-\frac{2\lambda}{s(s-1)}\right)\\
&\quad+\frac{2\lambda\xi_1}{(s-1)(s+\xi_2)^2}\left(\frac{s+\varkappa_2}{s}\right)^s\left[\left(y-\frac{\varkappa_1}{s+\varkappa_2}\right)^{s+1}-\left(\frac{s+\varkappa_1}{s+\varkappa_2}-y\right)^{s+1}\right]\\
&\quad+\frac{2\lambda\xi_1}{(s-1)(s+\xi_2)^2}\left(\frac{s+\varkappa_2}{s}\right)+\frac{1}{(s+\xi_2)}\frac{2s\xi_1}{(s+\xi_2)^2}+\frac{\xi_1^2}{(s+\xi_2)^2}.
\end{aligned}$$

Proof. We proof the equalities as follows:

$$\begin{aligned}
S_{s,\lambda}^{\varkappa_1,\varkappa_2}(1;y)&=\left(\frac{s+\varkappa_2}{s}\right)^s\sum_{i=0}^{s}\tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y)\\
&=B_{s,\lambda}^{\varkappa_1,\varkappa_2}(1;y)\\
&=1;
\end{aligned}$$

$$\begin{aligned}
S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y)&=\left(\frac{s+\varkappa_2}{s}\right)^s\sum_{i=0}^{s}\left(\frac{i+\xi_1}{s+\xi_2}\right)\tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y)\\
&=\left(\frac{s+\varkappa_2}{s}\right)^s\sum_{i=0}^{s}\frac{i}{(s+\xi_2)}\tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y)+\frac{\xi_1}{(s+\xi_2)}\left(\frac{s+\varkappa_2}{s}\right)^s\sum_{i=0}^{s}\tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y)\\
&=\frac{s}{(s+\xi_2)}\left(\frac{s+\varkappa_2}{s}\right)^s\sum_{i=0}^{s}\left(\frac{i}{s}\right)\tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y)+\frac{\xi_1}{(s+\xi_2)}B_{s,\lambda}^{\varkappa_1,\varkappa_2}(1;y)
\end{aligned}$$

$$= \frac{s}{(s+\xi_2)} B_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y) + \frac{\xi_1}{(s+\xi_2)};$$

$$\begin{aligned}
S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t^2;y) &= \left(\frac{s+\varkappa_2}{s}\right)^s \sum_{i=0}^{s} \left(\frac{i+\xi_1}{s+\xi_2}\right)^2 \tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y) \\
&= \left(\frac{s+\varkappa_2}{s}\right)^s \sum_{i=0}^{s} \frac{i^2}{(s+\xi_2)^2} \tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y) + \left(\frac{s+\varkappa_2}{s}\right)^s \sum_{i=0}^{s} \frac{2i\xi_1}{(s+\xi_2)^2} \tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y) \\
&+ \left(\frac{s+\varkappa_2}{s}\right)^s \sum_{i=0}^{s} \frac{\xi_1^2}{(s+\xi_2)^2} \tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y) \\
&= \frac{s^2}{(s+\xi_2)^2} \left(\frac{s+\varkappa_2}{s}\right)^s \sum_{i=0}^{s} \frac{i^2}{s^2} \tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y) + \frac{2s\xi_1}{(s+\xi_2)^2} \sum_{i=0}^{s} \frac{i}{s} \tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y) \\
&+ \frac{\xi_1^2}{(s+\xi_2)^2} \sum_{i=0}^{s} \tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y) \\
&= \frac{s^2}{(s+\xi_2)^2} B_{s,\lambda}^{\varkappa_1,\varkappa_2}(t^2;y) + \frac{2s\xi_1}{(s+\xi_2)^2} B_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y) + \frac{\xi_1^2}{(s+\xi_2)^2} B_{s,\lambda}^{\varkappa_1,\varkappa_2}(1;y).
\end{aligned}$$

Thus finally, we get $S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t^2;y)$. □

Lemma 2.3. *For the operators $S_{s,\lambda}^{\varkappa_1,\varkappa_2}$, we get the following central moments:*

$$\begin{aligned}
S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y;y) &= \left(\frac{s+\varkappa_2}{s+\xi_2} - \frac{2\lambda}{(s+\xi_2)(s-1)} - 1\right) y \\
&+ \frac{\lambda}{(s-1)(s+\xi_2)} \left(\frac{s+\varkappa_2}{s}\right)^s \left[\left(y - \frac{\varkappa_1}{s+\varkappa_2}\right)^{s+1} - \left(\frac{s+\varkappa_1}{s+\varkappa_2} - y\right)^{s+1}\right] \\
&+ \frac{\lambda}{(s-1)(s+\xi_2)} \left(\frac{s+\varkappa_2}{s}\right) + \frac{2\lambda\varkappa_1}{(s+\xi_2)(s+\varkappa_2)(s-1)} + \frac{1-\varkappa_1}{s+\xi_2}
\end{aligned}$$

$$\begin{aligned}
S_{s,\lambda}^{\varkappa_1,\varkappa_2}\left((t-y)^2;y\right) &= \frac{s}{(s+\xi_2)^2} \left[\left(\frac{s+\varkappa_2}{s}\right)^s + \frac{2\lambda}{s-1}\right] \left(y - \frac{\varkappa_1}{s+\varkappa_2}\right) \\
&+ \frac{s(s+\varkappa_2)}{(s+\xi_2)^2} \left[\frac{s-1}{s}\frac{s+\varkappa_2}{s} - \frac{4\lambda}{s^2}\right] \left(y - \frac{\varkappa_1}{s+\varkappa_2}\right)^2 \\
&+ \lambda \frac{s^2}{(s+\xi_2)^2} \left(\frac{s+\varkappa_2}{s}\right)^s \left[\frac{(s+1)^2}{(s-1)s^2} + \frac{1}{s+1}\right] \left(y - \frac{\varkappa_1}{s+\varkappa_2}\right)^{s+1} \\
&+ \frac{\lambda}{(s-1)(s+\xi_2)^2} \left(\frac{s+\varkappa_2}{s}\right) \left[\left(\frac{s+\varkappa_2}{s}\right)^{s-1} \left(\frac{s+\varkappa_1}{s+\varkappa_2} - y\right)^{s+1} - 1\right] \\
&+ \frac{2s\xi_1}{(s+\xi_2)^2} \left(y - \frac{\varkappa_1}{s+\varkappa_2}\right) \left(\frac{s+\varkappa_2}{s} - \frac{2\lambda}{s(s-1)}\right) \\
&+ \frac{2\lambda\xi_1}{(s-1)(s+\xi_2)^2} \left(\frac{s+\varkappa_2}{s}\right)^s \left[\left(y - \frac{\varkappa_1}{s+\varkappa_2}\right)^{s+1} - \left(\frac{s+\varkappa_1}{s+\varkappa_2} - y\right)^{s+1}\right] \\
&+ \frac{2\lambda\xi_1}{(s-1)(s+\xi_2)^2} \left(\frac{s+\varkappa_2}{s}\right) + \frac{1}{(s+\xi_2)} \frac{2s\xi_1}{(s+\xi_2)^2} + \frac{\xi_1^2}{(s+\xi_2)^2} + y^2 \\
&- 2\left(\frac{s+\varkappa_2}{s+\xi_2} - \frac{2\lambda}{(s+\xi_2)(s-1)}\right) \left(y - \frac{\varkappa_1}{s+\varkappa_2}\right) y \\
&+ \frac{2\lambda y}{(s-1)(s+\xi_2)} \left(\frac{s+\varkappa_2}{s}\right)^s \left[\left(\frac{s+\varkappa_1}{s+\varkappa_2} - y\right)^{s+1} - \left(y - \frac{\varkappa_1}{s+\varkappa_2}\right)^{s+1}\right] \\
&- \frac{2\lambda y}{(s-1)(s+\xi_2)} \left(\frac{s+\varkappa_2}{s}\right) - \frac{2y}{(s+\xi_2)}.
\end{aligned}$$

3. Convergence properties of operators $S_{s,\lambda}^{\varkappa_1,\varkappa_2}$

For operators (2.2), we derive various global and local approximation theorems in this section. Using the Ditzian–Totik uniform modulus of smoothness, we first define the uniform convergence property for our operators and then obtain the local and global approximations. Next, we derive some straightforward theorems based on the Lipschitz type maximal approximation property and Peetre's K-functional property. Let we denote $\mathcal{W} = C[0,1]$, the set of all continuous function on $[0,1]$. Given a continuous function f in $\mathcal{W}$ on $[0,1]$, we may replace ℓ with a real-valued function equipped by norm $\| \ell \|_{\mathcal{W}} = \sup_{y\in[0,1]} |\ell(y)|$.

Theorem 3.1 (Altomare, 2010; Korovkin, 1953)**.** *Let the sequences of positive linear operators $L_s : C[\mu_1,\mu_2] \to C[\mu_1,\mu_2]$ for all $j = 0,1,2$ $\lim_{s\to\infty} L_s(t^j;y) = y^j$, is uniformly on $[\mu_1,\mu_2]$. Then, for each $\ell \in C[\mu_1,\mu_2]$ we get $\lim_{s\to\infty} L_s(\ell) = \ell$ is uniformly converges for each compact subset contained in $[\mu_1,\mu_2]$.*

Theorem 3.2. *Regarding all $\ell \in C[0,1]$ we get*

$$\lim_{s\to\infty} S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\ell;y)=\ell(y)$$

is converges uniformly on $[0,1]$.

Proof. According to Lemma 2.2, it is evident that for any $j=0,1,2$,

$$\lim_{s\to\infty} S_{s,\lambda}^{\varkappa_1,\varkappa_2}\left(t^j;y\right)=y^j,$$

thus, by applying the famous Bohman–Korovkin–Popoviciu theorem, easy to get operators $S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\ell;y)$ are uniformly converge to set $\ell \in \mathcal{W}$. □

Theorem 3.3 (Gadziev, 1976; Gadjiev, 1974). *For the operators $\{L_s\}_{s\geq 1}:\mathcal{W}\to\mathcal{W}$, let $\lim_{s\to\infty}\|L_s(t^\beta)-y^\beta\|_{\mathcal{W}}=0,\ \beta=0,1,2$. We get for all $\ell\in\mathcal{W}$,*

$$\lim_{s\to\infty}\|L_s(\ell)-\ell\|_{\mathcal{W}}=0.$$

Theorem 3.4. *Suppose the operators $S_{s,\lambda}^{\varkappa_1,\varkappa_2}:\mathcal{W}\to\mathcal{W}$, which having $\lim_{s\to\infty}\|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t^j)-y^j\|_{\mathcal{W}}=0$. For all $\ell\in\mathcal{W}$ we obtain*

$$\lim_{s\to\infty}\|\ S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\ell)-\ell\ \|_{\mathcal{W}}=0.$$

Proof. We take in account Theorem 3.3 and famous Bohman–Korovkin-Popoviciu theorem then easily we lead to show that

$$\lim_{s\to\infty}\|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t^j)-y^j\|_{\mathcal{W}}=0,\quad j=0,1,2.$$

In the view of Lemma 2.2, easy to obtain $\|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(1)-1\|_{\mathcal{W}}=\sup_{y\in[0,1]}\left|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(1;y)-1\right|=0$. For $j=1$, easy to see

$$\begin{aligned}\|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t)-y\|_{\mathcal{W}} &= \sup_{y\in[0,1]}\left|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y)-y\right|\\ &\leq \max_{y\in[0,1]}\left|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y;y)\right|\\ &\leq \left|\left(\frac{s+\varkappa_2}{s+\xi_2}-\frac{2\lambda}{(s+\xi_2)(s-1)}-1\right)\right|\\ &\quad+\left|\frac{\lambda}{(s-1)(s+\xi_2)}\left(\frac{s+\varkappa_2}{s}\right)^s\left[\left(\frac{s+\varkappa_1}{s+\varkappa_2}-1\right)^{s+1}-\left(1-\frac{\varkappa_1}{s+\varkappa_2}\right)^{s+1}\right]\right|\\ &\quad+\left|\frac{\lambda}{(s-1)(s+\xi_2)}\left(\frac{s+\varkappa_2}{s}\right)+\frac{2\lambda\varkappa_1}{(s+\xi_2)(s+\varkappa_2)(s-1)}+\frac{1-\varkappa_1}{s+\xi_2}\right|\end{aligned}$$

Since $s\to\infty$ then $\frac{1}{s+\xi_2}\to 0,\ \frac{s+\varkappa_2}{s+\xi_2}\to 1,\ \frac{s+\varkappa_2}{s}\to 1$, therefore we get $\|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t)-y\|_{\mathcal{W}}\to 0$. Similarly for $j=2$, we see

$$\|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t^2)-y^2\|_{\mathcal{W}}=\sup_{y\in[0,1]}\left|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t^2;y)-y^2\right|,$$

which leads to get $\|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t^2)-y^2\|_{\mathcal{W}}\to 0$ as $s\to\infty$. These observations leads to get our proof. □

By employing the uniform modulus of smoothness of Ditzian–Totik, we present some results on global approximations. We go over the fundamental characteristic of the uniform modulus of smoothness for orders first and second, which is

$$\omega(\ell,\delta):=\sup_{0<|\rho|\leq\delta}\ \sup_{y,y+\rho\gamma(y)\in[0,1]}\{|\ell(y+\rho\gamma(y))-\ell(y)|\};$$

$$\omega_2^{\gamma}(\ell,\delta):=\sup_{0<|\rho|\leq\delta}\ \sup_{y,y\pm\rho\gamma(y)\in[0,1]}\{|\ell(y+\rho\gamma(y))-2\ell(y)+\ell(y-\rho\gamma(y))|\},$$

and the step-weight function γ on $[u,v]$ and suppose $\gamma(y)=[(v-y)(y-u)]^{1/2}$ if $y\in[u,v]$ (see Ditzian and Totik (1987)). The set of all continuously functions can be represented as C^*, then the Peetre's K-functional approximation property is given by

$$K_2^{\gamma}(\ell,\delta)=\inf_{\zeta\in\Theta^2(\gamma)}\left\{\delta\|\gamma^2\varphi''\|_{\mathcal{W}}+\|\ell-\zeta\|_{\mathcal{W}}\right\},\ \text{for all } \zeta\in C^2[0,1],$$

and for any $\delta>0$, $\Theta^2(\gamma)=\{\zeta\in\mathcal{W}$ such that $\zeta'\in C^*[0,1],\ \gamma^2\zeta''\in\mathcal{W}\}$ and $C^2[0,1]=\{\zeta\in\mathcal{W}:\zeta',\zeta''\in\mathcal{W}\}$.

Remark 3.5 (DeVore and Lorentz, 1993). For a positive real constant M one has the inequality

$$M^{-1}\omega_2^{\gamma}(\ell,\sqrt{\delta})\leq K_2^{\gamma}(\ell,\delta)\leq M\omega_2^{\gamma}(\ell,\sqrt{\delta}). \tag{3.1}$$

Theorem 3.6. *Let $\gamma(y)$ $(\gamma\neq 0)$ be any step-weight function γ^2 is concave, then, for all $\ell\in\mathcal{W}$ and $y\in[0,1]$ operators $S_{s,\lambda}^{\varkappa_1,\varkappa_2}$ satisfying*

$$|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\ell;y)-\ell(y)|\leq M\ \omega_2^{\gamma}\left(\ell,\frac{[\phi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)+\psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)]^{1/2}}{2\gamma(y)}\right)+\omega\left(\ell,\frac{\psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)}{\gamma(y)}\right),$$

where $\psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)=S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y;y)$ and $\phi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)=S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y)$.

Proof. Considering an auxiliary operators as follows

$$\Omega_{s,\lambda}^{\varkappa_1,\varkappa_2}(\ell;y)=\ell(y)+S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\ell;y)-\ell\left(\psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)+y\right), \tag{3.2}$$

where $\ell\in\mathcal{W},\ y\in[0,1]$, then by the virtue of Lemma 2.2 easy to get the following relations

$$\Omega_{s,\lambda}^{\varkappa_1,\varkappa_2}(1;y)=1 \text{ and } \Omega_{s,\lambda}^{\varkappa_1,\varkappa_2}(t;y)=y,$$

$\Omega_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)(t-y;y)=0.$

Let $\ell=\rho y+(1-\rho)t$ for $\rho\in[0,1]$. Following the property $\gamma^2(y)\ge\rho\gamma^2(y)+(1-\rho)\gamma^2(t)$ as γ^2 is concave on $[0,1]$ and

$$\frac{|t-\ell|}{\gamma^2(y)}\le\frac{\rho|y-t|}{\rho\gamma^2(y)+(1-\rho)\gamma^2(t)}\le\frac{|t-y|}{\gamma^2(y)}. \tag{3.3}$$

We get the following identities:

$$\begin{aligned}|\Omega_{s,\lambda}^{\varkappa_1,\varkappa_2}(\ell;y)-\ell(y)|&\le|\Omega_{s,\lambda}^{\varkappa_1,\varkappa_2}(\ell-\zeta;y)|+|\Omega_{s,\lambda}^{\varkappa_1,\varkappa_2}(\zeta;y)-\zeta(y)|+|\ell(y)-\zeta(y)|\\&\le4\|\ell-\zeta\|_{C[0,1]}+|\Omega_{s,\lambda}^{\varkappa_1,\varkappa_2}(\zeta;y)-\zeta(y)|.\end{aligned} \tag{3.4}$$

By use of Taylor's series, we can conclude that

$$\begin{aligned}|\Omega_{s,\lambda}^{\varkappa_1,\varkappa_2}(\zeta;y)-\zeta(y)|\ &\le\ S_{s,\lambda}^{\varkappa_1,\varkappa_2}\left(\left|\int_y^t|t-\ell|\ |\zeta''(\ell)|\mathrm{d}\ell\right|;y\right)\\&\quad+\left|\int_y^{\psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)+y}\left|\psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)+y-\ell\right|\ |\zeta''(\ell)|\ \mathrm{d}\ell\right|\\&\le\ \|\gamma^2\zeta''\|_{\mathcal{W}}S_{s,\lambda}^{\varkappa_1,\varkappa_2}\left(\left|\int_y^t\frac{|t-\ell|}{\gamma^2(\ell)}\mathrm{d}\ell\right|;y\right)+\|\gamma^2\zeta''\|_{\mathcal{W}}\\&\quad\times\left|\int_y^{\psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)+y}\left|\psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)+y-\ell\right|\ \frac{\mathrm{d}\ell}{\gamma^2(y)}\right|\\&\le\ \gamma^{-2}(y)\|\gamma^2\zeta''\|_{\mathcal{W}}S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y)\\&\quad+\gamma^{-2}(y)\psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)\|\gamma^2\zeta''\|_{\mathcal{W}}.\end{aligned} \tag{3.5}$$

We use the Peetre's K-functional properties and the relations (3.1), (3.4), and (3.5), then easy to get

$$\begin{aligned}\left|\Omega_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)(\ell;y)-\ell(y)\right|&\le4\|\ell-\zeta\|_{C[0,1]}+\gamma^{-2}(y)\|\gamma^2\zeta''\|_{\mathcal{W}}(\phi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)+\psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y))\\&\le M\ \omega_2^{\gamma}\left(\ell,\frac12\sqrt{\frac{\phi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)+\psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)}{\gamma(y)}}\right).\end{aligned}$$

It is obvious that

$$\left|\ell\left(\psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)+y\right)-\ell(y)\right|=\left|\ell\left(\psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)+y\right)-\ell(y)\right|\le\omega\left(\ell,\frac{\psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)}{\gamma(y)}\right).$$

Thus, finally, we get the inequality

$$\begin{aligned}|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\ell;y)-\ell(y)|&\le|\Omega_{s,\lambda}^{\varkappa_1,\varkappa_2}(\ell;y)-\ell(y)|+\left|\ell\left(\psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)+y\right)-\ell(y)\right|\\&\le M\ \omega_2^{\gamma}\left(\ell,\frac12\sqrt{\frac{\phi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)+\psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)}{(y-u)(v-y)}}\right)+\omega\left(\ell,\frac{\psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)}{\gamma(y)}\right),\end{aligned}$$

which completes the desired proof of Theorem 3.6. □

Theorem 3.7. *Assuming $y\in[0,1]$ and $\ell'\in\mathcal{W}$, the inequality is as follows:*

$$|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\ell;y)-\ell(y)|\le\phi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)\ |\ell'(y)|+2\sqrt{\phi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)}\omega\left(\ell',\sqrt{\phi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)}\ \right).$$

Proof. We know the relation

$$\ell(t)=\ell(y)+\ell'(y)(t-y)+\int_y^t(\ell'(z)-\ell'(y))\mathrm{d}z, \tag{3.6}$$

for all $t,y\in[0,1]$. On apply the operators $S_{s,\lambda}^{\varkappa_1,\varkappa_2}$ to equality (3.6), we obtain

$$S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\ell(t)-\ell(y);y)=\ell'(y)S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y;y)+S_{s,\lambda}^{\varkappa_1,\varkappa_2}\left(\int_y^t(\ell'(z)-\ell'(y))\mathrm{d}z;y\right).$$

For all $\ell\in C[0,1]$ and $y\in[0,1]$, one has

$$|\ell(t)-\ell(y)|\le\left(1+\frac{|t-y|}{\delta}\right)\omega(\ell,\delta),\quad\delta>0.$$

From the above inequality, we have

$$\left|\int_t^y(\ell'(y)-\ell'(z))\mathrm{d}z\right|\le\left(|t-y|+\frac{(t-y)^2}{\delta}\right)\omega(\ell',\delta).$$

Therefore, easy to obtain

$$|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\ell;y)-\ell(y)| \leq |\ell'(y)|\ |S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y;y)| + \omega(\ell',\delta)\left\{\frac{1}{\delta}S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y)+S_{s,\lambda}^{\varkappa_1,\varkappa_2}(|t-y|;y)\right\}.$$

The Cauchy–Schwarz inequality yields,

$$S_{s,\lambda}^{\varkappa_1,\varkappa_2}(|t-y|;y) \leq S_{s,\lambda}^{\varkappa_1,\varkappa_2}(1;y)^{\frac{1}{2}}S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y)^{\frac{1}{2}} = S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y)^{\frac{1}{2}}.$$

Thus, we have

$$|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\ell;y)-\ell(y)| \leq \ell'(y)S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y) + \left\{\frac{1}{\delta}\sqrt{S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y)}+1\right\}S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y)^{\frac{1}{2}}\omega(\ell',\delta),$$

by use of $\delta=\sqrt{S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y)}$, then we get the desired results. □

Next, we estimate the local direct approximation using a Lipschitz-type maximum function, which brings back memories from (Ozarslan and Aktuğlu, 2013).

$$Lip_K(\kappa) := \left\{\ell\in\mathcal{W} \text{ such that } |\ell(t)-\ell(y)| \leq K\frac{|t-y|^{\kappa}}{(\beta_1 y^2+\beta_2 y+t)^{\frac{\kappa}{2}}};\ y,t\in[0,1]\right\}$$

where $\beta_1 \geq 0,\ \beta_2 > 0,\ \kappa\in(0,1]$ and $K>0$ be any constant (see Ozarslan and Aktuğlu (2013)).

Theorem 3.8. *Let $\ell\in Lip_K(\kappa)$, then for any $\kappa\in(0,1]$ we obtain*

$$|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\ell;y)-\ell(y)| \leq K\sqrt{\frac{[\phi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)]^{\kappa}}{(\beta_1 y^2+\beta_2 y)^{\kappa}}}.$$

Proof. Take $\ell\in Lip_K(\kappa)$ for any $\kappa\in(0,1]$. First, we want to show that our result is valid for $\kappa=1$. Thus for any $\ell\in Lip_K(1)$ we have

$$\begin{aligned}|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\ell;y)-\ell(y)| &\leq |S_{s,\lambda}^{\varkappa_1,\varkappa_2}(|\ell(t)-\ell(y)|;y)|+\ell(y)\ |S_{s,\lambda}^{\varkappa_1,\varkappa_2}(1;y)-1|\\ &\leq \left(\frac{s+\varkappa_2}{s}\right)^s\sum_{i=0}^{s}\tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y)\Big|\ell(t)-\ell(y)\Big|\\ &\leq K\left(\frac{s+\varkappa_2}{s}\right)^s\sum_{i=0}^{s}\frac{\tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y)|t-y|}{(\beta_1 y^2+\beta_2 y+t)^{\frac{1}{2}}}.\end{aligned}$$

By using

$$(\beta_1 y^2+\beta_2 y+t)^{-1/2} \leq (\beta_1 y^2+\beta_2 y)^{-1/2} \quad (\beta_1\geq 0,\beta_2>0)$$

and by Cauchy–Schwarz inequality, we see

$$\begin{aligned}|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\ell;y)-\ell(y)| &\leq K\left(\frac{s+\varkappa_2}{s}\right)^s(\beta_1 y^2+\beta_2 y)^{-1/2}\sum_{i=0}^{s}\tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y)\Big|t-y\Big|\\ &= K(\beta_1 y^2+\beta_2 y)^{-1/2}|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y;y)|\\ &\leq K(\beta_1 y^2+\beta_2 y)^{-1/2}S_{s,\lambda}^{\varkappa_1,\varkappa_2}(|t-y|;y)\\ &\leq K\left(S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y)\right)^{\frac{1}{2}}(\beta_1 y^2+\beta_2 y)^{-1/2}.\end{aligned}$$

Consequently, for $\kappa=1$ our result valid. Moreover, take $\kappa\in(0,1]$, then the following requirements are also satisfied by applying the monotonicity property to operators $S_{s,\lambda}^{\varkappa_1,\varkappa_2}$ and introducing the Hölder's property:

$$\begin{aligned}\Big|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\ell;y)-\ell(y)\Big| &\leq \left(\frac{s+\varkappa_2}{s}\right)^s\sum_{i=0}^{s}\tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y)\Big|\ell(t)-\ell(y)\Big|\\ &\leq \left(\frac{s+\varkappa_2}{s}\right)^s\left(\sum_{i=0}^{s}\tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y)\Big|\ell(t)-\ell(y)\Big|\right)^{\frac{\kappa}{2}}\left(S_{s,\lambda}^{\varkappa_1,\varkappa_2}(1;y)\right)^{\frac{2-\kappa}{2}}\\ &\leq K\left(\frac{s+\varkappa_2}{s}\right)^s\left(\sum_{i=0}^{s}\frac{\tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y)\big(t-y\big)^2}{\beta_1 y^2+\beta_2 y+t}\right)^{\frac{\kappa}{2}}\\ &\leq K\left\{\sum_{i=0}^{s}\tilde{b}_{s,i}^{\varkappa_1,\varkappa_2}(\lambda;y)\Big(t-y\Big)^2\right\}^{\frac{\kappa}{2}}(\beta_1 y^2+\beta_2 y+t)^{-\kappa/2}\\ &\leq K\left(\frac{s+\varkappa_2}{s}\right)^s(\beta_1 y^2+\beta_2 y)^{-\kappa/2}\left[S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y)\right]^{\frac{\kappa}{2}}\\ &= K\sqrt{\frac{[\phi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)]^{\kappa}}{(\beta_1 y^2+\beta_2 y)^{\kappa}}}.\ \square\end{aligned}$$

However, we also prove another local approximation feature for the operators of $S^{\varkappa_1,\varkappa_2}_{s,\lambda}$ by using the Lipschitz maximum function. Let Φ be such that $\Phi \in C_B[0,1]$ and $t, y \in [0,1]$ belong to the same class of maximal functions of Lipschitz type (see Lenze (1988)).

$$\omega^*_{\vartheta}(\Phi; y) = \sup_{t\neq y,\, t\in[0,1]} \frac{\mid \Phi(t) - \Phi(y) \mid}{\mid t-y \mid^{\vartheta}}, \tag{3.7}$$

where $0 < \vartheta \le 1$.

Theorem 3.9. *Let $y \in [0,1]$, then for all $\hbar \in C_B[0,1]$ (the collection of all bounded and continuous functions on $[0,1]$), then operators $S^{\varkappa_1,\varkappa_2}_{s,\lambda}$ satisfying*

$$\left| S^{\varkappa_1,\varkappa_2}_{s,\lambda}(\hbar; y) - \hbar(y) \right| \le \left(\delta^{\varkappa_1,\varkappa_2}_{s,\lambda}(y) \right)^{\frac{\vartheta}{2}} \omega^*_{\vartheta}(\hbar; y),$$

where $\omega^*_{\vartheta}(\hbar; y)$ is given by (3.7) and $\delta^{\varkappa_1,\varkappa_2}_{s,\lambda}(y) = S^{\varkappa_1,\varkappa_2}_{s,\lambda}((t-y)^2; y)$.

Proof. With the help of the Höglider inequality, writing

$$\begin{aligned}\left| S^{\varkappa_1,\varkappa_2}_{s,\lambda}(\hbar; y) - \hbar(y) \right| &\le S^{\varkappa_1,\varkappa_2}_{s,\lambda} \left(|\hbar(t) - \hbar(y)| \,; y \right)\\ &\le \omega^*_{\vartheta}(\hbar; y) \mid S^{\varkappa_1,\varkappa_2}_{s,\lambda} \left(|t-y|^{\vartheta} \,; y \right)\\ &\le \omega^*_{\vartheta}(\hbar; y) \left(S^{\varkappa_1,\varkappa_2}_{s,\lambda}(|t-y|^2 \,; y) \right)^{\frac{\vartheta}{2}} \left(S^{\varkappa_1,\varkappa_2}_{s,\lambda}(1; y) \right)^{\frac{2-\vartheta}{2}}\\ &= \omega^*_{\vartheta}(\hbar; y) \left(S^{\varkappa_1,\varkappa_2}_{s,\lambda} \left((t-y)^2; y \right) \right)^{\frac{\vartheta}{2}}.\end{aligned}$$

This outcome give desired proof. □

4. Voronovskaja type asymptotic formula

Based on the article (Barbosu, 2002; Özger et al., 2020), we examine the quantitative Voronovskaja-type approximation theorem and derive the Voronovskaja-type approximation properties for our novel operators $S^{\varkappa_1,\varkappa_2}_{s,\lambda}$. By use of the modulus of smoothness properties allows here to see an explanation as follows:

$$\omega_{\Theta}(\Psi, \delta) := \sup_{0<|\rho|\le\delta} \left\{ \left| f\left(y + \frac{\rho\Theta(y)}{2} \right) - \Psi\left(y - \frac{\rho\Theta(y)}{2} \right) \right|, y \pm \frac{\rho\Theta(y)}{2} \in [0,1] \right\}.$$

Here $\Psi \in C[0,1]$ and $\Theta(y) = (y - y^2)^{1/2}$, and related Peetre's K-functional given by:

$$K_{\Theta}(\Psi, \delta) = \inf_{\ell\in\omega_{\Theta}[0,1]} \left\{ \|\ell - \Psi\| + \delta \|\Theta \ell'\| \,:\, \ell' \in C[0,1], \delta > 0 \right\},$$

where $\omega_{\Theta}[0,1] = \{\ell \,:\, \ell' \in C^*[0,1],\ \|\Theta\ell'\| < \infty\}$ and $C^*[0,1]$, which represents the whole functions that are entirely continuous on the intervals $[0,1]$. There is a positive constant M that allows for:

$$K_{\Theta}(\Psi, \delta) \le M\ \omega_{\Theta}(\Psi, \delta).$$

Theorem 4.1. *Regarding all $\zeta, \zeta', \zeta'' \in \mathcal{W}$, it verify that*

$$\left| S^{\varkappa_1,\varkappa_2}_{s,\lambda}(\zeta; y) - \zeta(y) - \psi^{\varkappa_1,\varkappa_2}_{s,\lambda}(y)\zeta'(y) - \frac{\delta^{\varkappa_1,\varkappa_2}_{s,\lambda}(y) + 1}{2} \zeta''(y) \right| \le M \frac{\Theta^2(y)}{s} \omega_{\Theta}\left(\zeta'', \frac{1}{\sqrt{s}} \right),$$

where $y \in [0,1]$, $C > 0$ a constant, $\psi^{\varkappa_1,\varkappa_2}_{s,\lambda}(y) = S^{\varkappa_1,\varkappa_2}_{s,\lambda}(t-y; y)$ and $\delta^{\varkappa_1,\varkappa_2}_{s,\lambda} = S^{\varkappa_1,\varkappa_2}_{s,\lambda}\left((t-y)^2; y \right)$, which defined by Lemma 2.3.

Proof. Given that $\zeta \in \mathcal{W}$, the Taylor series expansion can be allow:

$$\zeta(t) = \zeta(y) + \zeta'(y)(t-y) + \int_y^t \zeta''(\theta)(t-\theta)\mathrm{d}\theta,$$

then easy to get

$$\zeta(t) \le \zeta(y) + (t-y)\zeta'(y) + \frac{\zeta''(y)}{2}\left((t-y)^2 + 1 \right) + \int_y^t (t-\theta)[\zeta''(\theta) - \zeta''(y)]\mathrm{d}\theta. \tag{4.1}$$

Therefore, (4.1) give us,

$$\begin{aligned}&\left| S^{\varkappa_1,\varkappa_2}_{s,\lambda}(\zeta; y) - \zeta(y) - \zeta'(y) S^{\varkappa_1,\varkappa_2}_{s,\lambda}(t-y; y) - \left(S^{\varkappa_1,\varkappa_2}_{s,\lambda}((t-y)^2; y) \frac{\zeta''(y)}{2} + S^{\varkappa_1,\varkappa_2}_{s,\lambda}(1; y) \right) \right|\\ &\le S^{\varkappa_1,\varkappa_2}_{s,\lambda}\left(\left| \int_y^t |\zeta''(\theta)|t-\theta| - \zeta''(y)|\ \mathrm{d}\theta \right|; y \right).\end{aligned} \tag{4.2}$$

We can estimate the following from the right part of equality (4.2):

$$\left|\int_y^t |t-\theta|\ |\zeta''(\theta)-\zeta''(y)|\ \mathrm{d}\theta\right| \leq 2(t-y)^2\|\zeta''-\ell\| + 2|t-y|^3\|\Theta\ell'\|\Theta^{-1}(y), \tag{4.3}$$

where $\zeta \in \omega_\Theta[0,1]$. A positive constant suppose M exists such that

$$S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y) \leq \frac{M}{2s}\Theta^2(y) \quad \text{and} \quad S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^4;y) \leq \frac{M}{2s^2}\Theta^4(y) \tag{4.4}$$

We may determine by applying the Cauchy–Schwarz inequality that

$$\begin{aligned}
&\left|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\zeta;y) - \zeta(y) - \zeta'(y)S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y;y) - \left(S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y)\frac{\zeta''(y)}{2} + S_{s,\lambda}^{\varkappa_1,\varkappa_2}(1;y)\right)\right| \\
&\leq 2S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y)\|\zeta''-\ell\| + 2S_{s,\lambda}^{\varkappa_1,\varkappa_2}(|t-y|^3;y)\|\Theta(y)\ell'\|\Theta^{-1}(y) \\
&\leq \frac{M}{s}\Theta^2(y)\|\zeta''-\ell\| + 2\|\Theta(y)\ell'\|\Theta^{-1}(y)\{S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y)\}^{1/2}\{S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^4;y)\}^{1/2} \\
&\leq M\frac{\Theta^2(y)}{s}\left\{\|\zeta''-\ell\| + s^{-1/2}\|\Theta(y)\ell'\|\right\}.
\end{aligned}$$

We determine that by taking the infimum over all $\ell \in \omega_\Theta[0,1]$.

$$\left|S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\zeta;y) - \zeta(y) - \psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)\zeta'(y) - \frac{\delta_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)+1}{2}\zeta''(y)\right| \leq \frac{M}{s}\Theta^2(y)\omega_\Theta\left(\zeta'', \frac{1}{\sqrt{s}}\right),$$

which brings the results. □

Theorem 4.2. *For any $\hbar \in C_B[0,1]$, we examine*

$$\lim_{s\to\infty} s\left[S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\hbar;y) - \hbar(y) - \psi_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)\hbar'(y) - \frac{\delta_{s,\lambda}^{\varkappa_1,\varkappa_2}(y)}{2}\hbar''(y)\right] = 0.$$

Proof. If $\hbar \in C_B[0,1]$, we can write using Taylor's series expansion as follows:

$$\hbar(t) = \hbar(y) + (t-y)\hbar'(y) + \frac{1}{2}(t-y)^2\hbar''(y) + (t-y)^2\ Q_y(t). \tag{4.5}$$

Moreover, $Q_y(t) \to 0$ as $t \to y$, where $Q_y(t) \in \mathcal{W}$ and specified for Peano form of remainder. Using the operators $S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\cdot;y)$ to the equality (4.5), then easy to see

$$S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\hbar;y) - \hbar(y) = \hbar'(y)S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y;y) + \frac{\hbar''(y)}{2}S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y) + S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2Q_y(t);y).$$

Cauchy–Schwarz inequality gives us

$$S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2Q_y(t);y) \leq \sqrt{S_{s,\lambda}^{\varkappa_1,\varkappa_2}(Q_y^2(t);y)}\sqrt{S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^4;y)}. \tag{4.6}$$

We clearly observe here $\lim_{s\to\infty} S_{s,\lambda}^{\varkappa_1,\varkappa_2}(Q_y^2(t);y) = 0$ and therefore

$$\lim_{s\to\infty} s\{S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2Q_y(t);y)\} = 0.$$

Thus, we have

$$\begin{aligned}
\lim_{s\to\infty} s\{S_{s,\lambda}^{\varkappa_1,\varkappa_2}(\hbar;y) - \hbar(y)\} = \lim_{s\to\infty} s\Big\{&S_{s,\lambda}^{\varkappa_1,\varkappa_2}(t-y;y)\hbar'(y) + \frac{\hbar''(y)}{2}S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2;y) \\
&+ S_{s,\lambda}^{\varkappa_1,\varkappa_2}((t-y)^2Q_y(t);y)\Big\}.
\end{aligned}$$ □

5. Graphical analysis

This section will provide several MATLAB-assisted numerical examples with explanatory images.

Example 5.1. Let $g(y) = y^2 + \frac{3}{4}$, $\varkappa_1 = 3.5, \varkappa_2 = 5, \lambda = 0.6$ and $s \in \{15, 35, 75\}$. Fig. 1 illustrates how the operator converges to the function $g(y)$.

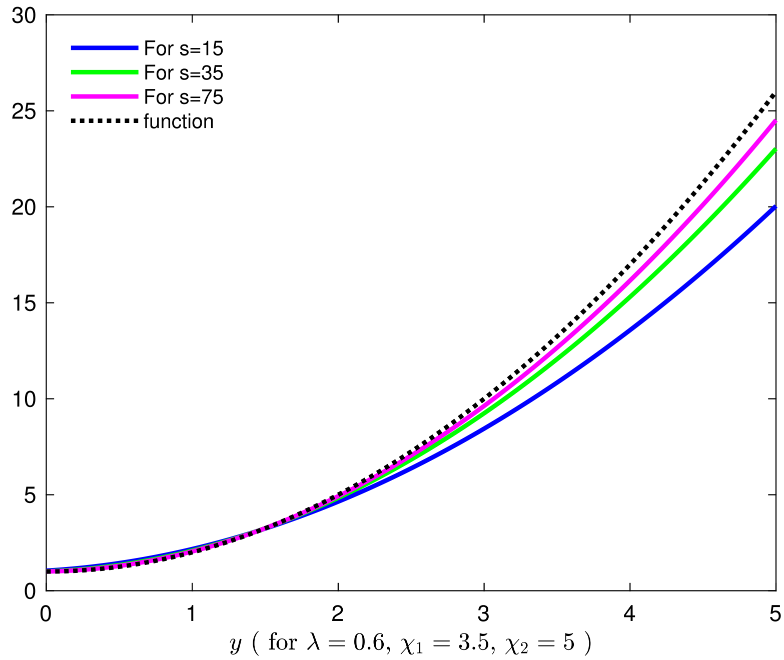

Fig. 1. Convergence of the operator towards the function $g(y) = y^2 + \frac{3}{4}$.

Example 5.2. Let $g(y) = (y - \frac{1}{3})(y - \frac{3}{5})$, $\varkappa_1 = 2, \varkappa_2 = 5, \lambda = 3$ and $s \in \{15, 45, 75\}$. Fig. 2 illustrates how the operator converges towards $g(y)$.

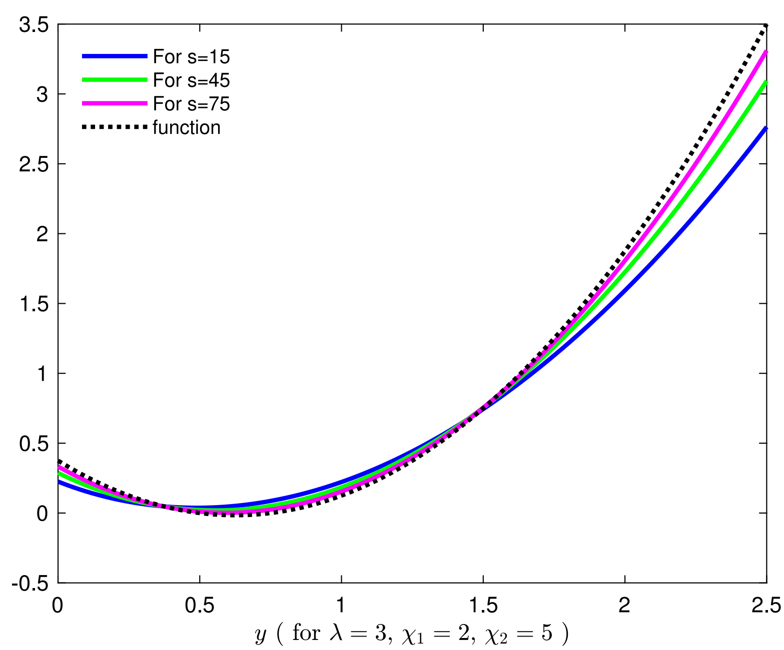

Fig. 2. Convergence of the operator towards $g(y) = (y - \frac{1}{3})(y - \frac{3}{5})$.

These examples show us that when we take higher values of s, the operators' approximations of the function get better. Observe that the operators (2.2) reduce to operators (1.2) for $\varkappa_1 = \varkappa_2 = 0$.

Example 5.3. Let $g(y) = y^2 - 5y + 9$. For $\lambda = 3.5, s = 35$ comparison of convergence of constructed operator (1.6) Fig. 3 displays (green and pink) with the previously defined operator (1.5)(blue). This image clearly shows that the created operator provides a more accurate approximation to $g(y)$ than the operator that was previously specified.

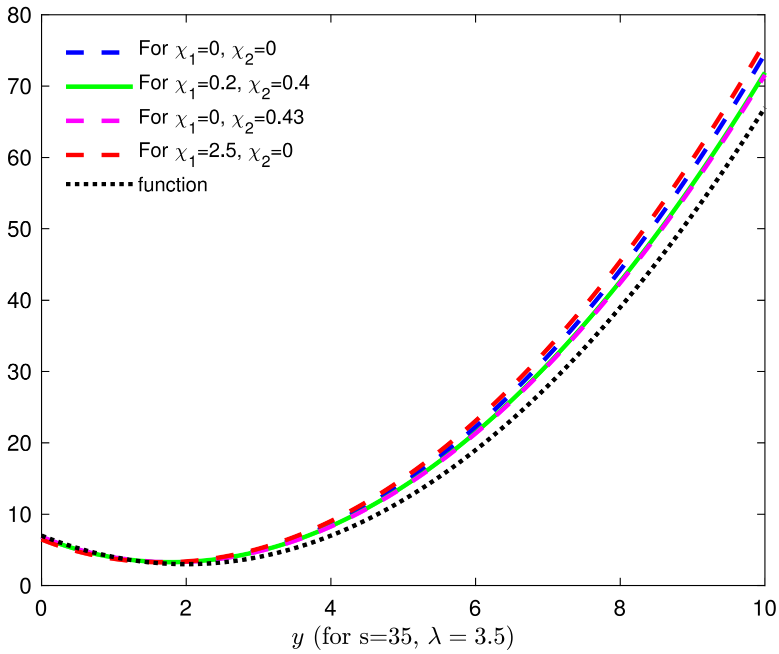

Fig. 3. Comparison of convergence of the operator with the previous operator.

6. Conclusion & observation

Our new operators (2.2) are the ones that we conclude in this article as the λ-Bernstein Stancu type generated operators associated by shifted knots of Bézier basis function. For the selection of $\xi_1 = \xi_2 = 0$ in the equality (2.2), it is evident that our new operators $S_{s,\lambda}^{\varkappa_1,\varkappa_2}$ decreased to the most recent operators by the equality (1.4) (see Ayman-Mursaleen et al. (2024)). Additionally, for the choice of $\xi_1 = \xi_2 = 0$ and $\varkappa_1 = \varkappa_2 = 0$ the equality (2.2) reduced to the cited article (Cai et al., 2018) by equality (1.2) given by Cai et al. Based on these findings, we conclude that rather than (Ayman-Mursaleen et al., 2024; Bernstein, 2012; Cai et al., 2018), our novel operators are more generalized previous sorts of published research articles.

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