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ORIGINAL ARTICLE

Traveling wave solutions of Whitham–Broer–Kaup equations by homotopy perturbation method

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Abstract The homotopy perturbation method (HPM) is employed to find the explicit and numerical traveling wave solutions of Whitham–Broer–Kaup (WBK) equations, which contain blow-up solutions and periodic solutions. The numerical solutions are calculated in the form of convergence power series with easily computable components. The homotopy perturbation method performs extremely well in terms of accuracy, efficiently, simplicity, stability and reliability.

1. Introduction

Calculating traveling wave solutions of nonlinear equations in mathematical physics plays an important role in soliton theory (Whitham, 1967; Ablowitz and Clarkson, 1991; Cox et al., 1991; Whitham, 1974). Many explicit exact methods have been introduced in literature (Whitham, 1967; Ablowitz and Clarkson, 1991; Cox et al., 1991; Whitham, 1974).

In this study, we consider coupled WBK equations which are introduced by Whitham, Broer and Kaup. The equation describes propagation of shallow water waves with different dispersion relation. The equation of the WBK

$$\begin{aligned} &u_t + uu_x + v_x + \beta u_{xx} = 0, \\ &v_t + (uv)_x + \alpha u_{xxx} - \beta v_{xx} = 0, \end{aligned} \tag{1}$$

where the field of horizontal velocity is represented by $u = u(x,t)$, $v = v(x,t)$ is the height that deviate from equilibrium position of liquid, and α, β are constants which represent different diffusion power. Xie et al. (2001) applied hyperbolic function method and found some new solitary wave solutions for the WBK Eq. (1). Recently (El-Sayed and Kaya, 2005) used Adomian decomposition method for solving the governing problem. System (1) is very good model to describe dispersive wave.

2In this paper, we will focus on finding analytical approximate and exact traveling wave solution of the system (1) using the homotopy perturbation method. The method provides the solutions in the form of a series with easily computable terms. The accuracy and rapid convergence of the solutions are demonstrated through some numerical examples. The homotopy perturbation method (HPM) was first proposed by the Chinese

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mathematician (He, 1999, 2000, 2003). The essential idea of this method is to introduce a homotopy parameter, say p, which takes values from 0 to 1. When $p = 0$, the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As p is gradually increased to 1, the system goes through a sequence of "deformations", the solution for each of which is "close" to that at the previous stage of "deformation". Eventually at $p = 1$, the system takes the original form of the equation and the final stage of "deformation" gives the desired solution. One of the most remarkable features of the HPM is that usually just a few perturbation terms are sufficient for obtaining a reasonably accurate solution. This technique has been employed to solve a large variety of linear and nonlinear problems (Yıldırım, 2008a, in press-a, 2008b, in press-b; Cveticanin, 2006; Achouri and Omrani, in press; Ghanmi et al., in press; Babolian et al., in press; Momani et al., in press; Zhu et al., in press; Inc, in press; Dehghan and Shakeri, 2007; Shakeri and Dehghan, 2008). The interested reader can see the references (He, 2008a,b, 2006a,b) for last development of HPM.

2. Implementation of the method

We first consider the application of the decomposition method to the WBK (Xie et al., 2001; El-Sayed and Kaya, 2005) equation with the initial conditions.

$$u(x,0) = \lambda - 2Bk\coth(k\xi),$$
$$v(x,0) = -2B\ (B+\beta)\ \ k^2 \csc h^2(k\xi), \tag{2}$$

where $B = \sqrt{\alpha+\beta^2}$ and $\xi = x + x_0$ and x_0, k, λ are arbitrary constants.

To solve Eq. (1) by the homotopy perturbation method, we construct the following homotopy.

$$u_t + p(uu_x + v_x + \beta u_{xx}) = 0, \tag{3}$$

$$v_t + p((uv)_x + \alpha u_{xxx} - \beta v_{xx}) = 0, \tag{4}$$

Assume the solution of Eqs. (3), (4) in the forms:

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots, \tag{5}$$

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \dots, \tag{6}$$

Substituting Eqs. (5), (6) into Eqs. (3), (4) and collecting terms of the same power of p give

$$p^0: \ (u_0)_t = 0,$$
$$p^1: \ (u_1)_t = u_0(u_0)_x + (v_0)_x + \beta(u_0)_{xx},$$
$$p^2: \ (u_2)_t = u_0(u_1)_x + u_1(u_0)_x + (v_1)_x + \beta(u_1)_{xx},$$
$$p^3: \ (u_3)_t = u_0(u_2)_x + u_1(u_1)_x + u_2(u_0)_x + (v_2)_x + \beta(u_2)_{xx},$$
$$\dots$$

and

$$p^0: \ (v_0)_t = 0,$$
$$p^1: \ (v_1)_t = (u_0v_0)_x + \alpha\ (v_0)_{xxx} - \beta(v_0)_{xx},$$
$$p^2: \ (v_2)_t = (u_0v_1 + u_1v_0)_x + \alpha(v_1)_{xxx} - \beta(v_1)_{xx},$$
$$p^3: \ (v_3)_t = (u_0v_2 + u_1v_1 + u_2v_0)_x + \alpha(v_2)_{xxx} - \beta(v_2)_{xx},$$
$$\dots$$

The given initial value admits the use of

$$u_0 = \lambda - 2Bk\coth(k\xi),$$
$$v_0 = -2B(B+\beta)k^2\csc h^2(k\xi),$$

If we solve the above equation system, we successively obtain

$$u_1 = -2Bk^2\lambda t\csc h^2(k(x+x_0)),$$

$$v_1 = -2B(B+\beta)k^2t(-\lambda + 2Bk\coth(k\xi))\csc h^2(k\xi)$$
$$- 4B(\alpha + B\beta + \beta^2)k^4t(2+\cosh(2k(x+x_0)))\csc h^4(k\xi),$$

$$u_2 = \frac{Bk^3t^2}{2}\csc h^5(k\xi)((-44\alpha k^2 - 44B\beta k^2 - 44\beta^2k^2 - B\lambda - \beta\lambda$$
$$+ \lambda^2)\cosh(k\xi) - (4\alpha k^2 + 4B\beta k^2 + 4\beta^2k^2 - B\lambda - \beta\lambda + \lambda^2)$$
$$\times \cosh(3k\xi) - (6B^2k + 6B\beta k - 6Bk\lambda - 6\beta k\lambda)\sinh(k\xi)$$
$$- (2B^2k + 2B\beta k - 2Bk\lambda - 2\beta k\lambda)\sinh(3k\xi),$$

$$v_2 = \frac{Bk^2t^2}{8}\csc h^6(k\xi)(4B^3k^2 + 4B^2\beta k^2 - 528\alpha\beta k^4 - 528B\beta^2k^4$$
$$- 528\beta^3k^4 - 12\alpha k^2\lambda + 8B^2k^2\lambda - 16B\beta k^2\lambda - 24\beta^2k^2\lambda$$
$$- 3B\lambda^2 - 3\beta\lambda^2 - (416\alpha\beta k^4 + 416B\beta^2k^4 + 416\beta^3k^4$$
$$- 8\alpha k^2\lambda + 8B^2k^2\lambda - 8B\beta k^2\lambda - 16\beta^2k^2\lambda - 4B\lambda^2 - 4\beta\lambda^2)$$
$$\times \cosh(2k\xi) - (4B^3k^2 + 4B^2\beta k^2 + 16\alpha\beta k^4 + 16B\beta^2k^4$$
$$+ 16\beta^3k^4 - 4\alpha k^2\lambda\cosh(4k\xi) + 8B\beta k^2\lambda - 8\beta^2k^2\lambda$$
$$+ B\lambda^2 + \beta\lambda^2)\cosh(4k\xi) - (32\alpha Bk^3 + 112B^2\beta k^3$$
$$+ 112B\beta^2k^3 + 8B^2k\lambda + 8B\beta k\lambda + 80\alpha k^3\lambda)\sinh(2k\xi)$$
$$- (8\alpha Bk^3 + 16B^2\beta k^3 + 16B\beta^2k^3 - 4B^2k\lambda - 4B\beta k\lambda$$
$$+ 8\alpha k^3\lambda)\sinh(4k\xi)),$$
$$\dots$$

and so on; in this manner, the rest of the components of the homotopy perturbation series can be obtained. Then the series solutions expression by HPM can be written in the form:

$$u = u_0 + u_1 + u_2 + u_3 + \dots, \tag{7}$$

$$v = v_0 + v_1 + v_2 + v_3 + \dots, \tag{8}$$

So we obtain the closed form solutions

$$u(x,t) = \lambda - 2Bk\coth(k(\xi - \lambda t)), \tag{9}$$

$$v(x,t) = -2B(B+\beta)k^2\csc h^2(k(\xi - \lambda t)), \tag{10}$$

where $B = \sqrt{\alpha+\beta^2}$ and $\xi = x + x_0$ and $x_0,\ k,\ \lambda$ are arbitrary constants. These solutions are constructed by Xie et al. (2001). As a special case, if $\alpha = 1$ and $\beta = 0$, WBK equations can be reduced to the modified Boussinesq (MB) equations. We shall second consider the initial conditions of the MB equations.

$$u(x,0) = \lambda - 2k\coth(k\xi), \quad v(x,0) = -2k^2\csc h^2(k\xi), \tag{11}$$

where $\xi = x + x_0$ being arbitrary constant. Using the similar homotopy procedure, we obtain components of the iteration. So we get exact solution as

$$u(x,t) = \lambda - 2k\coth(k\xi - \lambda t), v(x,t) = -2k^2\csc h^2(k\xi - \lambda t),$$

where $k,\ \lambda$ are constants to be determined and x_0 is an arbitrary constant.

Table 1 The numerical results for $\varphi_n(x, t)$ and $\varphi_n(x, t)$ in comparison with the exact solution for $u(x, t)$ and $v(x, t)$ when $k = 0.1$, $\lambda = 0.005$, $\alpha = 1.5$, $\beta = 1.5$ and $x_0 = 10$, for the approximate solution of the WBK equation.

t_i/x_i	0.1	0.2	0.3	0.4	0.5
$\lvert u - \phi_n \rvert$					
0.1	1.04892E-04	4.25408E-04	9.71992E-04	1.75596E-03	2.79519E-03
0.3	9.64474E-05	3.91098E-04	8.93309E-04	1.61430E-03	2.56714E-03
0.5	8.88312E-05	3.60161E-04	8.22452E-04	1.48578E-03	2.36184E-03
$\lvert v - \varphi_n \rvert$					
0.1	6.41419E-03	1.33181E-03	2.07641E-02	2.88100E-02	3.75193E-02
0.3	5.99783E-03	1.24441E-02	1.93852E-02	2.68724E-02	3.49617E-02
0.5	5.61507E-03	1.16416E-02	1.81209E-02	2.50985E-02	3.26239E-02

Table 2 The numerical results for $\varphi_n(x, t)$ and $\varphi_n(x, t)$ in comparison with the analytical solution for $u(x, t)$ and $v(x, t)$ when $k = 0.1$, $\lambda = 0.005$, $\alpha = 1$, $\beta = 0$ and $x_0 = 10$, for the approximate solution of the MB equation.

t_i/x_i	0.1	0.2	0.3	0.4	0.5
$\lvert u - \phi_n \rvert$					
0.1	8.16297E-07	3.26243E-06	7.33445E-06	1.30286E-05	2.03415E-05
0.3	7.64245E-07	3.05458E-06	6.86758E-06	1.22000E-05	1.90489E-05
0.5	7.16083E-07	2.86226E-06	6.43557E-06	1.14333E-05	1.78528E-05
$\lvert v - \varphi_n \rvert$					
0.1	5.88676E-05	1.18213E-04	1.78041E-04	2.38356E-04	2.99162E-04
0.3	5.56914E-05	1.11833E-04	1.68429E-04	2.25483E-04	2.83001E-04
0.5	5.27169E-05	1.05858E-04	1.59428E-04	2.13430E-04	2.67868E-04

Table 3 The numerical results for $\varphi_n(x, t)$ and $\varphi_n(x, t)$ in comparison with the analytical solution for $u(x, t)$ and $v(x, t)$ when $k = 0.1$, $\lambda = 0.005$, $\alpha = 0$, $\beta = 0.5$ and $x_0 = 10$, for the approximate solution of the ALW equation.

t_i/x_i	0.1	0.2	0.3	0.4	0.5
$\lvert u - \phi_n \rvert$					
0.1	8.02989E-06	3.23228E-05	7.32051E-05	1.31032E-04	2.06186E-04
0.3	7.38281E-06	2.97172E-05	6.73006E-05	1.20455E-04	1.89528E-04
0.5	6.79923E-06	2.73673E-05	6.19760E-05	1.10919E-04	1.74510E-04
$\lvert v - \varphi_n \rvert$					
0.1	4.81902E-04	9.76644E-04	1.48482E-03	2.00705E-03	2.54396E-03
0.3	4.50818E-04	9.13502E-04	1.38858E-03	1.87661E-03	2.37815E-03
0.5	4.22221E-04	8.55426E-04	1.30009E-03	1.75670E-03	2.22578E-03

In the last example, if $\alpha = 0$ and $\beta = 1/2$, WBK equations can be reduced to the approximate long wave (ALW) equation in shallow water. We can compute the ALW equation with the initial conditions.

$$u(x,0) = \lambda - k\coth(k\xi), \quad v(x,0) = -k^2 \operatorname{csc} h^2(k\xi),$$

where k is constant to be determined and $\xi = x + x_0$. Using the similar homotopy procedure, we obtain components of the iteration. So we get exact solution as

$$u(x,t) = \lambda - k\coth(k(\xi - \lambda t)),$$
$$v(x,t) = -2k^2 \operatorname{csc} h^2(k\xi - \lambda t),$$

where $k,\ \lambda$ are constants to be determined and $\xi = x + x_0$, x_0 is an arbitrary constant.

In order to verify numerically whether the proposed methodology lead to higher accuracy, we evaluate the numerical solutions using the n-term approximation. Tables 1–3 show the difference of analytical solution and numerical solution of the absolute error. We achieved a very good approximation with the actual solution of the equations by using five terms only of the homotopy perturbation series derived above.

3. Conclusion

In this study, we used the homotopy perturbation method for finding the exact and approximate traveling waves solutions of the WBK equation in shallow water. In addition, variant Boussinesq equation and the approximate equation for long water waves, as the special cases of WBK equation, also possess the corresponding many solitary wave solutions and periodic wave solutions. The method is extremely simple, easy to use and is very accurate for wide classes of problems. It is also worth noting to point out that the advantage of the HPM is the fast convergence of the solutions.

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