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ORIGINAL ARTICLE

Numerical study of time-fractional fourth-order differential equations with variable coefficients

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Abstract In this article, we study numerical solutions of time-fractional fourth-order partial differential equations with variable coefficients by introducing the fractional derivative in the sense of Caputo. We implement reliable series solution techniques namely Adomian decomposition method (ADM) and He's variational iteration method (HVIM). Some applications are presented to highlight the significant features of these techniques. The comparison shows that the solutions obtained are in good agreement with each other and with their respective exact solutions. Some of these types of differential equations arise practically in the theory of transverse vibrations.

1. Introduction

Fractional calculus is three centuries old as the conventional calculus, but not very popular among science and/or engineering community. The beauty of this subject is that fractional derivatives (and integral) are not a local (or point) property. Thereby this considers the history and non-local distributed effects. In other words, perhaps this subject translates the reality of nature better. Many physical problems (Khan et al., 2009; Mahmood et al., 2009; Yildirim and Koçak, 2009; Konuralp et al., 2009; Yildirim and Gülkanat, 2010; Momani and Yildirim, 2010) are governed by fractional differential equations (FDEs), and finding the solution of these equations has been the subject of many investigators in recent years. The main reason consists in the fact that the theory of derivatives of fractional (non-integers) stimulates considerable interest in the areas of mathematics, physics, engineering and other sciences. Most of the FDE cannot be solved exactly, approximate and numerical methods must be used. Numerical methods that are commonly used such as finite difference and characteristics approaches need large amount of computational work and usually the affect of rounding off error causes loss of accuracy in the results. The Adomian decomposition method (ADM) (Wazwaz, 2001, 2002; Adomian and Rach, 1996) and He's variational iteration method (HVIM) (He, 1997, 1998, 1999, 2006, 2007; He and Wu, 2007; Ates and Yildirim, 2009) are relatively new approaches to provide the

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analytical approximation to linear and nonlinear problems. In 1998, the variational iteration method was first proposed by He to give approximate solutions of the seepage flow problem in the porous media with fractional derivatives. These techniques are particularly valuable as tool for scientists and applied mathematicians, because they provide immediate and visible symbolic terms of analytic solutions (Biazar and Ghazvini, 2007; Momani and Odibat, 2007; Khan et al., 2010), as well as numerical approximate solutions to both linear and nonlinear differential equations without linearization and discretization (Khaliq and Twizell, 1963). The governing equation of motion of the beam can be written as

$$\frac{\partial^2 u(x,t)}{\partial t^2}+\eta(x)\frac{\partial^4 u(x,t)}{\partial x^4}=0,\quad l_0<x<l_1,\ \ \eta(x)>0,\ \ t>0,$$

where $\eta(x)$ is the ratio of flexural rigidity of the beam to its mass per unit length (Gorman, 1975). The initial and boundary conditions associated with above equation are of the form (Khaliq and Twizell, 1963)

$$u(x,0)=h_0(x),\quad \frac{\partial u}{\partial t}(x,0)=h_1(x),\quad l_0\leqslant x\leqslant l_l,$$
$$u(l_0,t)=f_0(t),\quad u(l_1,t)=f_1(t),\quad \frac{\partial^2 u}{\partial x^2}(l_0,t)=G_0(t),$$
$$\frac{\partial^2 u}{\partial x^2}(l_1,t)=G_1(t),\ \ t>0,$$

where the functions $h_0(x),\ h_1(x),\ f_0(t),\ f_1(t),\ G_0(t)$, and $G_1(t)$ are continuous functions.

In this work, the n-dimensional time-fractional fourth-order partial differential equation with variable coefficients will be approached analytically by Adomian decomposition method (ADM) and He's variational iteration method (HVIM). Three applications are given to assess the efficiency and convenience of the two methods.

2. Fractional calculus

We give some basic definitions, notations and properties of the fractional calculus theory used in this work.

Definition 2.1. The Riemann–Liouville fractional integral operator of order $\alpha\geqslant 0$, for a function $f\in C_\mu$ $(\mu\geqslant -1)$ is defined as

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt,\quad \alpha>0,\ \ x>0, \tag{1}$$
$$J^0 f(x)=f(x). \tag{2}$$

We will need the following basic properties:

For $f\in C_\mu(\mu\geqslant -1),\ \alpha,\beta\geqslant 0$ and $\gamma>1$:

$$J^\alpha J^\beta f(x)=J^{\alpha+\beta}f(x), \tag{3}$$
$$J^\alpha J^\beta f(x)=J^\beta J^\alpha f(x), \tag{4}$$
$$J^\alpha x^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}. \tag{5}$$

Definition 2.2. The fractional derivative of $f\in C^m_{-1}$, in the Caputo sense, is defined as Caputo (Caputo, 1967)

$$D^\alpha f(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}f^{(m)}(\tau)d\tau,$$
$$(m-1<Re(\alpha)\leqslant m, m\in N, t>0). \tag{6}$$

We mention some of its properties as follows:

$$D^\alpha K=0,$$

where K is a constant.

$$D^\alpha t^\mu=\begin{cases}0, & \mu\leqslant\alpha-1,\\ \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}, & \mu>\alpha-1.\end{cases} \tag{7}$$

Also, we need here two of its basic properties:

Lemma 2.1. *If $m-1<\alpha\leqslant m,\ m\in N$ and $f\in C^m_\mu,\ \mu>-1$, then*

$$D^\alpha J^\alpha f(x)=f(x)$$

and

$$J^\alpha D^\alpha f(x)=f(x)-\sum_{k=0}^{m-1}f^{(k)}(0^+)\frac{x^k}{k!},\quad x>0.$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem.

Definition 2.3. For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative of order $\alpha>0$, is defined as

$$D_t^\alpha u(x,t)=\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\frac{\partial^m u(x,\tau)}{\partial\tau^m}d\tau, & m-1<\alpha<m,\\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha=m\in N.\end{cases} \tag{8}$$

3. Methodologies

3.1. The Adomian decomposition method (ADM)

Consider the n-dimensional time-fractional differential equation of fourth-order with variable coefficients

$$\frac{\partial^\alpha u(x_1,x_2,\ldots,x_n,t)}{\partial t^\alpha}+A_1(x_1,x_2,\ldots,x_n)\frac{\partial^4 u(x_1,x_2,\ldots,x_n,t)}{\partial x_1^4}$$
$$+A_2(x_1,x_2,\ldots,x_n)\frac{\partial^4 u(x_1,x_2,\ldots,x_n,t)}{\partial x_2^4}+\cdots$$
$$+A_n(x_1,x_2,\ldots,x_n)\frac{\partial^4 u(x_1,x_2,\ldots,x_n,t)}{\partial x_n^4}$$
$$=H(x_1,x_2,\ldots,x_n,t),x_1,x_2,\ldots,x_n\in R,$$
$$t>0,\ \ 1<\alpha\leqslant 2. \tag{9}$$

The time-fractional differential equation (9) can be expressed in terms of operator form as

$$D_t^\alpha u(x_1,x_2,\ldots,x_n,t)+(A_1L_{4x_1}+A_2L_{4x_2}+\cdots+A_nL_{4x_n})$$
$$\times u(x_1,x_2,\ldots,x_n,t)=H. \tag{10}$$

where $D_t^\alpha=\frac{\partial^\alpha}{\partial t^\alpha}$ and $L_{4x_q}=\frac{\partial^4}{\partial x_q^4},\ q=1,2,3,\ldots,n$, while $H,A_1,A_2,A_3,\ldots,A_n$ are continuous function and α is the parameter describing the order of the time-fractional derivative.

On applying the operator J^α, on both sides of Eq. (10), we obtain

$$u(x_1,x_2,\ldots,x_n,t) = \sum_{k=0}^{m-1}\frac{\partial^k}{\partial t^k}u(x_1,x_2,\ldots,x_n,0^+)\frac{t^k}{k!} - J^\alpha[(A_1L_{4x_1}+A_2L_{4x_2}+\cdots + A_nL_{4x_n})u(x_1,x_2,\ldots,x_n,t)-H]. \quad (11)$$

The Adomian decomposition method assumes a series solution for $u(x_1,x_2,\ldots,x_n,t)$ given by an infinite series of components

$$u(x_1,x_2,\ldots,x_n,t) = \sum_{j=0}^{\infty}u_j(x_1,x_2,\ldots,x_n,t), \quad (12)$$

where the components $u_j(x_1,x_2,\ldots,x_n,t)$ will be determined recursively. Using Eq. (12) in Eq. (11), we get

$$\sum_{j=0}^{\infty}u_j(x_1,x_2,\ldots,x_n,t) = \sum_{k=0}^{m-1}\frac{\partial^k}{\partial t^k}u(x_1,x_2,\ldots,x_n,0^+)\frac{t^k}{k!} - J^\alpha\left[(A_1L_{4x_1}+A_2L_{4x_2}+\cdots+A_nL_{4x_n})\times\sum_{j=0}^{\infty}u_j(x_1,x_2,\ldots,x_n,t)-H\right] \quad (13)$$

Following the decomposition method, we introduce the recursive relation as

$$u_0(x_1,x_2,\ldots,x_n,t) = \sum_{k=0}^{m-1}\frac{\partial^k}{\partial t^k}u(x_1,x_2,\ldots,x_n,0^+)\frac{t^k}{k!} + J^\alpha(H). \quad (14.1)$$

$$u_{r+1}(x_1,x_2,\ldots,x_n,t) = -J^\alpha[(A_1L_{4x_1}+A_2L_{4x_2}+\cdots+A_nL_{4x_n})u_r(x_1,x_2,\ldots,x_n,t)], \quad r \geqslant 0. \quad (14.2)$$

where in above relation (14.1), $m=2$, since for our problem $1<\alpha\leqslant 2$.

3.2. He's variational iteration method (HVIM)

Consider again the time-fractional differential equation (9) of fourth order. The correction functional for it can be approximately expressed as follows:

$$u_{k+1}(x_1,x_2,\ldots,x_n,t) = u_k(x_1,x_2,\ldots,x_n,t) + \int_0^t \lambda(\zeta)\left(\frac{\partial^m u_k(x_1,x_2,\ldots,x_n,\zeta)}{\partial\zeta^m} + A_1\frac{\partial^4\tilde{u}_k(x_1,x_2,\ldots,x_n,\zeta)}{\partial x_1^4}+\cdots + A_n\frac{\partial^4\tilde{u}_k(x_1,x_2,\ldots,x_n,\zeta)}{\partial x_n^4} - H\right)d\zeta, \quad (15)$$

where λ is a general Lagrange multiplier, which can be identified optimally via variational theory, here $\frac{\partial^4\tilde{u}_k}{\partial x_1^4},\ldots,\frac{\partial^4\tilde{u}_k}{\partial x_n^4}$ are considered as restricted variations. Making the above functional stationary, noticing that $\delta\tilde{u}_k=0$,

$$\delta u_{k+1} = \delta u_k + \delta\int_0^t\lambda(\zeta)\left(\frac{\partial^m u_k}{\partial\zeta^m}-H\right)d\zeta \quad (16)$$

yields the following Lagrange multipliers

$$\lambda=-1, \quad \text{for } m=1. \quad (17)$$

$$\lambda=\zeta-t, \quad \text{for } m=2. \quad (18)$$

Therefore, for $m=1$, we obtain the following iteration formula:

$$u_{k+1}(x_1,x_2,\ldots,x_n,t) = u_k(x_1,x_2,\ldots,x_n,t) - \int_0^t\left(\frac{\partial^\alpha u_k}{\partial\zeta^\alpha}+A_1\frac{\partial^4u_k}{\partial x_1^4}+\cdots+A_n\frac{\partial^4u_k}{\partial x_n^4}-H\right)d\zeta \quad (19)$$

and for $m=2$, we obtain the following iteration formula:

$$u_{k+1} = u_k + \int_0^t(\zeta-t)\left(\frac{\partial^\alpha u_k}{\partial\zeta^\alpha}+A_1\frac{\partial^4u_k}{\partial x_1^4}+\cdots+A_n\frac{\partial^4u_k}{\partial x_n^4}-H\right)d\zeta. \quad (20)$$

4. Numerical applications

In this section, we apply the ADM and HVIM developed in Section 3 to solve one and two dimensional initial boundary value problems with variable coefficients. The methods may also be applicable for higher dimensional spaces. Numerical results reveal that the ADM and HVIM are easy to implement and reduce the computational work to a tangible level while still maintaining a higher level of accuracy. All the results for the following three applications are calculated by using the symbolic calculus software MATHEMATICA.

4.1. Application 1

Consider the following case of one-dimensional time-fractional fourth-order PDE

$$D_t^\alpha u(x,t) + \left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4u(x,t)}{\partial x^4} = 0, \quad \frac{1}{2}<x<1, \quad t>0, \ 1<\alpha\leqslant 2. \quad (21)$$

subject to the initial and boundary conditions:

$$u(x,0)=0, \quad \frac{\partial u}{\partial t}(x,0) = 1+\frac{x^5}{120}, \quad u\left(\frac{1}{2},t\right) = \left(1+\frac{0.5^5}{120}\right)\sin(t,\alpha), \quad \frac{\partial^2u}{\partial x^2}\left(\frac{1}{2},t\right) = \frac{1}{6}\frac{1}{2^3}\sin(t,\alpha), \quad u(1,t) = \frac{121}{120}\sin(t,\alpha), \quad \frac{\partial^2u}{\partial x^2}(1,t) = \frac{1}{6}\sin(t,\alpha). \quad (22)$$

where the function is defined as $\sin(t,\alpha) = \sum_{i=0}^{\infty}\frac{(-1)^i t^{i\alpha+1}}{\Gamma(i\alpha+2)}$

On applying ADM, the first component of the decomposition series solution is:

$$u_0(x,t) = \left(1+\frac{x^5}{120}\right)t \quad (23)$$

and the next few successive components are as follows:

$$u_1 = -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\left(1+\frac{x^5}{120}\right), \quad (24)$$

$$u_2 = \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\left(1+\frac{x^5}{120}\right), \quad (25)$$

$$u_3 = -\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\left(1+\frac{x^5}{120}\right), \quad (26)$$

and so on, in this manner the rest of the components of the decomposition series can be obtained. The ADM solution in series form is

$$u=\left(t-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}+\cdots\right)\left(1+\frac{x^5}{120}\right). \tag{27}$$

Now we solve the problem by HVIM. According to the iteration formula (20), the iteration formula for Eq. (21) is given by

$$u_{k+1}=u_k+\int_0^t(\zeta-t)\left(D_\zeta^\alpha u_k(x,\zeta)+\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4 u_k(x,\zeta)}{\partial x^4}\right)d\zeta. \tag{28}$$

Using the above iteration formula, we begin with $u_0(x,t)=\left(1+\frac{x^5}{120}\right)t$, and get the next approximations as follows:

$$u_1=\left(t-\frac{t^3}{3!}\right)\left(1+\frac{x^5}{120}\right), \tag{29}$$

$$u_2=\left(t-\frac{t^3}{3}+\frac{t^5}{5!}+t^{5-\alpha}\left(\frac{1}{\Gamma(5-\alpha)}-\frac{1}{(5-\alpha)\Gamma(4-\alpha)}\right)\right)\left(1+\frac{x^5}{120}\right), \tag{30}$$

$$u_3=\left(t-\frac{t^3}{2}+\frac{t^5}{40}-\frac{t^7}{7!}+t^{5-\alpha}\left(\frac{3}{\Gamma(5-\alpha)}-\frac{3}{(5-\alpha)\Gamma(4-\alpha)}\right)+t^{7-2\alpha}\left(\frac{-1}{\Gamma(7-2\alpha)}+\frac{1}{(7-2\alpha)\Gamma(6-2\alpha)}\right)+A_1(\alpha)t^{7-\alpha}\right)\times\left(1+\frac{x^5}{120}\right). \tag{31}$$

and so on, in the same manner the rest of the components of the iteration formula equation (28) can be obtained, where $A_1(\alpha)$ are given in Appendix. The fourth term approximate solution is

$$u=\left(t-\frac{t^3}{2}+\frac{t^5}{40}-\frac{t^7}{7!}+t^{5-\alpha}\left(\frac{3}{\Gamma(5-\alpha)}-\frac{3}{(5-\alpha)\Gamma(4-\alpha)}\right)+t^{7-2\alpha}\left(\frac{-1}{\Gamma(7-2\alpha)}+\frac{1}{(7-2\alpha)\Gamma(6-2\alpha)}\right)+A_1(\alpha)t^{7-\alpha}\right)\times\left(1+\frac{x^5}{120}\right). \tag{32}$$

When $\alpha=2$, the solution obtained by Wazwaz (2001, 2002) and Biazar and Ghazvini (2007) is recovered as a special case.

Table 1 shows the approximate solutions for Eqs. (21) and (22) obtained for different values of α using the ADM and HVIM. It is clear from the table that our approximate solutions using these methods are in good agreement with the exact values. It is note that only the fourth-order term of HVIM solution and only four terms of the ADM series used in evaluating the approximate solutions for Table 1. It is evident that the efficiency of these approaches can be dramatically enhanced by computing further terms or further components of $u(x,t)$.

4.2. Application 2

$$D_t^\alpha u(x,t)+\left(\frac{x}{\sin x}-1\right)\frac{\partial^4 u(x,t)}{\partial x^4}=0,\quad 0<x<1,\quad t>0,\quad 1<\alpha\leqslant 2, \tag{33}$$

subject to the initial and boundary conditions:

$$u(x,0)=x-\sin x,\quad \frac{\partial u}{\partial t}(x,0)=-x+\sin x,$$
$$u(0,t)=0,\quad \frac{\partial^2 u}{\partial x^2}(0,t)=0,\quad u(1,t)=Exp(t,\alpha)(1-\sin 1),$$
$$\frac{\partial^2 u}{\partial x^2}(1,t)=Exp(t,\alpha)\sin 1. \tag{34}$$

where the function $Exp(t,\alpha)$ is defined as $Exp(t,\alpha)=\sum_{i=0}^{\infty}(-1)^i\frac{t^{i\alpha/2}}{\Gamma\left(\frac{i\alpha}{2}+1\right)}$

On solving Eqs. (33) and (34) by ADM, the first few components are

$$u_0(x,t)=(1-t)(x-\sin x), \tag{35}$$

$$u_1=\left(\frac{t^\alpha}{\Gamma(\alpha+1)}-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)(x-\sin x), \tag{36}$$

$$u_2=\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)(x-\sin x), \tag{37}$$

$$u_3=\left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}-\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right)(x-\sin x). \tag{38}$$

and so on, in this manner the rest of the components of the decomposition series can be obtained.

Table 1 Comparison of the approximate solutions of equations (21) and (22) obtained by ADM and HVIM.

t	x	$\alpha=1.50$		$\alpha=1.75$		$\alpha=2$		
		u_{ADM}	u_{HVIM}	u_{ADM}	u_{HVIM}	u_{ADM}	u_{HVIM}	u_{Exact}
0.2	0.50	0.194734	0.196914	0.19736	0.197687	0.198721	0.198721	0.198721
	0.60	0.194309	0.196991	0.197437	0.1977 63	0.198798	0.198798	0.198798
	0.75	0.195068	0.197527	0.197699	0.198026	0.199062	0.199062	0.199062
	1.0	0.196306	0.198504	0.198953	0.199282	0.200325	0.200325	0.200325
0.4	0.50	0.370692	0.377682	0.382211	0.383217	0.38952	0.38952	0.38952
	0.60	0.370835	0.377828	0.382359	0.383366	0.389671	0.389671	0.389671
	0.75	0.3713 2 8	0.37833	0.382867	0.383875	0.390188	0.390188	0.390188
	1.0	0.373633	0.38073	0.38529 6	0.3363 1	0.392663	0.392663	0.392663
0.6	0.50	0.521419	0.531411	0.546537	0.547792	0.564789	0.564789	0.564789
	0.60	0.521621	0.531617	0.546748	0.548005	0.565008	0.565008	0.565008
	0.75	0.522314	0.532323	0.547475	0.548733	0.565759	0.565759	0.565759
	1.0	0.525627	0.5357	0.550947	0.552214	0.569348	0.569348	0.569348

The fourth term approximate solution is given by

$$u = \left(1 - t + \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right)(x - \sin x). \tag{39}$$

According to the iteration formula (20), the iteration formula for Eq. (33) is given by

$$u_{k+1} = u_k + \int_0^t (\zeta - t)\left(D_\zeta^\alpha u_k(x,\zeta) + \left(\frac{x}{\sin x} - 1\right)\frac{\partial^4 u_k(x,\zeta)}{\partial x^4}\right)d\zeta. \tag{40}$$

By the above iteration formula, starting with $u_0(x,t) = (1-t)(x - \sin x)$, we can obtain the following approximations:

$$u_1 = \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!}\right)(x - \sin x), \tag{41}$$

$$u_2 = \left(1 - t + t^2 - \frac{t^3}{3} + \frac{t^4}{4!} - \frac{t^5}{5!} + t^{4-\alpha}\left(\frac{1}{(4-\alpha)\Gamma(3-\alpha)} - \frac{1}{\Gamma(4-\alpha)}\right) - t^{5-\alpha}\left(\frac{1}{(5-\alpha)\Gamma(4-\alpha)} - \frac{1}{\Gamma(5-\alpha)}\right)\right) \times (x - \sin x), \tag{42}$$

$$\begin{aligned} u_3 = \Bigg(&1 - t + \frac{3t^2}{2} - \frac{t^3}{2} + \frac{t^4}{8} - \frac{t^5}{40} + \frac{t^6}{6!} - \frac{t^7}{7!} + A_2(\alpha)t^{6-\alpha} \\ &+ A_3(\alpha)t^{7-\alpha} + t^{4-\alpha}\left(\frac{3}{(4-\alpha)\Gamma(3-\alpha)} - \frac{3}{\Gamma(4-\alpha)}\right) \\ &+ t^{5-\alpha}\left(\frac{3}{\Gamma(5-\alpha)} - \frac{3}{(5-\alpha)\Gamma(4-\alpha)}\right) \\ &+ t^{6-2\alpha}\left(\frac{1}{\Gamma(6-2\alpha)} - \frac{1}{(6-2\alpha)\Gamma(5-2\alpha)}\right) \\ &+ t^{7-2\alpha}\left(\frac{-1}{\Gamma(7-2\alpha)} + \frac{1}{(7-2\alpha)\Gamma(6-2\alpha)}\right)\Bigg)(x - \sin x) \end{aligned} \tag{43}$$

and so on. When $\alpha = 2$, we recovered the solution obtained by Wazwaz (2001, 2002) and Biazar and Ghazvini (2007). The exact solution of Eq. (33) for $\alpha = 2$ is $u = (x - \sin x)e^{-t}$.

$$\begin{aligned} u_3 = 1 - t &+ \frac{3t^2}{2} - \frac{t^3}{2} + \frac{t^4}{8} - \frac{t^5}{40} + \frac{t^6}{6!} - \frac{t^7}{7!} \\ &+ t^{4-\alpha}\left(\frac{3}{(4-\alpha)\Gamma(3-\alpha)} - \frac{3}{\Gamma(4-\alpha)}\right) \\ &+ t^{5-\alpha}\left(\frac{3}{\Gamma(5-\alpha)} - \frac{3}{(5-\alpha)\Gamma(4-\alpha)}\right) \\ &+ A_2(\alpha)t^{6-\alpha} + A_3(\alpha)t^{7-\alpha} + t^{6-2\alpha} \\ &\times\left(\frac{1}{\Gamma(6-2\alpha)} - \frac{1}{(6-2\alpha)\Gamma(5-2\alpha)}\right) \\ &+ t^{7-2\alpha}\left(\frac{-1}{\Gamma(7-2\alpha)} + \frac{1}{(7-2\alpha)\Gamma(6-2\alpha)}\right)\Bigg)(x - \sin x). \end{aligned} \tag{44}$$

Table 2 shows the approximate solutions for Eqs. (33) and (34) obtained for different values of α using the ADM and HVIM. It is clear from Table 2 that our approximate solutions using the methods are in good agreement with the exact values. As in the previous example, only the fourth-order term of HVIM solution and only four terms of the ADM series were used in evaluating the approximate solutions for Table 2.

4.3. Application 3

We consider a two dimensional time-fractional fourth-order PDE

$$\begin{aligned} &D_t^\alpha u(x,y,t) + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\frac{\partial^4 u(x,y,t)}{\partial x^4} \\ &\quad + 2\left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\frac{\partial^4 u(x,y,t)}{\partial y^4} = 0, \\ &\frac{1}{2} < x,\ \ y < 1,\ \ t > 0,\ \ 1 < \alpha \leqslant 2, \end{aligned} \tag{45}$$

subject to the initial and boundary conditions:

$$u(x,y,0) = 0, \quad \frac{\partial u}{\partial t}(x,y,0) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!},$$

$$u\left(\frac{1}{2},y,t\right) = \left(2 + \frac{0.5^6}{6!} + \frac{y^6}{6!}\right)\sin(t,\alpha),$$

$$u(1,y,t) = \left(2 + \frac{1}{6!} + \frac{y^6}{6!}\right)\sin(t,\alpha),$$

Table 2 Comparison of the approximate solutions of equations (33) and (34) obtained by ADM and HVIM.

t	x	$\alpha = 1.50$		$\alpha = 1.75$		$\alpha = 2$		
		u_{ADM}	u_{HVIM}	u_{ADM}	u_{HVIM}	u_{ADM}	u_{HVIM}	u_{Exact}
0.2	0.25	0.0022379	0.00218727	0.00216712	0.00215504	0.00212546	0.00212545	0.00212546
	0.30	0.0177361	0.0173348	0.0171751	0.0171111	0.0168449	0.0168449	0.0168449
	0.75	0.058934	0.0575971	0.0570666	0.05568538	0.0559694	0.0559693	0.0559694
	0.9	0.100577	0.0983018	0.0973963	0.0970331	0.0955238	0.0955238	0.0955238
0.4	0.25	0.00195106	0.00192801	0.00184347	0.00183161	0.00174018	0.00173996	0.00174018
	0.50	0.0154628	0.0152801	0.0146101	0.0145161	0.0137915	0.0137897	0.0137915
	0.75	0.0513771	0.0507702	0.0485438	0.0482315	0.0458239	0.0458181	0.0458239
	0.9	0.0876861	0.0865502	0.0828503	0.0823173	0.0782083	0.0782083	0.0782083
0.6	0.25	0.001573	0.00170687	0.00158783	0.00157339	0.00142474	0.00142307	0.00142474
	0.50	0.0124666	0.0135275	0.012584	0.0124697	0.0112915	0.0112783	0.0112915
	0.75	0.0414217	0.0449467	0.041812	0.041432	0.0375174	0.0374735	0.0375174
	0.9	0.0706951	0.0767111	0.713612	0.0701726	0.0640315	0.0640315	0.0640315

$$\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2}, y, t\right) = \frac{0.5^4}{24}\sin(t,\alpha),$$
$$\frac{\partial^2 u}{\partial y^2}\left(x, \frac{1}{2}, t\right) = \frac{0.5^4}{24}\sin(t,\alpha),$$
$$\frac{\partial^2 u}{\partial x^2}(1,y,t) = \frac{1}{24}\sin(t,\alpha), \quad \frac{\partial^2 u}{\partial y^2}(x,1,t) = \frac{1}{24}\sin(t,\alpha). \tag{46}$$

By applying ADM, we have

$$u_0 = t\left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right), \tag{47}$$
$$u_1 = -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right), \tag{48}$$
$$u_2 = \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right), \tag{49}$$
$$u_3 = -\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right), \tag{50}$$

and so on, in this manner the rest of the components of the decomposition series can be obtained. The fourth term approximate solution is given by

$$u = \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right)\left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right). \tag{51}$$

According to the iteration formula (20), the iteration formula for Eq. (45) is given by

$$u_{k+1} = u_k + \int_0^t (\zeta - t)\left(D_\zeta^\alpha u_k(x,y,\zeta) + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\frac{\partial^4 u_k(x,y,\zeta)}{\partial x^4} + 2\left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\frac{\partial^4 u_k(x,y,\zeta)}{\partial y^4}\right)d\zeta. \tag{52}$$

Using the above iteration formula, starting with $u_0(x,t) = t\left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)$, we can obtain the following approximations:

$$u_1 = \left(t - \frac{t^3}{3!}\right)\left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right). \tag{53}$$
$$u_2 = \left(t - \frac{t^3}{3} + \frac{t^5}{5!} + t^{5-\alpha}\left(\frac{1}{\Gamma(5-\alpha)} - \frac{1}{(5-\alpha)\Gamma(4-\alpha)}\right)\right) \times \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right). \tag{54}$$

Table 3 Comparison of the approximate solutions of equations (45) and (46) obtained by ADM and HVIM ($y = 0.4$).

t	x	$\alpha = 1.50$		$\alpha = 1.75$		$\alpha = 2$		
		u_{ADM}	u_{HVIM}	u_{ADM}	u_{HVIM}	u_{ADM}	u_{HVIM}	u_{Exact}
0.2	0.60	0.389381	0.39374	0.394632	0.395285	0.397353	0.397353	0.397353
	0.70	0.3894	0.393759	0.34651	0.395304	0.397372	0.397372	0.397372
	0.80	0.389439	0.393799	0.394691	0.395344	0.397412	0.397412	0.397412
	1.00	0.389638	0.394001	0.394893	0.395546	0.397616	0.397616	0.397616
0.4	0.60	0.741216	0.755194	0 764250	0.766262	0.778864	0.778364	0.778864
	0.70	0.741253	0.755231	0.764288	0.7663	0.778903	0.778903	0.778903
	0.80	0.741327	0.755307	0.764365	0.766377	0.778981	0.778981	0.778981
	1.00	0.741707	0.755694	0.764756	0.766769	0.77938	0.77938	0.77938
0.6	0.60	1.0426	1.06258	1.09283	1.09534	1.12932	1.12932	1.12932
	0.70	1.04265	1.06263	1.09288	1.09539	1.12938	1.12938	1.12938
	0.80	1.04276	1.06274	1.09299	1.0955	1.12949	1.12949	1.12949
	1.00	1.04329	1.06229	1.09355	1.09606	1.13007	1.13007	1.13007

Table 4 Comparison of the approximate solutions of equations (45) and (46) obtained by ADM and HVIM ($y = 0.8$).

t	x	$\alpha = 1.50$		$\alpha = 1.75$		$\alpha = 2$		
		u_{ADM}	u_{HVIM}	u_{ADM}	u_{HVIM}	u_{ADM}	u_{HVIM}	u_{Exact}
0.2	0.6	0.389945	0.395355	0.394703	0.39536	0.397424	0.397424	0.397424
	0.7	0.389947	0.395375	0.394722	0.395379	0.397443	0.397443	0.397443
	0.8	0.389509	0.395414	0.394762	0.395429	0.397483	0.397483	0.397483
	1.0	0.389708	0.395617	0.394064	0.395622	0.397687	0.397687	0 397687
0.4	0.6	0.741349	0.766399	0.764387	0 766575	0.779004	0.779004	0.779004
	0.7	0.741386	0.766437	0.764325	0.766613	0.779042	0.779042	0.779042
	0.8	0.74146	0.766514	0.764501	0.76669	0.77912	0.77912	0.77912
	1.0	0.74184	0.766906	0.764893	0.767683	0.779519	0.779519	0.779519
0.6	0.6	1.04279	1.09553	1.09302	1.09702	1.12953	1.12953	1.12953
	0.7	1.04284	1.09559	1.09308	1.09707	1.12958	1.12958	1.12958
	0.8	1.04295	1.0957	1.09319	1.09718	1.1297	1.1297	1.1297
	1.0	1.04348	1.09626	1.09375	1.09774	1.13027	1.13027	1.13027

$$u_3 = \left(t - \frac{t^3}{2} + \frac{t^5}{40} - \frac{t^7}{7!} + t^{5-\alpha}\left(\frac{3}{\Gamma(5-\alpha)} - \frac{3}{(5-\alpha)\Gamma(4-\alpha)}\right) + t^{7-2\alpha}\left(\frac{1}{(7-2\alpha)\Gamma(6-2\alpha)} - \frac{1}{\Gamma(7-2\alpha)}\right) + A_1(\alpha)t^{7-\alpha}\right) \times \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right), \quad (55)$$

and so on, in the same manner the rest of the components of the iteration formula (52) can be obtained. The fourth term approximate solution is given by:

$$u = \left(t - \frac{t^3}{2} + \frac{t^5}{40} - \frac{t^7}{7!} + t^{5-\alpha}\left(\frac{3}{\Gamma(5-\alpha)} - \frac{3}{(5-\alpha)\Gamma(4-\alpha)}\right) + t^{7-2\alpha}\left(\frac{1}{(7-2\alpha)\Gamma(6-2\alpha)} - \frac{1}{\Gamma(7-2\alpha)}\right) + A_1(\alpha)t^{7-\alpha}\right) \times \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right). \quad (56)$$

It is obvious that for $\alpha = 2$, ADM solution (51) and HVIM solution (56) are identical. The exact solution of Eq. (43) for $\alpha = 2$ is $u = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\sin t$.

Table 3 for $y = 0.4$ and Table 4 for $y = 0.8$ show the approximate solutions for Eq. (45) obtained for different values of α using the ADM and HVIM. The values of $\alpha = 2$ is the only case for which we know the exact solution. It is clear from the table that our approximate solutions using the methods are in good agreement with the exact values. As in the previous applications only the fourth-order term of HVIM solution and only four terms of the ADM series were used in evaluating the approximate solutions for Tables 3 and 4.

5. Conclusions

In this paper, the ADM and HVIM were use to obtain the analytical/numerical solution of time-fractional fourth-order partial differential equations with variable coefficients. To illustrate the analytical and numerical results, we used MATHEMATICA. There are few important points to make here. Firstly, ADM and HVIM provide the solution in terms of easily computable components. These methods are powerful and efficient techniques in finding exact and approximate solutions for linear and nonlinear models. They provide more realistic solutions that converges very rapidly in real physical problems. The analytic solutions of three applications, found by these two methods, are compared with each other as well as with exact solutions. The numerical results show that the solutions are in good agreement with each other and with their respective exact solutions. Secondly, the methods were used in a direct way without using linearization, perturbation or restrictive assumption. Finally, the recent appearance of fractional partial differential equations as in applications 1 and 2 modeled in some fields as transverse vibrations (Gorman, 1975) make it necessary to investigate the method of solutions for such equations analytical and numerical. The selection of the initial approximation is both one of the simplest and one of the most important choices we can make when employing the ADM and HVIM. The initial approximation should satisfy the initial and/or the boundary data for the problem. We remark that the concept of a best initial guess is a bit superfluous. Indeed, the best initial approximation would simply be the exact solution.

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Appendix A

$$A_1(\alpha) = \left(\frac{1}{(7-\alpha)\Gamma(6-\alpha)} - \frac{1}{\Gamma(7-\alpha)} + \frac{1}{(7-\alpha)\Gamma(5-\alpha)} - \frac{1}{(6-\alpha)\Gamma(5-\alpha)} - \frac{1}{(7-\alpha)(5-\alpha)\Gamma(4-\alpha)} + \frac{1}{(6-\alpha)(5-\alpha)\Gamma(4-\alpha)}\right)$$

$$A_2(\alpha) = \left(\frac{1}{(6-\alpha)\Gamma(5-\alpha)} - \frac{1}{\Gamma(6-\alpha)} + \frac{1}{(6-\alpha)\Gamma(4-\alpha)} - \frac{1}{(5-\alpha)\Gamma(4-\alpha)} - \frac{1}{(6-\alpha)(4-\alpha)\Gamma(3-\alpha)} + \frac{1}{(5-\alpha)(4-\alpha)\Gamma(3-\alpha)}\right)$$

$$A_3(\alpha) = \left(\frac{1}{\Gamma(7-\alpha)} - \frac{1}{(7-\alpha)\Gamma(6-\alpha)} - \frac{1}{(7-\alpha)\Gamma(5-\alpha)} + \frac{1}{(6-\alpha)\Gamma(5-\alpha)} + \frac{1}{(7-\alpha)(5-\alpha)\Gamma(4-\alpha)} - \frac{1}{(6-\alpha)(5-\alpha)\Gamma(4-\alpha)}\right)$$

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