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Original article

Decomposition of zero divisor graph into cycles and stars

A.Kuppan [a], J.Ravi Sankar [a,*]

[a] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore, Tamil Nadu 632014, India

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ABSTRACT

For a graph G and a subgraph H of G, an H-decomposition of G is a partition of the edge set of G into subsets $E_i, 1 \leqslant i \leqslant k$, such that each E_i induces a graph isomorphic to H. A graph $\Gamma(\mathbb{R})$ is said to be non-zero zero divisor graph of commutative ring $\mathbb{R}$ with identity if $u, v \in V(\Gamma(\mathbb{R}))$ and $(u, v) \in E(\Gamma(\mathbb{R}))$ if and only if $uv = 0$. It is prove that complete decomposible into cycle of length 4 of an H-decomposition of the zero divisor graph $\Gamma(\mathbb{R})$ where H is any simple connected graph. In particular, we give a complete solution to the problem in the case $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times, \ldots, \times \mathbb{Z}_p$ (n times). For any positive integer $n > 2$, there exists a decomposition of $\Gamma(\mathbb{R})$ into cycle and stars in a commutative ring $\mathbb{R}$. We show that the obvious the graph $\Gamma(\mathbb{R})$ is decomposition into cycle and stars. Overall, the proposed of the graph $\Gamma(\mathbb{R})$ has significantly improved the decomposing to algebraic structure which can be useful for networking. In this paper we investigate the concept of $\Gamma(\mathbb{R})$ is decomposition into cycles and stars as a commutative rings $R = \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times, \ldots, \times \mathbb{Z}_p$ with p is a prime number. It is prove that the zero divisor graph $\Gamma(\mathbb{R})$ is complete decomposible into cycle of length 4 and star. In particular, we give a complete solution to the problem in the case $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times, \ldots, \times \mathbb{Z}_p$ (n times). For any positive integer $n > 2$, there exists a decomposition of $\Gamma(\mathbb{R})$ into cycle and stars in a commutative ring $\mathbb{R}$. We show that the obvious the graph $\Gamma(\mathbb{R})$ is decomposition into cycle and stars. Overall, the proposed of the graph $\Gamma(\mathbb{R})$ has significantly improved the decomposing to algebraic structure which can be useful for networking area.

1. Introduction

Throughout this paper, let us take only simple, finite, non-directed graphs. For further graph theory terminology in general, refer (Bondy and Murty, 1976). Let us consider the complete graph on $p-1$ vertices for the zero divisor graph as $\Gamma(Z_{p^2}) \cong K_{p-1}$ and also consider the complete bipartite graph on $p-1$ and $q-1$ vertices for the zero divisor graph $\Gamma(Z_{pq}) \cong K_{p-1,q-1}$. If R is the commutative ring and $\mathbb{R} = \mathbb{Z}_p \times \mathbb{Z}_p$ then the zero divisor graph $\Gamma(\mathbb{R})$ is said to be balanced complete bipartite graph with $2(p-1)$ vertices. Here C_k represents the cycle with vertices $v_0, v_1, \ldots, v_{k-1}$ and edges $v_0v_1, v_1v_2, \ldots, v_{k-2}v_{k-1}, v_{k-1}v_0$ as $(v_0, v_1, v_2, \ldots, v_{k-1}, v_0)$. Also S_k represents the star graph with a centre vertex v_0 and k end vertices $v_1, v_2, \ldots, v_k$ as $(v_0; v_1, v_2, \ldots, v_k)$. The edges can be partitioned into $E_1, E_2, \ldots, E_k$ such that subgraph induced by $E_i\ \forall i$, where i lies between 1 and k. Let the graph $\Gamma(\mathbb{R})$ is the decomposition of $D_1, D_2, \ldots, D_k$ and we write $\Gamma(\mathbb{R}) = D_1 \oplus D_2 \oplus, \ldots, \oplus D_k$. If $D_i \cong D$, then $\Gamma(\mathbb{R})$ is known as D-decomposition and it is denoted by $D|\Gamma$ where $1 \leqslant i \leqslant k, \forall i$. If the zero divisor graph $\Gamma(\mathbb{R})$ can be decomposed into a times of D_1 and b times of D_2, then we can say $\Gamma(\mathbb{R})$ as $\{aD_1, bD_2\}$-decomposition or (D_1, D_2)-multidecomposition. If this decomposition is true for every a and b with necessary conditions then $\Gamma(\mathbb{R})$ has $\{D_1, D_2\}_{\{a,b\}}$-decomposition or complete (D_1, D_2)-decomposition (Shyu, 2010, 2012, 2013).

The multidecomposition (L_1, L_2) was introduced by Abueida and Daven (2003) and they proved the existence of (H_1, H_2)-multidecomposition of $K_m(\lambda)$ when $(H_1, H_2) = (K_{1,n-1}, C_n)$, where n = 3,4 and 5 (Abueida and Daven, 2004, 2007, 2000). Priyadharsini and Muthusamy (2009) showed the necessary and sufficient conditions for λK_n when (H_1, H_2) exists, where $H_1, H_2 \in \{C_n, P_n, S_{n-1}\}$. Lee (2013) discussed the necessary and sufficient conditions for the multidecomposition of $K_{m,n}$ into mini-

* Corresponding author.
E-mail addresses: kuppanmyname@gmail.com (A.Kuppan), ravisankar.j@vit.ac.in (J. Sankar).

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mum one copy of C_k and S_k. Also the necessary and sufficient conditions for the existence of decomposition of product graphs into paths and cycles with 4-edges is derived by Jeevadoss and Muthusamy (2016). Moreover Ilayaraja et.al, enterded their results for the decomposition of product of graphs into paths and stars on 5-vertices. Many other results on decomposition of zero divisor graphs into distinct subgraphs involving cycles, complete and stars have been proved in Alspach and Marshall (1994), Bryant and Maenhaut (2004) and Huang (2015).

The zero divisor graph concept was initiated by I.Beck in 1988 (Beck, 1998) and he considered zero for constructing zero divisor graphs. Few years later Anderson and Livingston, 1999 rearranged I.Beck's definition by removing zero from his vertex set while constructing his graph. For further algebraic graph theory terminology in Kuppan and Ravi Sankar (2020)].

2. Decomposition of $(\mathbb{Z}_p \times \mathbb{Z}_p)$

Proposition 2.1. If $\Gamma(\mathbb{R})$ is a square graph, then the commutative ring $R = \mathbb{Z}_3 \times \mathbb{Z}_3$.

Proof. The vertices $V = \{(0,1),(0,2),(1,0),(2,0)\}$ are non-zero zero divisors. Therefore $\Gamma(\mathbb{R})$ is

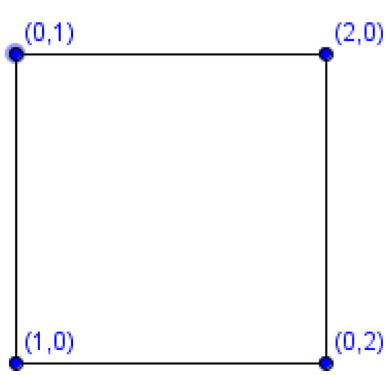

Fig. 1. $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$.

Theorem 2.2. The graph $\Gamma(\mathbb{R})$ is a balanced complete bipartite graph iff p is any prime and $\mathbb{R} = \mathbb{Z}_p \times \mathbb{Z}_p$.

Proof. Suppose $\Gamma(\mathbb{R})$ be the balanced complete bipartite graph, then the vertex set of $V(\Gamma(\mathbb{R})) = \{(u_1,0),(0,u_2)|u_1,u_2 \in \{1,2,3,\ldots,p-1\}\}$. Now consider the vertex subsets as $V_1 = \{(u_1,0)|u_1 \in \{1,2,3,\ldots,p-1\}\}$ and $V_2 = \{(0,u_2)|u_2 \in \{1,2,3,\ldots,p-1\}\}$. Let $0 \neq u \in V_1$ or V_2 with $u^2 \neq 0$ then no two vertices in V_1 or V_2 is non-adjacent. For any two vertices $u = (u_1,0) \in V_1$ and $v = (0,u_2) \in V_2$ such that $uv = 0$ we say that the edges from each vertex in V_1 to every vertices in V_2. Hence $\mathbb{R} = \mathbb{Z}_p \times \mathbb{Z}_p$. Conversely, consider the ring $\mathbb{Z}_p \times \mathbb{Z}_p$ with vertex set is non-zero zero divisors $V = \{(0,u_2),\ (u_1,0)|u_1,u_2 \in \{1,2,3,\ldots,p-1\}\}$. For all $u \in V_1, v \in V_2$ with $u^2 \neq 0$ and $v^2 \neq 0$. Suppose $uv = 0$ there exists an edge from u to v. Clearly, the commutative ring $\mathbb{R} = \mathbb{Z}_p \times \mathbb{Z}_p$ is a graph of $\Gamma(\mathbb{R})$ and it is a balanced complete bipartite graph.

Theorem 2.3. Let p be any odd prime then the graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$ is decomposition into $\frac{(p-1)^2}{4}$ copies of C_4.

Proof. The vertex set of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p) = \{(u_1,0),(0,u_2)\}$ such that u_1,u_2 lies between 1 and $p-1$. That is $|V(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p))| = 2(p-1)$. Let V_1 and V_2 be the partition of vertex subsets where $V_1 = \{(u_1,0)|u_1 \in \{0,1,2,\ldots,p-1\}\}$ and $V_2 = \{(0,u_2)|u_2 \in \{0,1,2,\ldots,p-1\}\}$. That is $|V_1| = p-1$ and $|V_2| = p-1$. Let any two vertices $u,v \in V_1$ or V_2 then $uv \neq 0$. Clearly u and v are disconnected.

Let us consider the any one member $u = (u_1,0)$ in V_1 and another member $v = (0,u_2)$ in V_2 then $(u_1,0).(0,u_2) = (0,0)$. Clearly u and v are connected.

Therefore, each vertex of V_1 is connected to every vertices in V_2. That is $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$ is balanced complete bipartite graph in the form of $K_{p-1,p-1}$.Clearly, $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$ can be decomposed into $\frac{(p-1)^2}{4}$ copies of C_4. That is

$$((0,1),(1,0),(0,p-1),(p-1,0),(0,1)),((0,1),(2,0),(0,p-1),(p-2,0),(0,1)),\ldots,\left((0,1),(\lfloor\tfrac{p}{2}\rfloor,0),(0,p-1),(\lfloor\tfrac{p}{2}\rfloor,0),(0,1)\right),$$

$$((0,2),(1,0),(0,p-2),(p-1,0),(0,2)),((0,2),(2,0),(0,p-2),(p-2,0),(0,2)),\ldots,\left((0,2),(\lfloor\tfrac{p}{2}\rfloor,0),(0,p-2),(\lfloor\tfrac{p}{2}\rfloor,0),(0,2)\right),$$

$$((0,3),(1,0),(0,p-3),(p-1,0),(0,3)),((0,3),(2,0),(0,p-3),(p-2,0),(0,3)),\ldots,\left((0,3),(\lfloor\tfrac{p}{2}\rfloor,0),(0,p-3),(\lfloor\tfrac{p}{2}\rfloor,0),(0,3)\right)$$

$$\vdots$$

$$\left((0,\lfloor\tfrac{p}{2}\rfloor),(\lfloor\tfrac{p}{2}\rfloor,0),(0,\lfloor\tfrac{p}{2}\rfloor+1),(\lfloor\tfrac{p}{2}\rfloor+1,0),(0,\lfloor\tfrac{p}{2}\rfloor)\right)$$

total values of each cycle is $\frac{(p-1)^2}{4}$. Therefore, $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$ can be decomposed into $\frac{(p-1)^2}{4}$ copies of C_4.

Example 1. Let us consider $p = 11$ and the vertex set of $\Gamma(\mathbb{Z}_{11} \times \mathbb{Z}_{11})$ as

$V = \{(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(0,7),(0,8),(0,9),(0,10),(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(7,0),(8,0),(9,0),(10,0)\}$.

Then $V(\Gamma(\mathbb{Z}_{11} \times \mathbb{Z}_{11}))$ can be split into two parts, namely V_1 and V_2.

$V_1 = \{(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(0,7),(0,8),(0,9),(0,10)\}$ and.

$V_2 = \{(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(7,0),(8,0),(9,0),(10,0)\}$.

Clearly, $\Gamma(\mathbb{Z}_{11} \times \mathbb{Z}_{11})$ is isomorphic to $K_{10,10}$. The Figs. 1,2 indicates the decomposition of the graph $K_{10,10}$ into 25 copies of C_4.

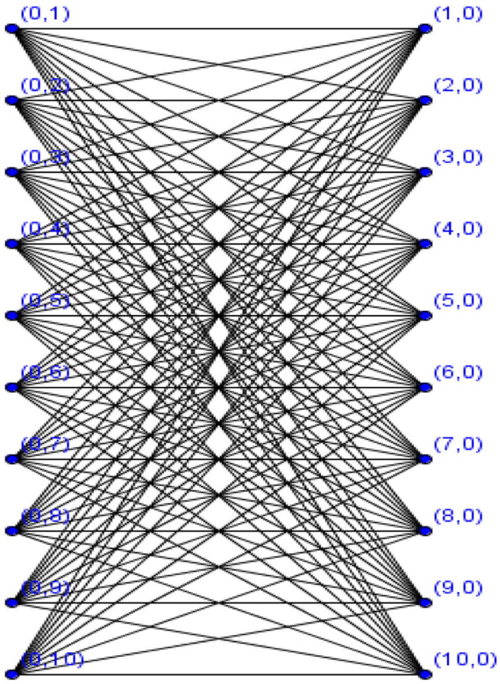

Fig. 2. $\Gamma(\mathbb{Z}_{11} \times \mathbb{Z}_{11})$.

Theorem 2.4. Let p be odd prime and a, b are the non-negative integers. If there exists (C_4, S_4)-decomposition of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$, then $(p-1)^2 \equiv 0 \ (mod\ 4)$.

Theorem 2.5. Let p be odd prime, the graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$ is S_4-decomposable. Then

If $p - 1 \equiv 0 \ (mod\ 4)$ then S_4 is decomposable

If $p - 1 \not\equiv 0 \ (mod\ 4)$ then S_4 is non – decomposable

Proof. We shall prove this theorem by using the following two cases.

Case(i): Let $p - 1 \equiv 0 \ (mod\ 4)$. Assume that $p - 1$ is divided by 4. Let us consider the vertex set of non-zero zero-divisors $V(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)) = \{(0,x),(x,0) | x \in \{0,1,2,\ldots,p-1\}\}$. Here $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$ be the zero divisor graph of a ring $\mathbb{Z}_p \times \mathbb{Z}_p$. If there is a member $0 \neq (0,x) \in V$ then $(0,x),(0,x) \neq 0$ such that the two zero divisors are non-adjacent. If there is a member $0 \neq (0,x) \in V$ and $0 \neq (x,0) \in V$ then, $(0,x),(x,0) = 0 \in V$ such that two zero divisors are adjacent. That is $|V(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p))| = 2(p-1)$. Clearly, the graph completely decomposed star with size 4.**Case(ii):** Let $p - 1 \not\equiv 0 \ (mod\ 4)$. Assume that $p - 1$ is not divided by 4. If there is a member $0 \neq (0,x) \in V$ then $(0,x),(0,x) \neq 0$ such that two zero divisors are non-adjacent. Clearly, the vertex $(0,x) \in V$ is not completely decomposed into the star graph of size 4.

Lemma 2.6. If a and b are any two positive integers, then there exists a complete $\{C_4, S_4\}$-decomposition of $\Gamma(\mathbb{Z}_5 \times \mathbb{Z}_5)$.

Proof. Let $V(\Gamma(\mathbb{Z}_5 \times \mathbb{Z}_5)) = \{(0,1),(0,2),(0,3),(0,4),(1,0),(2,0),(3,0),(4,0)\}$. Then the required entire decomposition of $\{C_4, S_4\}$ are

1. $a = 4$ and $b = 0$. The required cycles are
$((0,1),(1,0),(0,4),(4,0),(0,1)), ((0,1),(2,0),(0,4),(3,0),(0,1)),$
$((0,2),(1,0),(0,3),(4,0),(0,2)), ((0,2),(2,0),(0,3),(3,0),(0,2))$.
2. $a = 2$ and $b = 2$. The required cycles and stars are
$((0,1),(1,0),(0,4),(4,0),(0,1)), ((0,1),(2,0),(0,4),(3,0),(0,1)),$
$((0,2),(1,0),(2,0),(3,0),(4,0)), ((0,3),(1,0),(2,0),(3,0),(4,0))$.
3. $a = 0$ and $b = 4$. The required stars are
$((0,1),(1,0),(2,0),(3,0),(4,0)), ((0,2),(1,0),(2,0),(3,0),(4,0)),$
$((0,3),(1,0),(2,0),(3,0),(4,0)), ((0,4),(1,0),(2,0),(3,0),(4,0))$.

Lemma 2.7. If a and b are any two positive integers, then there exists a complete $\{C_4, S_6\}$-decomposition of $\Gamma(\mathbb{Z}_7 \times \mathbb{Z}_7)$.

Proof. Let $V(\Gamma(\mathbb{Z}_7 \times \mathbb{Z}_7)) = \{(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(0,1),(0,2),(0,3),(0,4),(0,5),(0,6)\}$. Then the required entire decomposition of $\{C_4, S_6\}$ are

1. $a = 9$ and $b = 0$ the required cycles are
$((0,1),(1,0),(0,6),(6,0),(0,1)), ((0,1),(2,0),(0,6),(5,0),$
$(0,1)), ((0,1),(3,0),(0,6),(4,0),(0,1)),$
$((0,2),(1,0),(0,5),(6,0),(0,2)), ((0,2),(2,0),(0,5),(5,0),$
$(0,2)), ((0,2),(3,0),(0,5),(4,0),(0,2)),$
$((0,3),(1,0),(0,4),(6,0),(0,3)), ((0,3),(2,0),(0,4),(5,0),$
$(0,3)), ((0,3),(3,0),(0,4),(4,0),(0,3))$.
2. $a = 6$ and $b = 2$ the required cycles and stars are
$((0,1),(1,0),(0,6),(6,0),(0,1)), ((0,1),(2,0),(0,6),(5,0),$
$(0,1)), ((0,1),(3,0),(0,6),(4,0),(0,1)),$
$((0,2),(1,0),(0,5),(6,0),(0,2)), ((0,2),(2,0),(0,5),(5,0),$
$(0,2)), ((0,2),(3,0),(0,5),(4,0),(0,2)),$
$((0,3);(1,0),(2,0),(3,0),(4),(5,0),(6,0)),$
$((0,4);(1,0),(2,0),(3,0),(4),(5,0),(6,0))$.
3. $a = 3$ and $b = 4$ the required cycles and stars are
$((0,1),(1,0),(0,6),(6,0),(0,1)), ((0,1),(2,0),(0,6),(5,0),$
$(0,1)), ((0,1),(3,0),(0,6),(4,0),(0,1)),$
$((0,2);(1,0),(2,0),(3,0),(4),(5,0),(6,0)),$
$((0,3);(1,0),(2,0),(3,0),(4),(5,0),(6,0)),$
$((0,4);(1,0),(2,0),(3,0),(4),(5,0),(6,0)),$
$((0,5);(1,0),(2,0),(3,0),(4),(5,0),(6,0))$.
4. $a = 0$ and $b = 6$ the required cycles and stars are
$((0,1);(1,0),(2,0),(3,0),(4),(5,0),(6,0)),$
$((0,2);(1,0),(2,0),(3,0),(4),(5,0),(6,0)),$
$((0,3);(1,0),(2,0),(3,0),(4),(5,0),(6,0)),$
$((0,4);(1,0),(2,0),(3,0),(4),(5,0),(6,0)),$
$((0,5);(1,0),(2,0),(3,0),(4),(5,0),(6,0)),$
$((0,6);(1,0),(2,0),(3,0),(4),(5,0),(6,0))$.

Theorem 2.8. Let a and b be any two positive integers and p is any prime number, then there exists a complete (C_4, S_4)-decomposition of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$ for all $p \geqslant 5$.

Proof. If $p = 5, 7$, then the result follows by Lemmas 2.6 and 2.7. For $p > 7$, we write, $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p) = \{(u_i, 0), (0, u_i)\}$ where u_i from 1 to $p - 1$. Let $p - 1 \equiv 0 (mod\ 4)$. By Lemma 2.6 and 2.7, the graphs $K_{4,4}, K_{6,6}$ have a complete $\{C_4, S_4\}$-decomposition. Clearly, the graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$ has the desired decomposition.

3. Decomposition of $(\boldsymbol{Z_p} \times \boldsymbol{Z_p} \times \boldsymbol{Z_p})$

Theorem 3.1. For any odd prime p then $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ is decomposition into 3 copies of $K_{(p-1)^2, p-1}$ and 3 copies of $K_{p-1, p-1}$.

Proof. Let $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ be the zero divisor graph of a commutative ring $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. The vertex set $V = \{(a,0,0),(0,b,0),(0,0,c),(a,b,0), (a,0,c),(0,b,c)|a,b,c \in \{1,2,3,\ldots,p-1\}\}$ where p is any odd prime. The vertex set can be partitioned into several disjoint subsets $V_1 = \{(a,0,0)|a \in \{1,2,3,\ldots,p-1\}\}$, $V_2 = \{(0,b,0)|b \in \{1,2,3,\ldots,p-1\}\}$, $V_3 = \{(0,0,c)|c \in \{1,2,3,\ldots,p-1\}\}, V_4 = \{(a,b,0)|a,b \in \{1,2,3,\ldots,p-1\}\}$, $V_5 = \{(a,0,c)|a,c \in \{1,2,3,\ldots,p-1\}\}$ and $V_6 = \{(0,b,c)|b,c \in \{1,2,3,\ldots,p-1\}\}$. That is $|V_1| = |V_2| = |V_3| = p-1$ and $|V_4| = |V_5| = |V_6| = (p-1)^2$.

Case(i): Consider the vertex subsets V_1, V_2 and V_3. If $u \in V_1$ with $u^2 \neq 0$ then the vertices are non-adjacent. If $v \in V_2$ with $uv = 0$ then there is an edge between every pair of vertices. Clearly, the above subsets are adjacent to each other. By continuing the same process we can prove the same with (V_2, V_3) and (V_1, V_3). Hence, 3 copies of $K_{p-1,p-1}$.

Case(ii): Consider the vertex subsets V_4, V_5 and V_6. If $u \in V_4$ with $u^2 \neq 0$ then the vertices are non-adjacent. If $v \in V_3$ with $uv = 0$ then there is an edge between every pair of vertices. Clearly, the above subsets are adjacent to each other. Continuing the same process we can prove the same with (V_1, V_6) and (V_2, V_5). Hence, 3 copies of $K_{p-1,(p-1)^2}$. Therefore, from the above two cases, indicate the graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ is decomposition into 3 copies of $K_{p-1,p-1}$ and 3 copies of $K_{p-1,(p-1)^2}$.

Theorem 3.2. For any odd prime p, then $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ is decomposed into $\frac{3}{4}((p-1)^3 + (p-1)^2)$ copies of C_4.

Proof. If p is an odd prime, then the vertices are non-zero zero divisors $V(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)) = \{(a_i,0,0),(0,b_i,0),(0,0,c_i),(a_i,b_i,0),(a_i,0,c_i),(0,b_i,c_i)\}$ where $1 \leqslant a_i,b_i,c_i \leqslant p-1$. That is $|V(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p))| = 3(p-1) + 3(p-1)^2$. Let us take the vertex subsets are $V_1 = \{(a_i,0,0)|1 \leqslant a_i \leqslant p-1\}, V_2 = \{(0,b_i,0)|1 \leqslant b_i \leqslant p-1\}$, $V_3 = \{(0,0,c_i)|1 \leqslant c_i \leqslant p-1\}, V_4 = \{(a_i,b_i,0)|1 \leqslant a_i,b_i \leqslant p-1\}, V_5 = \{(a_i,0,c_i)|1 \leqslant a_i,c_i \leqslant p-1\}$ and $V_6 = \{(0,b_i,c_i)|1 \leqslant b_i,c_i \leqslant p-1\}$. That is $|V_1| = |V_2| = |V_3| = p-1$ and $|V_4| = |V_5| = |V_6| = (p-1)^2$. Let us prove this by the following two cases.

Case(i): Let us take the pairs of vertex subsets $(V_1, V_6), (V_2, V_5)$ and (V_3, V_4). Here $(V_1, V_6) \cong K_{p-1,(p-1)^2}$. Similarly we say that $(V_2, V_5) \cong K_{p-1,(p-1)^2}$ and $(V_3, V_4) \cong K_{p-1,(p-1)^2}$.

Case(ii): Let us take the vertex subsets are V_1, V_2 and V_3. Then every pair of above vertex subsets are isomorphic to tri-partite graph of $K_{p-1,p-1,p-1}$.

$$
\begin{aligned}
&D\left(K^{V_1,V_2}_{p-1,p-1}\right) = \\
&\left((a_1,0,0),(0,b_1,0),(a_{p-1},0,0),(0,b_{p-1},0),(a_1,0,0)\right),\\
&\left((a_1,0,0),(0,b_2,0),(a_{p-1},0,0),(0,b_{p-2},0),(a_1,0,0)\right),\\
&\left((a_1,0,0),(0,b_3,0),(a_{p-1},0,0),(0,b_{p-3},0),(a_1,0,0)\right),\\
&\left((a_2,0,0),(0,b_1,0),(a_{p-2},0,0),(0,b_{p-1},0),(a_2,0,0)\right),\\
&\left((a_2,0,0),(0,b_2,0),(a_{p-2},0,0),(0,b_{p-2},0),(a_2,0,0)\right),\\
&\left((a_2,0,0),(0,b_3,0),(a_{p-2},0,0),(0,b_{p-3},0),(a_2,0,0)\right),\\
&\left((a_{\frac{p-1}{2}},0,0),(0,b_{\frac{p-1}{2}},0),(a_{\frac{p-1}{2}+1},0,0),(0,b_{\frac{p-1}{2}+1},0),(a_{\frac{p-1}{2}},0,0)\right).
\end{aligned}
$$

On the whole there are $(\frac{p-1}{2})^2$ copies of C_4. If we continue the same process we get $(\frac{p-1}{2})^2$ copies for $D(K^{V_1,V_3}_{p-1,p-1})$ and $D(K^{V_2,V_3}_{p-1,p-1})$. Clearly, from the above cases we get $\frac{3}{4}((p-1)^3 + (p-1)^2)$ copies of C_4. Therefore, $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ can be decomposed into $\frac{3}{4}((p-1)^3 + (p-1)^2)$ copies of C_4.

Theorem 3.3. Let p be any odd prime, then $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ is decomposed into $(p-1)^2$ copies of S_{p-1} and $\frac{3(p-1)^2}{4}$ copies of C_4.

Proof. Consider the vertex set of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ is $\{(x,0,0),(0,y,0),(0,0,z),(0,y,z),(x,0,z),(x,y,0)|x,y,z \in \{1,2,3,\ldots,p-1\}\}$. Let us consider the following subsets V_1, V_2, V_3, V_4, V_5 and V_6 are in $V(\Gamma(Z_p \times Z_p \times Z_p))$.

Case(i): Let us take the vertex subsets V_1, V_2 and V_3. Then every pair of subsets are isomorphic to $K_{p-1,p-1}$. Cardinality of above all subsets is $p-1$. If $x \in V_1$ or V_2 or V_3 and $x^2 \neq 0$ then they are non adjacent to each other. If every distinct pair of vertices are connected then there exists a tripartite graph. If the size of the pair (V_1, V_2) is $(p-1)^2$ then the decomposition of the graph $K_{p-1,p-1}$ yields $\frac{(p-1)^2}{4}$ copies. Similarly we can say that, if the sizes of (V_2, V_3) and (V_1, V_3) is $(p-1)^2$ then the decomposition of the graph $K_{p-1,p-1}$ has $\frac{(p-1)^2}{4}$ copies.**Case(ii):** Let us take the pairs of vertex subsets $(V_1, V_4), (V_2, V_5)$ and (V_3, V_6). Then every pair of subsets are isomorphic to $K_{p-1,(p-1)^2}$. Cardinality of above all pairs of subsets is $(p-1) + (p-1)^2$. If $x \in V$ and $x^2 \neq 0$ then they are non adjacent to each other. If the size of the pair (V_1, V_6) is $(p-1)^3$ then the decomposition of the graph $K_{p-1,(p-1)^2}$ yields $\frac{(p-1)^3}{4}$ copies. Similarly we can say that, if the sizes of (V_2, V_5) and (V_3, V_4) is $(p-1)^3$ then the decomposition of the graph $K_{p-1,(p-1)^2}$ has $\frac{(p-1)^3}{4}$ copies. Clearly, from the above cases, we get $(p-1)^2$ copies of S_{p-1} and $\frac{3(p-1)^2}{4}$ copies of C_4.

Theorem 3.4. Let p be any odd prime, then $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ can be decomposed into $\binom{p-1}{2}$ copies of $C_6, p-1$ copies of C_3 and $\frac{3(p-1)^3}{4}$ copies of C_4.

Proof. Let the vertex set of non-zero zero divisors is $V(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)) = \{(a_i,0,0),(0,b_i,0),(0,0,c_i),(a_i,b_i,0),(a_i,0,c_i),(0,b_i,c_i)\}$ where $1 \leqslant a_i,b_i,c_i \leqslant p-1$. That is $|V(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p))| = 3(p-1) + 3(p-1)^2$. The partition of vertex subsets are $V_1 = \{(a_i,0,0)|1 \leqslant a_i \leqslant p-1\} = \{u_1,u_2,u_3,\ldots,u_{p-1}\}, V_2 = \{(0,b_i,0)|1 \leqslant b_i \leqslant p-1\} = \{v_1,v_2,v_3,\ldots,v_{p-1}\}, V_3 = \{(0,0,c_i)|1 \leqslant c_i \leqslant p-1\} = \{w_1,w_2,w_3,\ldots,w_{p-1}\}, V_4 = \{(a_i,b_i,0)|1 \leqslant a_i,b_i \leqslant p-1\}, V_5 = \{(a_i,0,c_i)|1 \leqslant a_i,c_i \leqslant p-1\}$ and $V_6 = \{(0,b_i,c_i)|1 \leqslant b_i,c_i \leqslant p-1\}$. That is $|V_1| = |V_2| = |V_3| = p-1$ and $|V_4| = |V_5| = |V_6| = (p-1)^2$. We can prove this proof by the following cases.

Case(i): Consider the vertex subsets V_1, V_2 and V_3. If we decompose the tripartite graph then we get $\binom{p-1}{2}$ copies of C_6 and $p-1$ copies of C_3. Which are written as follows.

$$
\begin{aligned}
&(u_1,v_2,w_1,u_2,v_1,w_2,u_1),(u_1,v_3,w_1,u_3,v_1,w_3,u_1),\\
&\ldots,(u_1,v_{p-2},w_1,u_{p-2},v_1,w_{p-2},u_1).\\
&(u_2,v_3,w_2,u_3,v_2,w_3,u_2),(u_2,v_4,w_2,u_4,v_2,w_4,u_2),\\
&\ldots,(u_2,v_{p-2},w_2,u_{p-2},v_2,w_{p-2},u_2).\\
&\vdots\\
&(u_{p-3},v_{p-2},w_{p-3},u_{p-2},v_{p-3},w_{p-2},u_{p-3}).
\end{aligned}
$$

If we remove $\binom{p-1}{2}$ copies of C_6 then the resultant graph will have $(u_1,v_1,w_1),(u_2,v_2,w_2)$ and (u_3,v_3,w_3) copies of C_3 edges only. Here the degree of each vertex is of even degree.

Case(ii): Consider the following vertex subsets V_4, V_5 and V_6. That is $|V_4| = |V_5| = |V_6| = (p-1)^2$. By Theorem 3.3 we say that $\frac{3(p-1)^3}{4}$ copies of C_4. Therefore, from the above two cases, we get $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ is decomposition into $\binom{p-1}{2}$ copies of $C_6, p-1$ copies of C_3 and $\frac{3(p-1)^3}{4}$ copies of C_4.

4. Decomposition of $(Z_p \times Z_p \times Z_p \times Z_p)$

Theorem 4.1. If p is any odd prime, then $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ is decomposed into 3-copies of $K_{(p-1)^2,(p-1)^2}$, 4-copies of $K_{(p-1)^3,(p-1)}$, 6-copies of $K_{(p-1),(p-1)}$ and 12-copies of $K_{(p-1)^2,(p-1)}$.

Proof. Let the vertex set $V(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p))$ is $V = \{(k,0,0,0),(0,l,0,0),(0,0,m,0),(0,0,0,n),(k,l,0,0),(k,0,m,0),(k,0,0,n),(0,l,m,0),(0,l,0,n),(0,0,m,n),(k,l,m,0),(k,0,m,n),(k,l,0,n),(0,l,m,n)|k,l,m,n \in \{1,2,3,\ldots,p-1\}\}$. The vertex set can be split into disjoint subsets, such as $A_i, B_i, C_i \in V$, where $A_1 = \{(k,0,0,0)|1 \leqslant k \leqslant p-1\}, A_2 = \{(0,l,0,0)|1 \leqslant l \leqslant p-1\}$, $A_3=\{(0,0,m,0)|1\leqslant m\leqslant p-1\}, A_4=\{(0,0,0,n)|1\leqslant n\leqslant p-1\}, B_1=\{(k,l,0,0)|1 \leqslant k,l \leqslant p-1\}, B_2 = \{(k,0,m,0)|1 \leqslant k,m \leqslant p-1\}, B_3 = \{(k,0,0,n)|1\leqslant k,n\leqslant p-1\}, B_4=\{(0,l,m,0)|1\leqslant l,m\leqslant p-1\}, B_5=\{(0,l,0,n)|1\leqslant l,n\leqslant p-1\}, B_6=\{(0,0,m,n)|1\leqslant m,n\leqslant p-1\}, C_1=\{(k,l,m,0)|1\leqslant k,l,m\leqslant p-1\}$, $C_2=\{(k,0,m,n)|1\leqslant k,m,n\leqslant p-1\}, C_3=\{(k,l,0,n)|1\leqslant k,l,n\leqslant p-1\}, C_4=\{(0,l,m,n)|1\leqslant l,m,n\leqslant p-1\}$. That is $|\cup_{i=1}^{p-1C_1} A_i|=4(p-1), |\cup_{i=1}^{p-1C_2} B_i|=6(p-1)^2$ and $|\cup_{i=1}^{p-1C_3} C_i|=4(p-1)^3$.

Case(i): Consider the vertex subsets A_i and C_i. If $u \in C_i$ with $u^2 \neq 0$ then the vertices are non-adjacent. If $v \in A_i$ with $uv = 0$ then there is an edge between every pairs of vertices. Clearly, the above subsets are adjacent to each other. Clearly, we get 4 copies of $K_{p-1,(p-1)^2}$.

Case(ii): Consider the vertex subsets A_i and B_i. If $u \in B_i$ with $u^2 \neq 0$ then the vertices are non-adjacent. If $v \in A_i$ with $uv = 0$ then every pair of vertices are connected. Clearly, the above subsets (A_i, B_i) are isomorphic to $K_{p-1,(p-1)^2}$. Therefore, we get 4 copies of $K_{(p-1)^3,p-1}$.

Case(iii): Consider the vertex subset of B_i. If $u \in B_i$ with $u^2 = 0$ then the vertices are adjacent. Clearly, every pair of B_i is adjacent to itself. Therefore, we get 3 copies of $K_{(p-1)^2,(p-1)^2}$.

Case(iv): Consider the vertex subset A_i. If $u \in A_i$ with $u^2 = 0$ then the vertices are adjacent. Clearly, the every pair of A_i is adjacent to itself. Therefore, we get 6 copies of $K_{(p-1),(p-1)}$.

From the above four cases we say that $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ is decomposed into 3-copies of $K_{(p-1)^2,(p-1)^2}$, 4-copies of $K_{(p-1)^3,(p-1)}$, 6-copies of $K_{(p-1),(p-1)}$ and 12-copies of $K_{(p-1)^2,(p-1)}$.

Theorem 4.2. If p is any odd prime, then $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ is decomposed into $p-1$ copies of K_4 and $\frac{7(p-1)^4+12(p-1)^3+6(p-1)^2}{4}$ copies of C_4.

Proof. The vertex set of $V(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)) = \{(k,0,0,0),(0,l,0,0),(0,0,m,0),(0,0,0,n),(k,l,0,0),(k,0,m,0),(k,0,0,n),(0,l,m,0),(0,l,0,n),(0,0,m,n),(k,l,m,0),(k,0,m,d),(k,l,0,d),(0,l,m,n)|k,l,m,n \in \{1,2,3,\ldots,p-1\}\}$. The vertex set can be split into disjoint subsets, such as $A_i, B_i, C_i \in V$, where $A_1 = \{(k,0,0,0)|1 \leqslant k \leqslant p-1\}, A_2 = \{(0,l,0,0)|1 \leqslant l \leqslant p-1\}$, $A_3 = \{(0,0,m,0)|1 \leqslant m \leqslant p-1\}, A_4 = \{(0,0,0,n)|1 \leqslant n \leqslant p-1\}, B_1 = \{(k,l,0,0)|1 \leqslant k,l \leqslant p-1\}$, $B_2 = \{(k,0,m,0)|1 \leqslant k,m \leqslant p-1\}, B_3 = \{(k,0,0,n)|1 \leqslant k,n \leqslant p-1\}$, $B_4 = \{(0,l,m,0)|1 \leqslant l,m \leqslant p-1\}, B_5 = \{(0,l,0,n)|1 \leqslant l,n \leqslant p-1\}$, $B_6 = \{(0,0,m,n)|1 \leqslant m,n \leqslant p-1\}, C_1 = \{(k,l,m,0)|1 \leqslant k,l,m \leqslant p-1\}, C_2 = \{(k,0,m,n)|1 \leqslant k,m,n \leqslant p-1\}, C_3 = \{(k,l,0,n)|1 \leqslant k,l,n \leqslant p-1\}, C_4 = \{(0,l,m,n)|1 \leqslant l,\ m,n \leqslant p-1\}$. That is $|\cup_{i=1}^{p-1C_1} A_i| = 4(p-1), |\cup_{i=1}^{p-1C_2} B_i| = 6(p-1)^2$ and $|\cup_{i=1}^{p-1C_3} C_i| = 4(p-1)^3$.

Case(i): Let us consider the vertex subsets.

$$A_i = \{(k,0,0,0),(0,l,0,0),(0,0,m,0),(0,0,0,n)|k,l,m,n \in \{1,2,3,\ldots,p-1\}\}.$$

If $u, v \in A_i$ with $uv = 0$ then there is an edge between every pair of vertices. Clearly, we get 4 copies of complete graph K_{p-1}.

Case(ii): Let us consider the pair of subsets $(A_i, C_i), (A_i, B_i)$ and (B_i, B_i)(using Theorem 4.1). The resultant of decomposition of (A_i, C_i) is isomorphic to $K_{p-1,(p-1)^3}$. Similarly the resultant of decomposition of (A_i, B_i) and (B_i, B_i) is isomorphic to $K_{p-1,(p-1)^2}$ and $K_{(p-1)^2,(p-1)^2}$. Then we get cycles of length 4 and described as follows.

$$D(4K_{p-1,(p-1)^3}) = \frac{4(p-1)(p-1)^3}{4}$$
$$D(12K_{p-1,(p-1)^2}) = \frac{12(p-1)(p-1)^2}{4}$$
$$D(6K_{p-1,p-1}) = \frac{6(p-1)(p-1)}{4}$$
$$D(3K_{(p-1)^2,(p-1)^2}) = \frac{3(p-1)^2(p-1)^2}{4}$$

Sum of above decomposition $= \frac{4(p-1)(p-1)^3}{4} + \frac{12(p-1)(p-1)^2}{4} + \frac{6(p-1)(p-1)}{4} + \frac{3(p-1)^2(p-1)^2}{4}$.

Therefore, from the above two cases we get $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ is decomposed into $p-1$ copies of K_4 and $\frac{7(p-1)^4+12(p-1)^3+6(p-1)^2}{4}$ copies of C_4.

5. Decomposition of $(Z_p \times Z_p \times Z_p \times \ldots \times Z_p)$

Theorem 5.1. If p is odd prime and m is any non-negative integer, then $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots \times \mathbb{Z}_p)$ has $m-1$ partitions.

Proof. The vertex set of

$$\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots \times \mathbb{Z}_p) = \{((u_1,0,0,\ldots,0),(0,u_2,0,\ldots,0),(0,0,u_3,\ldots,0),\ldots,(0,0,0,\ldots,u_m)), ((u_1,u_2,0,\ldots,0),(u_1,0,u_3,\ldots,0),\ldots,(\text{upto } (p-1)\tbinom{m}{2})), ((u_1,u_2,u_3,\ldots,0),\ldots,(\text{upto } (p-1)\tbinom{m}{3})), \ldots,((u_1,u_2,u_3,\ldots,u_{m-1},0),\ldots,(\text{upto } (p-1)\tbinom{m}{m-1}))\}.$$

The vertex set can be split into.

$V_1 = \{(u_1,0,0,\ldots,0),(0,u_2,0,\ldots,0),(0,0,u_3,\ldots,0),\ldots,(0,0,0,\ldots,u_m)\}$,

$V_2 = \{(u_1,u_2,0,\ldots,0),(u_1,0,u_3,\ldots,0),\ldots,(\text{upto } (p-1)\tbinom{m}{2})\}$,

$V_3 = \{(u_1,u_2,u_3,\ldots,0),\ldots,(\text{upto } (p-1)\tbinom{m}{3})\}\vdots$

$V_{m-1} = \{(u_1,u_2,u_3,\ldots,u_{m-1},0),\ldots,(\text{upto } (p-1)\tbinom{m}{m-1})\}$where $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots \times \mathbb{Z}_p$(up to m times). That is $|V_1| = (p-1)\binom{m}{1}, |V_2| = (p-1)\binom{m}{2}, |V_3| = (p-1)\binom{m}{3}$, similarly $|V_{m-1}| = (p-1)\binom{m}{m-1}$.

Let the edge set

$$E(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \ldots \times \mathbb{Z}_p)) = \begin{cases} V_i \text{ is adjacent to } V_j; & i+j \leqslant m \\ V_i \text{ is non-adjacent to } V_j; & i+j > n \end{cases}$$

Hence, the result.

It is obtained from the above results, we can find the following theorem.

Theorem 5.2. For any odd prime p, then $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots \times \mathbb{Z}_p)$ is complete decomposible into cycle of length 4.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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